# OLYMON

Produced by the **Canadian Mathematical Society** and the **Department of Mathematics of the University of Toronto**.

**Issue 5:9**

**October, 2004**

## PROBLEMS FOR OCTOBER

Please send your solution to
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no later than December 5, 2004. It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

339. Let $a, b, c$ be integers with $abc \neq 0$, and $u, v, w$ be integers, not all zero, for which

$$au^2 + bv^2 + cw^2 = 0 \ .$$

Let $r$ be any rational number. Prove that the equation

$$ax^2 + by^2 + cz^2 = r$$

is solvable.

340. The lock on a safe consists of three wheels, each of which may be set in eight different positions. Because of a defect in the safe mechanism, the door will open if any two of the three wheels is in the correct position. What is the smallest number of combinations which must be tried by someone not knowing the correct combination to guarantee opening the safe?

341. Let $s, r, R$ respectively specify the semiperimeter, inradius and circumradius of a triangle $ABC$.

(a) Determine a necessary and sufficient condition on $s, r, R$ that the sides $a, b, c$ of the triangle are in arithmetic progression.

(b) Determine a necessary and sufficient condition on $s, r, R$ that the sides $a, b, c$ of the triangle are in geometric progression.

342. Prove that there are infinitely many solutions in positive integers of the system

$$a + b + c = x + y$$
$$a^3 + b^3 + c^3 = x^3 + y^3 \ .$$

343. A sequence $\{a_n\}$ of integers is defined by

$$a_0 = 0 \ , \quad a_1 = 1 \ , \quad a_n = 2a_{n-1} + a_{n-2}$$

for $n > 1$. Prove that, for each nonnegative integer $k$, $2^k$ divides $a_n$ if and only if $2^k$ divides $n$.

344. A function $f$ defined on the positive integers is given by

$$f(1) = 1 \ , \quad f(3) = 3 \ , \quad f(2n) = f(n) \ ,$$

$$f(4n+1) = 2f(2n+1) - f(n)$$
$$f(4n+3) = 3f(2n+1) - 2f(n) \ ,$$

for each positive integer $n$. Determine, with proof, the number of positive integers no exceeding 2004 for which $f(n) = n$.

345. Let $\mathfrak{C}$ be a cube with edges of length 2. Construct a solid figure with fourteen faces by cutting off all eight corners of $\mathfrak{C}$, keeping the new faces perpendicular to the diagonals of the cuhe and keeping the newly formed faces identical. If the faces so formed all have the same area, determine the common area of the faces.

## Solutions

318. Solve for integers $x, y, z$ the system

$$1 = x + y + z = x^3 + y^3 + z^2 \ .$$

[Note that the exponent of $z$ on the right is 2, not 3.]

*Solution 1.* Substituting the first equation into the second yields that

$$x^3 + y^3 + [1 - (x+y)]^2 = 1$$

which holds if and only if

$$\begin{aligned} 0 &= (x+y)(x^2 - xy + y^2) + (x+y)^2 - 2(x+y) \\ &= (x+y)(x^2 - xy + y^2 + x + y - 2) \\ &= (1/2)(x+y)[(x-y)^2 + (x+1)^2 + (y+1)^2 - 6] \ . \end{aligned}$$

It is straightforward to check that the only possibilities are that either $y = -x$ or $(x, y) = (0, -2), (-2, 0)$ or $(x, y) = (-3, -2), (-2, -3)$ or $(x, y) = (1, 0), (0, 1)$. Hence

$$(x, y, z) = (t, -t, 1), (1, 0, 0), (0, 1, 0), (-2, -3, 6), (-3, -2, 6), (-2, 0, 3), (0, -2, 3)$$

where $t$ is an arbitrary integer. These all check out.

*Solution 2.* As in Solution 1, we find that either $x + y = 0, z = 1$ or $x^2 + (1-y)x + (y^2 + y - 2) = 0$. The discriminant of the quadratic in $x$ is

$$-3y^2 - 6y + 9 = -3(y+1)^2 + 12 \ ,$$

which is nonnegative when $|y + 1| \leq 4$. Checking out the possibilities leads to the solution.

*Solution 3.*
$$\begin{aligned} (1-z)(1+z) &= 1 - z^2 = x^3 + y^3 \\ &= (x+y)[(x+y)^2 - 3xy] = (1-z)[(1-z)^2 - 3xy] \ , \end{aligned}$$

whence either $z = 1$ or $3xy = (1 - 2z + z^2) - (1 + z) = z(z-3)$. The former case yields $(x, y, z) = (x, -x, 1)$ while the latter yields

$$x + y = 1 - z \qquad xy = \frac{1}{3}z(z-3) \ .$$

Thus, we must have that $z \equiv 0 \pmod 3$ and that $x, y$ are roots of the quadratic equation

$$t^2 - (1-z)t + \frac{z(z-3)}{3} = 0 \ .$$

The discriminant of this equation is $[12-(z-3)^2]/3$. Thus, the only possibilities are that $z = 0, 3, 6$; checking these gives the solutions.

319. Suppose that $a, b, c, x$ are real numbers for which $abc \neq 0$ and

$$\frac{xb + (1-x)c}{a} = \frac{xc + (1-x)a}{b} = \frac{xa + (1-x)b}{c} \ .$$

Is it true that, necessarily, $a = b = c$?

*Comment.* There was an error in the original formulation of this problem, and it turns out that the three numbers $a, b, c$ are not necessarily equal. Note that in the problem, $a, b, c, x$ all have the same status. Some solvers, incorrectly, took the given conditions as an identity in $x$, so that they assumed that the equations held for some $a, b, c$ and all $x$.

*Solution 1.* Suppose first that $a + b + c \neq 0$. Then the three equal fractions are equal to the sum of their numerators divided by the sum of the denominators [why?]:

$$\frac{x(a+b+c) + (1-x)(a+b+c)}{a+b+c} = 1 \ .$$

Hence $a = xb + (1-x)c$, $b = xc + (1-x)a$, $c = xa + (1-x)b$, from which $x(b-c) = (a-c)$, $x(c-a) = (b-a)$, $x(a-b) = (c-b)$. Multiplying these three equations together yields that $x^3(b-c)(c-a)(a-b) = (a-c)(b-a)(c-b)$. Therefore, either $x = -1$ or at least two of $a, b, c$ are equal.

If $x = -1$, then $a + b = 2c$, $b + c = 2a$ and $c + a = 2b$. This implies for example that $a - c = 2(c - a)$, whence $a = c$. Similarly, $a = b$ and $b = c$. Suppose on the other hand that, say, $a = b$; then $b = c$ and $c = a$.

The remaining case is that $a + b + c = 0$. Then each entry and sum of pairs of entries is nonzero, and

$$\frac{xa + (1-x)b}{-(a+b)} = \frac{x(-a-b) + (1-x)a}{b}$$

$$\Longrightarrow xab + (1-x)b^2 = x(a+b)^2 - (1-x)(a^2 + ab)$$
$$\Longrightarrow (1-x)(a^2 + ab + b^2) = x(a^2 + ab + b^2) \ .$$

Since $2(a^2 + ab + b^2) = (a+b)^2 + a^2 + b^2 > 0$, $1 - x = x$ and $x = 1/2$. But in this case, the equations become

$$\frac{b+c}{2a} = \frac{c+a}{2b} = \frac{a+b}{2c}$$

each member of which takes the value $-1/2$ for all $a, b, c$ for which $a + b + c = 0$.

Hence, the equations hold if and only if either $a = b = c$ and $x$ is arbitrary, or $x = 1/2$ and $a + b + c = 0$.

*Comment.* On can get the first part another way. If $d$ is the common value of the three fractions, then

$$xb + (1-x)c = da \ ; \quad xc + (1-x)a = db \ ; \quad xa + (1-x)b = dc \ .$$

Adding these yeilds that $a + b + c = d(a + b + c)$, whence $d = 1$ or $a + b + c = 0$.

*Solution 2.* The first inequality leads to

$$xb^2 + (1-x)bc = xac + (1-x)a^2$$

or

$$x(a^2+b^2)-x(a+b)c=a^2-bc\ .$$

Similarly

$$x(c^2+a^2)-x(c+a)b=b^2-ca\ ;$$

$$x(b^2+c^2)-x(b+c)a=c^2-ab\ .$$

Adding these three equations together leads to

$$2x[(a-b)^2+(b-c)^2+(c-a)^2]=(a-b)^2+(b-c)^2+(c-a)^2\ .$$

Hence, either $a=b=c$ or $x=1/2$.

If $x=1/2$, then for some constant $k$,

$$\frac{b+c}{a}=\frac{c+a}{b}=\frac{a+b}{c}=k\ ,$$

whence

$$-ka+b+c=a-kb+c=a+b-kc=0\ .$$

Add the three left members to get

$$(2-k)(a+b+c)=0\ .$$

Therefore, $k=2$ or $a+b+c=0$. If $k=2$, then $a=b=c$, as in Solution 1. If $a+b+c=0$, then $k=-1$ for any relevant values of $a,b,c$. Hence, either $a=b=c$ or $x=1/2$ and $a+b+c=0$.

320. Let $L$ and $M$ be the respective intersections of the internal and external angle bisectors of the triangle $ABC$ at $C$ and the side $AB$ produced. Suppose that $CL=CM$ and that $R$ is the circumradius of triangle $ABC$. Prove that

$$|AC|^2+|BC|^2=4R^2\ .$$

*Solution 1.* Since $\angle LCM=90^\circ$ and $CL=CM$, we have that $\angle CLM=\angle CML=45^\circ$. Let $\angle ACB=2\theta$. Then $\angle CAB=45^\circ-\theta$ and $\angle CBA=45^\circ+\theta$. It follows that

$$\begin{aligned}|BC|^2+|AC|^2&=(2R\sin\angle CAB)^2+(2R\sin\angle CBA)^2\\&=4R^2(\sin^2(45^\circ-\theta)+\sin^2(45^\circ+\theta))\\&=4R^2(\sin^2(45^\circ-\theta)+\cos^2(45^\circ-\theta))=4R^2\ .\end{aligned}$$

*Solution 2.* [B. Braverman] $\angle ABC$ is obtuse [why?]. Let $AD$ be a diameter of the circumcircle of triangle $ABC$. Then $\angle ADC=\angle CBM=45^\circ+\angle LCB$ (since $ABCD$ is concyclic). Since $\angle ACD=90^\circ$, $\angle DAC=45^\circ-\angle LCB=\angle CAB$. Hence, chords $DC$ and $CB$, subtending equal angles at the circumference of the circumcircle, are equal. Hence

$$4R^2=|AC|^2+|CD|^2=|AC|^2+|BC|^2\ .$$

321. Determine all positive integers $k$ for which $k^{1/(k-7)}$ is an integer.

*Solution.* When $k=1$, the number is an integer. Suppose that $2\le k\le 6$. Then $k-7<0$ and so

$$0<k^{1/(k-7)}=1/(k^{1/7-k})<1$$

and the number is not an integer. When $k=7$, the expression is undefined.

When $k = 8$, the number is equal to 8, while if $k = 9$, the number is equal to 3. When $k = 10$, the number is equal to $10^{1/3}$, which is not an integer [why?].

Suppose that $k \geq 11$. We establish by induction that $k < 2^{k-7}$. This is clearly true when $k = 11$. Suppose it holds for $k = m \geq 11$. Then

$$m + 1 < 2^{m-7} + 2^{m-7} = 2^{(m+1)-7} ;$$

the desired result follows by induction. Thus, when $k \geq 11$, $1 < k^{1/(k-7)} < 2$ and the number is not an integer.

Thus, the number is an integer if and only if $k = 1, 8, 9$.

322. The real numbers $u$ and $v$ satisfy

$$u^3 - 3u^2 + 5u - 17 = 0$$

and

$$v^3 - 3v^2 + 5v + 11 = 0 .$$

Determine $u + v$.

*Solution 1.* The equations can be rewritten

$$u^3 - 3u^2 + 5u - 3 = 14 ,$$

$$v^3 - 3v^2 + 5v - 3 = -14 .$$

These can be rewritten as

$$(u - 1)^3 + 2(u - 1) = 14 ,$$
$$(v - 1)^3 + 2(v - 1) = -14 .$$

Adding these equations yields that

$$\begin{aligned} 0 &= (u - 1)^3 + (v - 1)^3 + 2(u + v - 2) \\ &= (u + v - 2)[(u - 1)^2 - (u - 1)(v - 1) + (v - 1)^2 + 2] . \end{aligned}$$

Since the quadratic $t^2 - st + s^2$ is always positive [why?], we must have that $u + v = 2$.

*Solution 2.* Adding the two equations yields

$$\begin{aligned} 0 &= (u^3 + v^3) - 3(u^2 + v^2) + 5(u + v) - 6 \\ &= (u + v)[(u + v)^2 - 3uv] - 3[(u + v)^2 - 2uv] + 5(u + v) - 6 \\ &= [(u + v)^3 - 3(u + v)^2 + 5(u + v) - 6] - 3uv(u + v - 2) \\ &= \frac{1}{2}(u + v - 2)[(u - v)^2 + (u - 1)^2 + (v - 1)^2 + 4] . \end{aligned}$$

Since the second factor is positive, we must have that $u + v = 2$.

*Solution 3.* [N. Horeczky] Since $x^3 - 3x^2 + 5x = (x - 1)^3 + 2(x - 1) + 3$ is an increasing function of $x$ (since $x - 1$ is increasing), the equation $x^3 - 3x^2 + 5x - 17 = 0$ has exactly one real solution, namely $x = u$. But

$$\begin{aligned} 0 &= v^3 - 3v^2 + 5v + 11 \\ &= (v - 2)^3 + 3(v - 2)^2 + 5(v - 2) + 17 \\ &= -[(2 - v)^3 - 3(2 - v)^2 + 5(2 - v) - 17] . \end{aligned}$$

Thus $x = 2 - v$ satisfies $x^3 - 3x^2 + 5x - 17 = 0$, so that $2 - v = u$ and $u + v = 2$.

*Comment.* One can see also that each of the two given equations has a unique real root by noting that the sum of the squares of the roots, given by the cofficients, is equal to $3^2 - 2 \times 5 = -1$.

*Solution 4.* [P. Shi] Let $m$ and $n$ be determined by $u + v = 2m$ and $u - v = 2n$. Then $u = m + n$, $v = m - n$, $u^2 + v^2 = 2m^2 + 2n^2$, $u^2 - v^2 = 4mn$, $u^2 + uv + v^2 = 3m^2 + n^2$, $u^2 - uv + v^2 = m^2 + 3n^2$, $u^3 + v^3 = 2m(m^2 + 3n^2)$ and $u^3 - v^3 = 2n(3m^2 + n^2)$. Adding the equations yields that

$$\begin{aligned} 0 &= (u^3 + v^3) - 3(u^2 + v^2) + 5(u + v) - 6 \\ &= 2m^3 + 6mn^2 - 6m^2 - 6n^2 + 10m - 6 \\ &= 6(m - 1)n^2 + 2(m^3 - 3m^2 + 5m - 3) \\ &= 6(m - 1)n^2 + 2(m - 1)(m^2 - 2m + 3) \\ &= 2(m - 1)[3n^2 + (m - 1)^2 + 2] \ . \end{aligned}$$

Hence $m = 1$.

323. Alfred, Bertha and Cedric are going from their home to the country fair, a distance of 62 km. They have a motorcycle with sidecar that together accommodates at most 2 people and that can travel at a maximum speed of 50 km/hr. Each can walk at a maximum speed of 5 km/hr. Is it possible for all three to cover the 62 km distance within 3 hours?

*Solution 1.* We consider the following regime. $A$ begins by walking while $B$ and $C$ set off on the motorcycle for a time of $t_1$ hours. Then $C$ dismounts from the motorcycle and continues walking, while $B$ drives back to pick up $A$ for a time of $t_2$ hours. Finally, $B$ and $A$ drive ahead until they catch up with $C$, taking a time of $t_3$ hours. Suppose that all of this takes $t = t_1 + t_2 + t_3$ hours.

The distance from the starting point to the point where $B$ picks up $A$ is given by

$$5(t_1 + t_2) = 50(t_1 - t_2)$$

km, and the distance from the point where $B$ drops off $C$ until the point where they all meet again is given by

$$5(t_2 + t_3) = 50(t_3 - t_2) \ .$$

Hence $45t_3 = 45t_1 = 55t_2$, so that $t_1 = t_3 = (11/9)t_2$ and so $t = (31/9)t_2$ and

$$t_1 = \frac{11}{31}t \ , \quad t_2 = \frac{9}{31}t \ , \quad t_3 = \frac{11}{31}t \ .$$

The total distance travelled in the $t$ hours is equal to

$$50t_1 + 5(t_2 + t_3) = \frac{650}{31}$$

kilometers. In three hours, they can travel $1950/31 = 60 + (90/31) > 62$ kilometers in this way, so that all will reach the fair before the three hours are up.

*Solution 2.* Follow the same regime as in Solution 1. Let $d$ be the distance from the start to the point where $B$ drops $C$ in kilometers. The total time for for $C$ to go from start to finish, namely

$$\frac{d}{50} + \frac{62 - d}{5}$$

hours, and we wish this to be no greater than 3. The condition is that $d \geq 470/9$.

The time for $B$ to return to pick up $A$ after dropping $C$ is $9d/550$ hours in which he covers a distance of $9d/11$ km. The total distance travelled by the motorcycle is

$$d + \frac{9d}{11} + (62 - \frac{2d}{11}) = \frac{18d + 682}{11}$$

km, and this is covered in

$$\frac{18d+682}{550}$$

hours. To get $A$ and $B$ to their destinations on time, we wish this to not exceed 3; the condition for this is that $d \leq 484/9$. Thus, we can get everyone to the fair on time if

$$\frac{470}{9} \leq d \leq \frac{484}{9} \ .$$

Thus, if $d = 53$, for example, we can achieve the desired journey.

*Solution 3.* [D. Dziabenko] Suppose that $B$ and $C$ take the motorcycle for exactly 47/45 hours while $A$ walks after them. After 47/45 hours, $B$ leaves $C$ to walk the rest of the way, while $B$ drives back to pick up $A$. $C$ reaches the destination in exactly

$$\frac{62-(47/45)50}{5} + \frac{47}{45} = 3$$

hours. Since $B$ and $A$ start and finish at the same time, it suffices to check that that $B$ reaches the fair on time. When $B$ drops $C$ off, $B$ and $A$ are 47 km apart. It takes $B$ 47/55 hours to return to pick up $A$. At this point, they are now

$$62 - 5\left(\frac{47}{45}+\frac{47}{55}\right) = 62 - 47\left(\frac{20}{99}\right) = \frac{5198}{99}$$

km from the fair, which they will reach in a further

$$\frac{5198}{99\times 50} = \frac{2599}{2475}$$

hours. The total travel time for $A$ and $B$ is

$$\frac{47}{45}+\frac{47}{55}+\frac{1}{50}\left[62-5\left(\frac{47}{45}+\frac{47}{55}\right)\right]$$
$$= \frac{9\times 47}{10\times 5}\left[\frac{1}{9}+\frac{1}{11}\right] + \frac{31}{25} = \frac{517+423+682}{550} = \frac{811}{275}$$

hours. This is less than three hours.

324. The base of a pyramid $ABCDV$ is a rectangle $ABCD$ with $|AB| = a$, $|BC| = b$ and $|VA| = |VB| = |VC| = |VD| = c$. Determine the area of the intersection of the pyramid and the plane parallel to the edge $VA$ that contains the diagonal $BD$.

*Solution 1.* A dilation with centre C and factor 1/2 takes $A$ to $S$, the centre of the square and $V$ to $M$, the midpoint of $VC$. The plane of intersection is the plane that contains triangle $BMD$. Since $BM$ is a median of triangle $BVC$ with sides $c, c, b$, its length is equal to $\frac{1}{2}\sqrt{2b^2+c^2}$ [why?]; similarly, $|DM| = \frac{1}{2}\sqrt{2a^2+c^2}$. Also, $|BD| = \sqrt{a^2+b^2}$. Let $\theta = \angle BMD$. Then, by the law of Cosines,

$$\cos\theta = \frac{c^2-a^2-b^2}{\sqrt{2b^2+c^2}\sqrt{2a^2+c^2}} \ ,$$

whence

$$\sin\theta = \frac{\sqrt{4c^2(a^2+b^2)-(a^2-b^2)^2}}{\sqrt{2b^2+c^2}\sqrt{2a^2+c^2}} \ .$$

The required area is

$$\frac{1}{2}|BM||DM|\sin\theta = \frac{1}{8}\sqrt{4c^2(a^2+b^2)-(a^2-b^2)^2} \ .$$

*Comment.* One can also use Heron's formula to get the area of the triangle, but this is more labourious. Another method is to calculate $(1/2)|BD||MN|$, where $N$ is the foot of the perpendicular from $M$ to $BD$, Note that, when $a \neq b$, $N$ is not the same as $S$ [do you see why?]. If $d = |BD|$ and $x = |SN|$ and, say $|MB| \leq |MD|$, then

$$|MN|^2 = |MB|^2 - \left(\frac{d}{2} - x\right)^2 = |MD|^2 - \left(\frac{d}{2} + x\right)^2$$

whence

$$x = \frac{|MD|^2 - |MB|^2}{2d} \ .$$

If follows that

$$|MN|^2 = \frac{2a^2b^2 - a^4 - b^4 + 4a^2c^2 + 4b^2c^2}{16(a^2 + b^2)} \ .$$