OLYMON

VOLUME 6

2005

Problems 353-422

This is the *Mathematical Olympiads Correspondence Program* sponsored by the Canadian Mathematical Society and the University of Toronto Department of Mathematics. The organizer and editor is Edward J. Barbeau of the University of Toronto Department of Mathematics, and the problems and solutions for this volume of Olymon were prepared by **Edward J. Barbeau** of the University of Toronto, **Valeria Pandelieva** in Ottawa and **Lily Yen** in Burnaby, BC.

Notes: The absolute value $|x|$ is equal to x when x is nonnegative and $-x$ when x is negative; always $|x| \geq 0$. The *floor of* x, denoted by $\lfloor x \rfloor$ is equal to the greatest integer that does not exceed x. For example, $\lfloor 5.34 \rfloor = 5$, $\lfloor -2.3 \rfloor - -3$ and $\lfloor 5 \rfloor = 5$. A geometric figure is said to be *convex* if the segment joining any two points inside the figure also lies inside the figure.

353. The two shortest sides of a right-angled triangle, a and b, satisfy the inequality:

$$\sqrt{a^2 - 6a\sqrt{2} + 19} + \sqrt{b^2 - 4b\sqrt{3} + 16} \leq 3 \ .$$

Find the perimeter of this triangle.

354. Let ABC be an isosceles triangle with $AC = BC$ for which $|AB| = 4\sqrt{2}$ and the length of the median to one of the other two sides is 5. Calculate the area of this triangle.

355. (a) Find all natural numbers k for which $3^k - 1$ is a multiple of 13.

(b) Prove that for any natural number k, $3^k + 1$ is not a multiple of 13.

356. Let a and b be real parameters. One of the roots of the equation $x^{12} - abx + a^2 = 0$ is greater than 2. Prove that $|b| > 64$.

357. Consider the circumference of a circle as a set of points. Let each of these points be coloured red or blue. Prove that, regardless of the choice of colouring, it is always possible to inscribe in this circle an isosceles triangle whose three vertices are of the same colour.

358. Find all integers x which satisfy the equation

$$\cos\left(\frac{\pi}{8}(3x - \sqrt{9x^2 + 160x + 800})\right) = 1 \ .$$

359. Let ABC be an acute triangle with angle bisectors AA_1 and BB_1, with A_1 and B_1 on BC and AC, respectively. Let J be the intersection of AA_1 and BB_1 (the incentre), H be the orthocentre and O the circumcentre of the triangle ABC. The line OH intersects AC at P and BC at Q. Given that C, A_1, J and B_1 are vertices of a concyclic quadrilateral, prove that $PQ = AP + BQ$.

360. Eliminate θ from the two equations

$$x = \cot\theta + \tan\theta$$

$$y = \sec\theta - \cos\theta \ ,$$

to get a polynomial equation satisfied by x and y.

361. Let $ABCD$ be a square, M a point on the side BC, and N a point on the side CD for which $BM = CN$. Suppose that AM and AN intersect BD and P and Q respectively. Prove that a triangle can be constructed with sides of length $|BP|$, $|PQ|$, $|QD|$, one of whose angles is equal to $60°$.

362. The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B, C meet the circle again at U, V, W, respectively. Prove that the area of triangle UVW is not less than the area of triangle ABC.

363. Suppose that x and y are positive real numbers. Find all real solutions of the equation

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} = \sqrt{xy} + \frac{x+y}{2} \ .$$

364. Determine necessary and sufficient conditions on the positive integers a and b such that the vulgar fraction a/b has the following property: *Suppose that one successively tosses a coin and finds at one time, the fraction of heads is less than a/b and that at a later time, the fraction of heads is greater than a/b; then at some intermediate time, the fraction of heads must be exactly a/b.*

365. Let $p(z)$ be a polynomial of degree greater than 4 with complex coefficients. Prove that $p(z)$ must have a pair u, v of roots, not necessarily distinct, for which the real parts of both u/v and v/u are positive. Show that this does not necessarily hold for polynomials of degree 4.

366. What is the largest real number r for which

$$\frac{x^2+y^2+z^2+xy+yz+zx}{\sqrt{x}+\sqrt{y}+\sqrt{z}} \geq r$$

holds for all positive real values of x, y, z for which $xyz = 1$.

367. Let a and c be fixed real numbers satisfying $a \leq 1 \leq c$. Determine the largest value of b that is consistent with the condition

$$a + bc \leq b + ac \leq c + ab \ .$$

368. Let A, B, C be three distinct points of the plane for which $AB = AC$. Describe the locus of the point P for which $\angle APB = \angle APC$.

369. $ABCD$ is a rectangle and APQ is an inscribed equilateral triangle for which P lies on BC and Q lies on CD.

(a) For which rectangles is the configuration possible?

(b) Prove that, when the configuration is possible, then the area of triangle CPQ is equal to the sum of the areas of the triangles ABP and ADQ.

370. A deck of cards has nk cards, n cards of each of the colours C_1, C_2, $\cdots$, C_k. The deck is thoroughly shuffled and dealt into k piles of n cards each, P_1, P_2, $\cdots$, P_k. A game of solitaire proceeds as follows: The top card is drawn from pile P_1. If it has colour C_i, it is discarded and the top card is drawn from pile P_i. If it has colour C_j, it is discarded and the top card is drawn from pile P_j. The game continues in this way, and will terminate when the nth card of colour C_1 is drawn and discarded, as at this point, there are no further cards left in pile P_1. What is the probability that every card is discarded when the game terminates?

371. Let X be a point on the side BC of triangle ABC and Y the point where the line AX meets the circumcircle of triangle ABC. Prove or disprove: if the length of XY is maximum, then AX lies between the median from A and the bisector of angle BAC.

372. Let b_n be the number of integers whose digits are all $1, 3, 4$ and whose digits sum to n. Prove that b_n is a perfect square when n is even.

373. For each positive integer n, define

$$a_n = 1 + 2^2 + 3^3 + \cdots + n^n \ .$$

Prove that there are infinitely many values of n for which a_n is an odd composite number.

374. What is the maximum number of numbers that can be selected from $\{1, 2, 3, \cdots, 2005\}$ such that the difference between any pair of them is not equal to 5?

375. Prove or disprove: there is a set of concentric circles in the plane for which both of the following hold:

(i) each point with integer coordinates lies on one of the circles;

(ii) no two points with integer coefficients lie on the same circle.

376. A soldier has to find whether there are mines buried within or on the boundary of a region in the shape of an equilateral triangle. The effective range of his detector is one half of the height of the triangle. If he starts at a vertex, explain how he can select the shortest path for checking that the region is clear of mines.

377. Each side of an equilateral triangle is divided into 7 equal parts. Lines through the division points parallel to the sides divide the triangle into 49 smaller equilateral triangles whose vertices consist of a set of 36 points. These 36 points are assigned numbers satisfying both the following conditions:

(a) the number at the vertices of the original triangle are 9, 36 and 121;

(b) for each rhombus composed of two small adjacent triangles, the sum of the numbers placed on one pair of opposite vertices is equal to the sum of the numbers placed on the other pair of opposite vertices.

Determine the sum of all the numbers. Is such a choice of numbers in fact possible?

378. Let $f(x)$ be a nonconstant polynomial that takes only integer values when x is an integer, and let P be the set of all primes that divide $f(m)$ for at least one integer m. Prove that P is an infinite set.

379. Let n be a positive integer exceeding 1. Prove that, if a graph with $2n + 1$ vertices has at least $3n + 1$ edges, then the graph contains a circuit (*i.e.*, a closed non-self-intersecting chain of edges whose terminal point is its initial point) with an even number of edges. Prove that this statement does not hold if the number of edges is only $3n$.

380. Factor each of the following polynomials as a product of polynomials of lower degree with integer coefficients:

(a) $(x + y + z)^4 - (y + z)^4 - (z + x)^4 - (x + y)^4 + x^4 + y^4 + z^4$;

(b) $x^2(y^3 - z^3) + y^2(z^3 - x^3) + z^2(x^3 - y^3)$;

(c) $x^4 + y^4 - z^4 - 2x^2y^2 + 4xyz^2$;

(d) $(yz + zx + xy)^3 - y^3z^3 - z^3x^3 - x^3y^3$;

(e) $x^3y^3 + y^3z^3 + z^3x^3 - x^4yz - xy^4z - xyz^4$;

(f) $2(x^4 + y^4 + z^4 + w^4) - (x^2 + y^2 + z^2 + w^2)^2 + 8xyzw$;

(g) $6(x^5 + y^5 + z^5) - 5(x^2 + y^2 + z^2)(x^3 + y^3 + z^3)$.

381. Determine all polynomials $f(x)$ such that, for some positive integer k,

$$f(x^k) - x^3 f(x) = 2(x^3 - 1)$$

for all values of x.

382. Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of these intervals. Show that S is a finite union of disjoint intervals of total length not less than 1.

383. Place the numbers $1, 2, \cdots, 9$ in a 3×3 unit square so that

(a) the sums of numbers in each of the first two rows are equal;

(b) the sum of the numbers in the third row is as large as possible;

(c) the column sums are equal;

(d) the numbers in the last row are in descending order.

Prove that the solution is unique.

384. Prove that, for each positive integer n,

$$(3 - 2\sqrt{2})(17 + 12\sqrt{2})^n + (3 + 2\sqrt{2})(17 - 12\sqrt{2})^n - 2$$

is the square of an integer.

385. Determine the minimum value of the product $(a + 1)(b + 1)(c + 1)(d + 1)$, given that $a, b, c, d \geq 0$ and

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 1 \ .$$

386. In a round-robin tournament with at least three players, each player plays one game against each other player. The tournament is said to be *competitive* if it is impossible to partition the players into two sets, such that each player in one set beat each player in the second set. Prove that, if a tournament is not competitive, it can be made so by reversing the result of a single game.

387. Suppose that a, b, u, v are real numbers for which $av - bu = 1$. Prove that

$$a^2 + u^2 + b^2 + v^2 + au + bv \geq \sqrt{3} \ .$$

Give an example to show that equality is possible. (Part marks will be awarded for a result that is proven with a smaller bound on the right side.)

388. A class with at least 35 students goes on a cruise. Seven small boats are hired, each capable of carrying 300 kilograms. The combined weight of the class is 1800 kilograms. It is determined that any group of 35 students can fit into the boats without exceeding the capacity of any one of them. Prove that it is unnecessary to leave any student off the cruise.

389. Let each of m distinct points on the positive part of the $x-$axis be joined by line segments to n distinct points on the positive part of the $y-$axis. Obtain a formula for the number of intersections of these segments (exclusive of endpoints), assuming that no three of the segments are concurrent.

390. Suppose that $n \geq 2$ and that $x_1, x_2, \cdots, x_n$ are positive integers for which $x_1 + x_2 + \cdots + x_n = 2(n + 1)$. Show that there exists an index r with $0 \leq r \leq n - 1$ for which the following $n - 1$ inequalities hold:

$$x_{r+1} \leq 3$$

$$x_{r+1} + x_{r+2} \leq 5$$

$$\cdots$$

$$x_{r+1} + x_{r+2} + \cdots + r_{r+i} \leq 2i + 1$$

$$\cdots$$

$$x_{r+1} + x_{r+2} + \cdots + x_n \leq 2(n - r) + 1$$

$$\cdots$$

$$x_{r+1} + \cdots + x_n + x_1 + \cdots + x_j \leq 2(n + j - r) + 1$$

$$\cdots$$

$$x_{r+1} + x_{r+2} + \cdots + x_n + x_1 + \cdots + x_{r-1} \le 2n - 1$$

where $1 \le i \le n - r$ and $1 \le j \le r - 1$. Prove that, if all the inequalities are strict, then r is unique, and that, otherwise, there are exactly two such r.

391. Show that there are infinitely many nonsimilar ways that a square with integer side lengths can be partitioned into three nonoverlapping polygons with integer side lengths which are similar, but no two of which are congruent.

392. Determine necessary and sufficient conditions on the real parameter a, b, c that

$$\frac{b}{cx+a} + \frac{c}{ax+b} + \frac{a}{bx+c} = 0$$

has exactly one real solution.

393. Determine three positive rational numbers x, y, z whose sum s is rational and for which $x - s^3$, $y - s^3$, $z - s^3$ are all cubes of rational numbers.

394. The average age of the students in Ms. Ruler's class is 17.3 years, while the average age of the boys is 17.5 years. Give a cogent argument to prove that the average age of the girls cannot also exceed 17.3 years.

395. None of the nine participants at a meeting speaks more than three languages. Two of any three speakers speak a common language. Show that there is a language spoken by at least three participants.

396. Place 32 white and 32 black checkers on a 8×8 square chessboard. Two checkers of different colours form a *related pair* if they are placed in either the same row or the same column. Determine the maximum and the minimum number of related pairs over all possible arrangements of the 64 checkers.

397. The altitude from A of triangle ABC intersects BC in D. A circle touches BC at D, intersectes AB at M and N, and intersects AC at P and Q. Prove that

$$(AM + AN) : AC = (AP + AQ) : AB\ .$$

398. Given three disjoint circles in the plane, construct a point in the plane so that all three circles subtend the same angle at that point.

399. Let n and k be positive integers for which $k < n$. Determine the number of ways of choosing k numbers from $\{1, 2, \cdots, n\}$ so that no three consecutive numbers appear in any choice.

400. Let a_r and b_r $(1 \le r \le n)$ be real numbers for which $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$ and

$$b_1 \ge a_1\ , \quad b_1 b_2 \ge a_1 a_2\ , \quad b_1 b_2 b_3 \ge a_1 a_2 a_3\ , \quad \cdots \quad , b_1 b_2 \cdots b_n \ge a_1 a_2 \cdots a_n\ .$$

Show that

$$b_1 + b_2 + \cdots + b_n \ge a_1 + a_2 + \cdots + a_n\ .$$

401. Five integers are arranged in a circle. The sum of the five integers is positive, but at least one of them is negative. The configuration is changed by the following moves: at any stage, a negative integer is selected and its sign is changed; this negative integer is added to each of its neighbours (*i.e.*, its absolute value is subtracted from each of its neighbours).

Prove that, regardless of the negative number selected for each move, the process will eventually terminate with all integers nonnegative in exactly the same number of moves with exactly the same configuration.

402. Let the sequences $\{x_n\}$ and $\{y_n\}$ be defined, for $n \geq 1$, by $x_1 = x_2 = 10$, $x_{n+2} = x_{n+1}(x_n + 1) + 1$ $(n \geq 1)$ and $y_1 = y_2 = -10$, $y_{n+2} = y_{n+1}(y_n + 1) + 1$ $(n \geq 1)$. Prove that there is no number that is a term of both sequences.

403. Let $f(x) = |1 - 2x| - 3|x + 1|$ for real values of x.

(a) Determine all values of the real parameter a for which the equation $f(x) = a$ has two different roots u and v that satisfy $2 \leq |u - v| \leq 10$.

(b) Solve the equation $f(x) = \lfloor x/2 \rfloor$.

404. Several points in the plane are said to be *in general position* if no three are collinear.

(a) Prove that, given 5 points in general position, there are always four of them that are vertices of a convex quadrilateral.

(b) Prove that, given 400 points in general position, there are at least 80 nonintersecting convex quadrilaterals, whose vertices are chosen from the given points. (Two quadrilaterals are nonintersecting if they do not have a common point, either in the interior or on the perimeter.)

(c) Prove that, given 20 points in general position, there are at least 969 convex quadrilaterals whose vertices are chosen from these points. (**Bonus:** Derive a formula for the number of these quadrilaterals given n points in general position.)

405. Suppose that a permutation of the numbers from 1 to 100, inclusive, is given. Consider the sums of all triples of consecutive numbers in the permutation. At most how many of these sums can be odd?

406. Let a, b. c be natural numbers such that the expression

$$\frac{a+1}{b} + \frac{b+1}{c} + \frac{c+1}{a}$$

is also equal to a natural number. Prove that the greatest common divisor of a, b and c, $\gcd(a,b,c)$, does not exceed $\sqrt[3]{ab + bc + ca}$, *i.e.*,

$$\gcd(a,b,c) \leq \sqrt[3]{ab + bc + ca}\ .$$

407. Is there a pair of natural numbers, x and y, for which

(a) $x^3 + y^4 = 2^{2003}$?

(b) $x^3 + y^4 = 2^{2005}$?

Provide reasoning for your answers to (a) and (b).

408. Prove that a number of the form $a000 \cdots 0009$ (with $n + 2$ digits for which the first digit a is followed by n zeros and the units digit is 9) cannot be the square of another integer.

409. Find the number of ways of dealing n cards to two persons $(n \geq 2)$, where the persons may receive unequal (positive) numbers of cards. Disregard the order in which the cards are received.

410. Prove that $\log n \geq k \log 2$, where n is a natural number and k the number of distinct primes that divide n.

411. Let b be a positive integer. How many integers are there, each of which, when expressed to base b, is equal to the sum of the squares of its digits?

412. Let A and B be the midpoints of the sides, EF and ED, of an equilateral triangle DEF. Extend AB to meet the circumcircle of triangle DEF at C. Show that B divides AC according to the golden section. (That is, show that $BC : AB = AB : AC$.)

413. Let I be the incentre of triangle ABC. Let A', B' and C' denote the intersections of AI, BI and CI, respectively, with the incircle of triangle ABC. Continue the process by defining I' (the incentre of triangle $A'B'C'$), then $A''B''C''$, *etc.*. Prove that the angles of triangle $A^{(n)}B^{(n)}C^{(n)}$ get closer and closer to $\pi/3$ as n increases.

414. Let $f(n)$ be the greatest common divisor of the set of numbers of the form $k^n - k$, where $2 \leq k$, for $n \geq 2$. Evaluate $f(n)$. In particular, show that $f(2n) = 2$ for each integer n.

415. Prove that

$$\cos\frac{\pi}{7} = \frac{1}{6} + \frac{\sqrt{7}}{6}\left(\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right) + \sqrt{3}\sin\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right)\right) .$$

416. Let P be a point in the plane.

(a) Prove that there are three points A, B, C for which $AB = BC$, $\angle ABC = 90°$, $|PA| = 1$, $|PB| = 2$ and $|PC| = 3$.

(b) Determine $|AB|$ for the configuration in (a).

(c) A rotation of $90°$ about B takes C to A and P to Q. Determine $\angle APQ$.

417. Show that for each positive integer n, at least one of the five numbers 17^n, 17^{n+1}, 17^{n+2}, 17^{n+3}, 17^{n+4} begins with 1 (at the left) when written to base 10.

418. (a) Show that, for each pair m,n of positive integers, the minimum of $m^{1/n}$ and $n^{1/m}$ does not exceed $3^{1/2}$.

(b) Show that, for each positive integer n,

$$\left(1 + \frac{1}{\sqrt{n}}\right)^2 \geq n^{1/n} \geq 1 .$$

(c) Determine an integer N for which

$$n^{1/n} \leq 1.00002005$$

whenever $n \geq N$. Justify your answer.

419. Solve the system of equations

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = t$$

for x, y, z not all equal. Determine xyz.

420. Two circle intersect at A and B. Let P be a point on one of the circles. Suppose that PA meets the second circle again at C and PB meets the second circle again at D. For what position of P is the length of the segment CD maximum?

421. Let $ABCD$ be a tetrahedron. Prove that

$$|AB| \cdot |CD| + |AC| \cdot |BD| \geq |AD| \cdot |BC| .$$

422. Determine the smallest two positive integers n for which the numbers in the set $\{1, 2, \cdots, 3n-1, 3n\}$ can be partitioned into n disjoint triples $\{x, y, z\}$ for which $x + y = 3z$.

Solutions.

353. The two shortest sides of a right-angled triangle, a and b, satisfy the inequality:

$$\sqrt{a^2 - 6a\sqrt{2} + 19} + \sqrt{b^2 - 4b\sqrt{3} + 16} \leq 3 \ .$$

Find the perimeter of this triangle.

Solution. The equation can be rewritten as

$$\sqrt{(a - 3\sqrt{2})^2 + 1} + \sqrt{(b - 2\sqrt{3})^2 + 4} \leq 3 \ .$$

Since the left side is at least equal to $1 + 2 = 3$, we must have equality and so $a = 3\sqrt{2}$ and $b = 2\sqrt{3}$. The hypotenuse of the triangle equal to $\sqrt{a^2 + b^2} = \sqrt{30}$, and so the perimeter is equal to $3\sqrt{2} + 2\sqrt{3} + \sqrt{30}$. ♠

354. Let ABC be an isosceles triangle with $AC = BC$ for which $|AB| = 4\sqrt{2}$ and the length of the median to one of the other two sides is 5. Calculate the area of this triangle.

Solution. Let M be the midpoint of BC, $\theta = \angle AMB$ and $x = |BM| = |MC|$. Then $|AC| = 2x$. By the Law of Cosines, $4x^2 = x^2 + 25 + 10x \cos \theta$ and $32 = x^2 + 25 - 10x \cos \theta$. Adding these two equations yields that $x^2 = 9$, so that $x = 3$. The height of the triangle from C is $\sqrt{4x^2 - 8} = \sqrt{28} = 2\sqrt{7}$. Hence the area of the triangle is $4\sqrt{14}$. ♠

355. (a) Find all natural numbers k for which $3^k - 1$ is a multiple of 13.

(b) Prove that for any natural number k, $3^k + 1$ is not a multiple of 13.

Solution 1. Let $k = 3q + r$. Since $3^3 \equiv 1 \pmod{13}$, $3^k - 1 \equiv 3^r - 1 \pmod{13}$ and $3^k + 1 \equiv 3^r + 1$ (mod 13). Since $3^0 = 1$, $3^2 = 9$, we see that only $3^k - 1$ is a multiple of 13 when k is a multiple of 3. ♠

Solution 2. Let p be a prime and $N = d_0 + d_1 p + \cdots + d_r p^r = (d_r d_{r-1} \cdots d_1 d_0)_p$ be an integer written to base p. Then $p^k = (100 \cdots 00)_p$, $p^k + 1 = (1000 \cdots 01)_p$ and $p^k - 1 = (\overline{p-1}, \overline{p-1}, \cdots \overline{p-1})_p$ where the first two have $k + 1$ digits and the last has k digits. Let $p = 3$, we see that $3^k - 1 = (222 \cdots 22)_3$ and $3^k + 1 = (100 \cdots 01)_3$. Since $13 = (111)_3$, we see that $3^k + 1$ is never a multiple of 13 and $3^k - 1$ is a multiple of 13 if and only if k is a multiple of 3. ♠

356. Let a and b be real parameters. One of the roots of the equation $x^{12} - abx + a^2 = 0$ is greater than 2. Prove that $|b| > 64$.

Solution 1. Clearly, $a \neq 0$. The equation can be rewritten $b = (x^{12} + a^2)/(ax)$. If $x > 2$, then

$$|b| = \frac{x^{12} + a^2}{|a|x} \geq \frac{2|a|x^6}{|a|x} = 2x^5 > 64 \ ,$$

by the arithmetic-geometric means inequality. ♠

Solution 2. [V. Krakovna] The equation can be rewritten

$$x^{12} + \left(\frac{bx}{2} - a\right)^2 = \frac{b^2x^2}{4} \ ,$$

whence $b^2x^2 = 4x^{12} + (bx - 2a)^2 \geq 4x^{12}$ and $b^2 \geq 4x^{10}$. If $|x| > 2$, then $b^2 \geq 2^{12}$ and so $|b| \geq 2^6$. ♠

357. Consider the circumference of a circle as a set of points. Let each of these points be coloured red or blue. Prove that, regardless of the choice of colouring, it is always possible to inscribe in this circle an isosceles triangle whose three vertices are of the same colour.

Solution 1. Consider any regular pentagon inscribed in the given circle. Since there are five vertices and only two options for their colours. the must be three vertices of the same colour. If they are adjacent, then two of the sides of the triangle they determine are sides of the pentagon, and so equal. If two are adjacent and the third opposite to the side formed by the first two, then once again they determine an isosceles triangle. As this covers all the possibilities, the result follows.

Solution 2. If at most finitely many points on the circumference are red, then it is possible to find an isosceles triangle with green vertices. (Why?) Suppose that there are infinitely many red points. Then there are two red points, P and Q, that are neither at the end of a diagonal nor two vertices of an inscribed equilateral triangle. Let U, V, W be three distinct points on the circumference of the circle unequal to P and Q for which $|UP| = |PQ| = |QV|$ and $|PW| = |QW|$. Then the triangles PQU, PQV, PQW and UVW are isoceles. Either one of the first three has red vertices, or the last one has green vertices. ♠

Rider. Can one always find both a red and a green isosceles triangle if there are infinitely many points of each colour?

358. Find all integers x which satisfy the equation

$$\cos\left(\frac{\pi}{8}(3x - \sqrt{9x^2 + 160x + 800})\right) = 1 \ .$$

Solution. We must have that

$$\frac{\pi}{8}(3x - \sqrt{9x^2 + 160x + 800}) = 2k\pi$$

for some integer k, whence

$$3x - \sqrt{9x^2 + 160x + 800} = 16k \ .$$

Multiplying by the surd conjugate of the left side yields

$$-160x - 800 = 16k(3x + \sqrt{9x^2 + 160x + 800})$$

so that

$$3x + \sqrt{9x^2 + 160x + 800} = \frac{1}{k}(-10x - 50) \ .$$

Therefore, $6x = 16k - (1/k)(10x + 50)$, whereupon $(3k + 5)x = 8k^2 - 25$. Multiplying by 9 yields that

$$9x(3k + 5) = 8(9k^2 - 25) - 25 = 8(3k - 5)(3k + 5) - 25 \ ,$$

whereupon $3k + 5$ is a divisor of 25, *i.e.*, one of the six numbers $\pm 1, \pm 5, \pm 25$. This leads to the three possibilities $(k, x) = (-2, -7), (0, -5), (-10, -31)$. The solution $x = -5$ is extraneous, so the given equation has only two integers solutions, $x = -31, -7$. ♠

359. Let ABC be an acute triangle with angle bisectors AA_1 and BB_1, with A_1 and B_1 on BC and AC, respectively. Let J be the intersection of AA_1 and BB_1 (the incentre), H be the orthocentre and O the circumcentre of the triangle ABC. The line OH intersects AC at P and BC at Q. Given that C, A_1, J and B_1 are vertices of a concyclic quadrilateral, prove that $PQ = AP + BQ$.

Solution. [Y. Zhao] Since CA_1JB_1 is concyclic, we have that

$$\angle C = 180^\circ - \angle AJB = \angle JAB + \angle JBA = \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90^\circ - \frac{1}{2}\angle C$$

so that $\angle C = 60°$. (Here we used the fact that the sum of opposite angles of a concyclic quadrilateral is 180° and the sum-of-interior-angles theorem for triangle AJB.) Now, applying the same reasoning to $\angle AHB$ and using the fact that H is the orthocentre of triangle ABC, we find that

$$\begin{aligned}\angle AHB &= 180° - \angle HAB - \angle HBA == 180° - \angle HAB - \angle HCA \\ &= 180° - (90° - \angle ABC) - (90° - \angle BAC) \\ &= \angle ABC + \angle BAC = 180° - \angle ACB = 120° \ .\end{aligned}$$

On the other hand, since O is the circumcentre of triangle ABC, $\angle AOB = 2\angle ACB = 120°$. Therefore, $AOHB$ is concyclic. Now, $\angle PHA + \angle AHO = 180°$ (supplementary angles) and $\angle OBA + \angle AHO = 180°$ (opposite angles of concyclic quadrilateral), so that $\angle PHA = \angle OBA$. Next, in triangle AOB with $AO = OB$,

$$\angle OBA = \frac{1}{2}(180° - \angle AOB) = 90° - \frac{1}{2}\angle AOB = 90° - \angle C = \angle PAH \ .$$

So, $\angle PHA = \angle PAH$; thus, triangle APH is isosceles and $AP = PH$. Similarly, $QB = QH$. Therefore $PQ = PH + QH = AP + BQ$ as desired. ♠

360. Eliminate θ from the two equations

$$x = \cot\theta + \tan\theta$$

$$y = \sec\theta - \cos\theta \ ,$$

to get a polynomial equation satisfied by x and y.

Solution 1. We have that

$$x = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1}{\sin\theta\cos\theta} \Longrightarrow \sin\theta\cos\theta = \frac{1}{x} \ .$$

$$y = \frac{1}{\cos\theta} - \cos\theta = \frac{\sin^2\theta}{\cos\theta} \ .$$

Hence

$$\sin^3\theta = \frac{y}{x} \qquad \text{and} \qquad \cos^3\theta = \frac{1}{x^2y}$$

so that

$$\begin{aligned}&\left(\frac{y}{x}\right)^{\frac{2}{3}} + \left(\frac{1}{x^2y}\right)^{\frac{2}{3}} = 1 \Longrightarrow (xy^2)^{\frac{2}{3}} + 1 = (x^2y)^{\frac{2}{3}} \\ &\Longrightarrow (x^2y)^2 = 1 + (xy^2)^2 + 3(xy^2)^{\frac{2}{3}}(x^2y)^{\frac{2}{3}} = 1 + x^2y^4 + 3x^2y^2 \ .\end{aligned}$$

Hence

$$x^4y^2 = 1 + x^2y^4 + 3x^2y^2 \ .♠$$

Solution 2. [D. Dziabenko] Since $\sin^3\theta = y/x$ and $\cos^3\theta = 1/(x^2y)$,

$$\begin{aligned}\frac{y^2}{x^2} + \frac{1}{x^4y^2} &= \sin^6\theta + \cos^6\theta \\ &= (\sin^2\theta + \cos^2\theta)^3 - 3(\sin^2\theta + \cos^2\theta)\sin^2\theta\cos^2\theta \\ &= 1 - \frac{3}{x^2} \ .\end{aligned}$$

Hence $x^2y^4 + 1 = x^4y^2 - 3x^2y^2$. ♠

Solution 3. [P. Shi] Using the fact that $\cos^3\theta = 1/(x^2y)$ in the expression for y, we find that

$$y^3 = x^2y - 3y - \frac{1}{x^2y} \Longrightarrow x^2y^4 = x^4y^2 - 3x^2y^2 - 1 \;. \spadesuit$$

Solution 4. [Y. Zhao] Since $\sin^3\theta = y/x$ and $\cos^3\theta = 1/(x^2y)$, we have that

$$\sqrt[3]{\frac{y^2}{x^2}} + \sqrt[3]{\frac{1}{x^4y^2}} - 1 = 0 \;.$$

Using the identity $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ with the substitution $a = \sqrt[3]{y^2/x^2}$, $b = \sqrt[3]{1/(x^4y^2)}$, $c = -1$, we obtain that

$$\frac{y^2}{x^2} + \frac{1}{x^4y^2} - 1 + \frac{3}{x^2} = 0$$

or

$$x^2y^4 + 1 - x^4y^2 + 3x^2y^2 = 0 \;. \spadesuit$$

Comment. In a question like this, it is very easy to make a mechanical slip. Accordingly, it is prudent to make a convenient substitution of values to see if your identity works. For example, when $\theta = \pi/4$, $x = 2$ and $y = 1/\sqrt{2}$, and we find that the identity checks out.

361. Let $ABCD$ be a square, M a point on the side BC, and N a point on the side CD for which $BM = CN$. Suppose that AM and AN intersect BD and P and Q respectively. Prove that a triangle can be constructed with sides of length $|BP|$, $|PQ|$, $|QD|$, one of whose angles is equal to 60°.

Solution 1. Let the sides of the square have length 1 and let $|BM| = u$. Then $|NC| = u$ and $|MC| = |ND| = 1-u$. Let $|BP| = a$, $|PQ| = b$ and $|QD| = c$. Since triangles APD and MPB are similar, $(a/u) = b + c$. Since triangle DQN and BQA are similar, $(c/(1-u))) = a + b$. Hence

$$(1-u+u^2)a = (2u-u^2)b \qquad \text{and} \qquad (1-u+u^2)c = (1-u^2)b$$

so that

$$a : b : c = (2u-u^2) : (1-u+u^2) : (1-u^2) \;.$$

Now

$$\begin{aligned}(2u-u^2)^2 &+ (1-u^2)^2 - 2(2u-u^2)(1-u^2)\cos 60^\circ \\ &= (4u^2-4u^3+u^4) + (1-2u^2+u^4) - (2u-2u^3-u^2+u^4) \\ &= u^4 - 2u^3 + 3u^2 - 2u + 1 = (1-u+u^2)^2 \;.\end{aligned}$$

Thus $b^2 = a^2 + c^2 - ac$. Note that this implies that $(a-c)^2 < b^2 < (a+c)^2$, whence $a < b + c$, $b < a + c$ and $c < a + b$. Accordingly, a, b, c are the sides of a triangle and $b^2 = a^2 + c^2 - \frac{1}{2}ac\cos 60^\circ$. From the law of cosines, the result follows. $\spadesuit$

Solution 2. [F. Barekat] *Lemma.* Let PQR be a right triangle with $\angle R = 90^\circ$, $|QR| = p$ and $|PR| = q$. Then the length m of the bisector RT of angle R (with T on PQ) is equal to $(\sqrt{2}pq)/(p+q)$.

Proof. $[PQR] = [QRT] + [PRT] \Longrightarrow \frac{1}{2}pq = \frac{1}{2}pm\sin 45^\circ + \frac{1}{2}qm\sin 45^\circ$, from which the result follows. $\clubsuit$

Using the same notation as in Solution 1, the above lemma and the fact that BD bisects the right angles at B and D, we find that

$$a = \frac{\sqrt{2}u}{1+u} \;, \quad c = \frac{\sqrt{2}(1-u)}{2-u} \;, \quad b = \sqrt{2} - a - c \;.$$

Hence

$$\begin{aligned}(a^2+c^2-2ac\cos 60°)-b^2 &= (a^2+c^2-ac)-(2+a^2+c^2)+2\sqrt{2}(a+c)-2ac \\ &= 2\sqrt{2}(a+c)-3ac-2 \\ &= \frac{2}{(1+u)(2-u)}[2u(2-u)+2(1+u)(1-u)-3u(1-u)-(1+u)(2-u)] \\ &= \frac{2}{(1+u)(2-u)}[4u-2u^2+2-2u^2-3u+3u^2-2-u+u^2]=0\ .\end{aligned}$$

Hence $b^2=a^2+c^2-2ac\cos 60°$ and the result follows from the law of cosines. ♠

Solution 3. [V. Krakovna; R. Shapiro] Let a, b, c, u be as in Solution 1. Then $u=a/(b+c)$ and $1-u=c/(a+b)$. Hence

$$1=\frac{a}{b+c}+\frac{c}{a+b}=\frac{a^2+c^2+b(a+c)}{b^2+b(a+c)+ac}\ ,$$

which is equivalent to $b^2=a^2+c^2-ac$. As in Solution 1, we find that a,b,c are sides of a triangle and $b^2=a^2+c^2-\frac{1}{2}ac\cos 60°$. ♠

Comment. P. Shi used Menelaus' Theorem with triangle BOC and transversal APM and with triangle COD and transversal AQN to get

$$\frac{|BP|}{|PO|}\cdot\frac{|OA|}{|AC|}\cdot\frac{|CM|}{|MB|}=\frac{|DQ|}{|QO|}\cdot\frac{|OA|}{|AC|}\cdot\frac{|CN|}{|ND|}=1\ ,$$

where O is the centre of the square. Noting that $|PO|=(1/\sqrt{2})-|BP|$ and $|QO|=(1/\sqrt{2})-|DQ|$, we can determine the lengths of BP and DQ.

362. The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B, C meet the circle again at U, V, W, respectively. Prove that the area of triangle UVW is not less than the area of triangle ABC.

Solution 1. Let R be the common circumradius of the triangles ABC and UVW. Observe that $\angle WUA=\angle WCA=\frac{1}{2}\angle ACB$ and $\angle VUA=\angle VBA=\frac{1}{2}\angle ABC$, whence

$$U=\angle WUV=\frac{1}{2}(\angle ACB+\angle ABC)=\frac{1}{2}(B+C)\ ,$$

et cetera. Now

$$[ABC]=\frac{abc}{4R}=2R^2\sin A\sin B\sin C$$

and

$$[UVW]=\frac{uvw}{4R}=2R^2\sin U\sin V\sin W\ .$$

Since by the arithmetic-geometric means inequality,

$$\begin{aligned}\sqrt{\sin A\sin B} &= 2\sqrt{(\sin\frac{A}{2}\cos\frac{B}{2})(\cos\frac{A}{2}\sin\frac{B}{2})} \\ &\le \sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2} \\ &= \sin\frac{A+B}{2}=\sin W\ ,\end{aligned}$$

et cetera, it follows that $[ABC]\le[UVW]$ with equality if and only if ABC is an equilateral triangle. ♠

Second solution. Since $\frac{1}{2}(A+B)=90°-\frac{C}{2}$, we find that

$$[ABC]=2R^2\sin A\sin B\sin C=16R^2\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}$$

and

$$[UVW] = 2R^2 \sin U \sin V \sin W = 2R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \ ,$$

so that

$$\begin{aligned}\frac{[ABC]}{[UVW]} &= 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ &= 8\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}} \\ &= \frac{8(s-a)(s-b)(s-c)}{abc} = \frac{8[ABC]^2}{sabc} = \frac{2r}{R} \le 1\end{aligned}$$

bu Euler's inequality for the inradius and the circumradius. The result follows. ♠

Third solution. Let I be the incentre and H the orthocentre of triangle ABC. Suppose the respective altitudes from A, B, C meet the circumcircle at P, Q, R. We have that

$$\angle BHP = \angle AHQ = 90^\circ - \angle HAC = 90^\circ - \angle PAC = 90^\circ - \angle PBC = \angle BPH$$

so that $BH = BP$. Similarly, $CH = CP$. Hence $\Delta HBC \equiv \Delta PBC$ (SSS). Similarly $\Delta HAC \equiv \Delta QAC$ and $\Delta HAB \equiv \Delta RAB$, whence $[ARBPCQ] = 2[ABC]$.

Let AU intersect VW at X. Then

$$\begin{aligned}\angle VXA &= \angle XWA + \angle XAW = \angle VWA + \angle UAB + \angle WAB \\ &= \angle VBA + \angle UAB + \angle WCB = \frac{1}{2}(\angle B + \angle A + \angle C) = 90^\circ \ ,\end{aligned}$$

so that UA is an altitude of triangle UVW, as similarly are VB and WC (so that I is the orthocentre of triangle UVW). Therefore, we have, as above, that $[UCVAWB] = 2[UVW]$.

But U is the midpoint of the arc BC, V the midpoint of the arc CA and W the modpoint of the orc AB. Thus, $[CPB] \le [CUB]$, $[AQC] \le [AVC]$ and $[BRA] \le [BWA]$. Therefore, $[ARBPCQ] \le [UCVAWB]$ and so $[ABC] \le [UVW]$. ♠

Comment. F. Barekat noted that for $0 < x < \pi$, the function $\log \sin x$ is concave so that $\sqrt{\sin u \sin v} \le \sin(\frac{1}{2}(u+v))$ for $0 \le u, v \le \pi$. (This can be seen by noting that the second derivative of $\log \sin x$ is $-\csc^2 x < 0$.) Then the solution can be completed as in Solution 1.

363. Suppose that x and y are positive real numbers. Find all real solutions of the equation

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} = \sqrt{xy} + \frac{x+y}{2} \ .$$

Preliminaries. It is clear that if one of x and y vanishes, then so must the other. Otherwise, there are two possibilities, according as x and y are both negative or both positive ($\sqrt{xy}$ needs to make sense). If x and y are both negative, then the only solution is $x = y$ as the equation asserts that the sum of the harmonic and geometric means of $-x$ and $-y$ is equal to the sum of the arithmetic mean and root-mean-square of these quantities. For unequal positive reals, each of the first two is less than each of the second two. Henceforth, we assume that x and y are positive. Let $h = 2xy/(x+y)$, $g = \sqrt{xy}$, $a = \frac{1}{2}(x+y)$ and $r = \sqrt{\frac{1}{2}(x^2+y^2)}$.

Solution 1. It is straightforward to check that $2a^2 = r^2 + g^2$ and that $g^2 = ah$. Suppose that $h + r = a + g$. Then

$$\begin{aligned}& r = a + g - h \\ &\Longrightarrow 2a^2 - g^2 = r^2 = a^2 + g^2 + h^2 - 2ah - 2gh + 2ag \\ &\Longrightarrow (a+h)(a-h) = a^2 - h^2 = 2(g^2 - ah) + 2g(a-h) = 0 + 2g(a-h) \\ &\Longrightarrow a = h \quad \text{or} \quad a + h = 2g \ .\end{aligned}$$

In the latter case, $g = \frac{1}{2}(a+h) = \sqrt{ah}$, so that both possibilities entail $a = g = h$. But equality of these means occur if and only if $x = y$. ♠

Solution 2. [A. Cornuneanu] Observe that

$$r^2 - g^2 = \frac{1}{2}(x-y)^2 = 2a(a-h) \ .$$

Since, from the given equation,

$$r^2 - g^2 = (r-g)(r+g) = (a-h)(r+g) \ ,$$

it follows that $a = \frac{1}{2}(r+g)$. However, it can be checked that, in general,

$$a = \sqrt{\frac{r^2+g^2}{2}} \ ,$$

so that a is at once the arithmetic mean and the root-mean-square of r and g. But this can occur if and only if $r = g = a$ if and only if $x = y$. ♠

Solution 3. [R. O'Donnell] From the homogeneity of the given equation, we can assume without loss of generality that $g = 1$ so that $y = 1/x$. Then $h = 1/a$ and $r = \sqrt{2a^2-1}$. The equation becomes

$$r + \frac{1}{a} = a + 1 \qquad \text{or} \qquad \sqrt{2a^2-1} = 1 + a - \frac{1}{a} \ .$$

Squaring and manipulating leads to

$$0 = a^4 - 2a^3 + 2a - 1 = (a-1)^3(a+1)$$

whence $a = 1 = g$ and so $x = y = 1$. The main result follows from this. ♠

Solution 4. [D. Dziabenko] Let $2a = x+y$ and $2b = x-y$, so that $x = a+b$ and $y = a-b$, Then $a \neq 0$, $a \geq b$ and the equation becomes

$$\frac{a^2-b^2}{a} + \sqrt{a^2+b^2} = \sqrt{a^2-b^2} + a \Longleftrightarrow \sqrt{a^2+b^2} - \sqrt{a^2-b^2} = \frac{b^2}{a} \ .$$

Multiplying by $\sqrt{a^2+b^2} + \sqrt{a^2-b^2}$ yields that

$$2b^2 = (a^2+b^2) - (a^2-b^2) = \frac{b^2}{a}(\sqrt{a^2+b^2} + \sqrt{a^2-b^2}) \ ,$$

from which

$$\frac{\sqrt{a^2+b^2} + \sqrt{a^2-b^2}}{2} = a = \sqrt{\frac{(a^2+b^2)+(a^2-b^2)}{2}} \ .$$

The left side is the arithmetic mean and the right the root-mean-square of $\sqrt{a^2+b^2}$ and $\sqrt{a^2-b^2}$. These are equal if and only if $a^2+b^2 = a^2-b^2 \Leftrightarrow b = 0 \leftrightarrow x = y$. ♠

Solution 5. [M. Elqars] With $2a = x+y$ and $2b = x-y$, we have that $x^2+y^2 = 2(a^2+b^2)$ and $xy = a^2-b^2$. The equation becomes

$$\frac{a^2-b^2}{a} + \sqrt{a^2+b^2} = \sqrt{a^2-b^2} + a$$

$$\Longleftrightarrow \sqrt{a^2+b^2} - \sqrt{a^2-b^2} = a - \frac{a^2-b^2}{a} = \frac{b^2}{a}$$

$$\Longleftrightarrow 2a^2 - 2\sqrt{a^4 - b^4} = \frac{b^4}{a^2}$$

$$\Longleftrightarrow \sqrt{a^4 - b^4} = a^2 - \frac{b^4}{2a^2} = \frac{2a^4 - b^4}{2a^2} = \frac{a^4 - b^4}{2a^2} + \frac{a^2}{2}$$

$$\Longleftrightarrow 0 = (a^4 - b^4) - 2a^2\sqrt{a^4 - b^4} + a^4 = [\sqrt{a^4 - b^4} - a^2]^2$$

$$\Longleftrightarrow b = 0 \longleftrightarrow x = y\ .\spadesuit$$

364. Determine necessary and sufficient conditions on the positive integers a and b such that the vulgar fraction a/b has the following property: *Suppose that one successively tosses a coin and finds at one time, the fraction of heads is less than a/b and that at a later time, the fraction of heads is greater than a/b; then at some intermediate time, the fraction of heads must be exactly a/b.*

Solution. Consider the situation in which a tail is tossed first, and then a head is tossed thereafter. Then the fraction of heads after n tosses is $(n-1)/n$. Since any positive fraction a/b less than 1 exceeds this for $n = 1$ and is less than this for n sufficiently large, a/b can be realized as a fraction of head tosses only if it is of this form (*i.e.* $a = b - 1$).

On the other hand, suppose that $a/b = (n-1)/n$ for some positive integer n. There must exist numbers p and q for which the fraction p/q of heads at one toss is less than a/b and the fraction $(p+1)/(q+1)$ at the next toss is not less than a/b. Thus

$$\frac{p}{q} < \frac{n-1}{n} \le \frac{p+1}{q+1}\ .$$

Hence $np < nq - q$ and $nq - q + n - 1 \le np + n$, so that

$$np < nq - q \le np + 1\ .$$

Since the three members of this inequality are integers and the outer two are consecutive, we must have $nq - q = np + 1$, whence

$$\frac{n-1}{n} = \frac{p+1}{q+1}\ .$$

Hence the necessary and sufficient condition is that $a/b = (n-1)/n$ for some positive integer n. ♠

Rider. What is the situation when the fraction of heads moves from a number greater than a/b to a number less than a/b?

365. Let $p(z)$ be a polynomial of degree greater than 4 with complex coefficients. Prove that $p(z)$ must have a pair u, v of roots, not necessarily distinct, for which the real parts of both u/v and v/u are positive. Show that this does not necessarily hold for polynomials of degree 4.

Solution. Since the degree of the polynomial exceeds 4, there must be two roots u, v in one of the four quadrants containing a ray from the origin along either the real or the imaginary axis along with all the points within the region bounded by this ray and the next such ray in the counterclockwise direction. The difference in the arguments between two such numbers must be strictly between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since $\arg(u/v) = \arg u - \arg v$ and $\arg(v/u) = \arg v - \arg u$ both lie in this range, both u/v and v/u lie to the right of the imaginary axis, and so have positive real parts.

This result does not necessarily hold for a polynomial of degree 4, as witnessed by $z^4 - 1$ whose roots are $1, -1, i, -i$.

366. What is the largest real number r for which

$$\frac{x^2 + y^2 + z^2 + xy + yz + zx}{\sqrt{x} + \sqrt{y} + \sqrt{z}} \ge r$$

holds for all positive real values of x, y, z for which $xyz = 1$.

Solution 1. Let u, v, w be positive reals for which $u^2 = yz$, $v^2 = zx$ and $w^2 = xy$. Then $\sqrt{x} = x\sqrt{yz} = xu$, $\sqrt{y} = yv$ and $\sqrt{z} = zw$, so that

$$\begin{aligned}(x^2+y^2+z^2)+(xy+yz+zx) &= (x^2+y^2+z^2)+(u^2+v^2+w^2)\\ &\geq 2\sqrt{x^2+y^2+z^2}\sqrt{u^2+v^2+w^2} \geq 2(xu+yv+zw)\ ,\end{aligned}$$

from the arithmetic-geometric means and the Cauchy-Schwarz inequalities. Hence, the inequality is always valid when $r \leq 2$. When $(x, y, z) = (1, 1, 1)$, the left side is equal to 2, so the inequality does not always hold when $r > 2$. Hence the largest value of r is 2. ♠

Solution 2. Applying the arithmetic-geometric means inequality to the left side yields

$$\begin{aligned}\frac{(x^2+yz)+(y^2+zx)+(z^2+xy)}{\sqrt{x}+\sqrt{y}+\sqrt{z}} &\geq \frac{2(\sqrt{x^2yz}+\sqrt{y^2zx}+\sqrt{z^2xy}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\\ &= \frac{2\sqrt{xyz}(\sqrt{x}+\sqrt{y}+\sqrt{z})}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = 2\ .\end{aligned}$$

Equality occurs when $(x, y, z) = (1, 1, 1)$. Hence the largest value of r for which the inequality always holds is 2. ♠

Solution 3. [V. Krakovna]

$$\begin{aligned}\frac{x^2+y^2+z^2+xy+yz+zx}{\sqrt{x}+\sqrt{y}+\sqrt{z}} &= \frac{\frac{1}{2}(x^2+y^2)+\frac{1}{2}(y^2+z^2)+\frac{1}{2}(z^2+x^2)+xy+yz+zx}{x\sqrt{yz}+y\sqrt{xz}+z\sqrt{xy}}\\ &\geq \frac{(xy+yz+zx)+(xy+yz+zx)}{x(y+z)/2+y(x+z)/2+z(x+y)/2}\\ &= \frac{2(xy+yz+zx)}{xy+yz+zx} = 2\ .\end{aligned}$$

Equality occurs if and only if $x = y = z$.

Solution 4. [R. Shapiro] Applying the arithmetic-geometric means and Cauchy-Schwarz inequalities, we have that

$$\begin{aligned}x^2+y^2+z^2+xy+yz+zx &= (x^2+y^2+z^2)+(xy+yz+zx)\\ &\geq 2\sqrt{(x^2+y^2+z^2)(xy+yz+zx)}\\ &\geq 2(x\sqrt{yz}+y\sqrt{zx}+z\sqrt{xy}) = 2(\sqrt{x}+\sqrt{y}+\sqrt{z})\ ,\end{aligned}$$

with equality if and only if $x = \sqrt{yz}$, $y = \sqrt{zx}$, $z = \sqrt{xy}$, if and only if $x = y = z = 1$. ♠

367. Let a and c be fixed real numbers satisfying $a \leq 1 \leq c$. Determine the largest value of b that is consistent with the condition

$$a + bc \leq b + ac \leq c + ab\ .$$

Solution. Since $(b + ac) - (a + bc) = (a - b)(c - 1)$ and $(c + ab) - (b + ac) = (c - b)(1 - a)$, the given inequalities are equivalent to $(a - b)(c - 1) \geq 0$ and $(c - b)(1 - a) \geq 0$.

If $a = c = 1$, then the inequalites hold for any value of b, and there is no maximum value. If $a < 1 = c$, then the first inequality is automatic and the inequalities hold if and only if $b \leq c = 1$. If $1 < c$, then $a \leq 1 < c$ and the two inequalities are equivalent to $a \geq b$ and $c \geq b$, which both hold if and only if $a \geq b$. Thus, when $1 < c$, the maximum value of b for which the inequalities hold is a. ♠

368. Let A, B, C be three distinct points of the plane for which $AB = AC$. Describe the locus of the point P for which $\angle APB = \angle APC$.

Solution 1. We observe that P cannot be any of A, B, C, as the angles become degenerate, so that A, B and C must be excluded from the locus. Suppose first that A, B, C are collinear, so that A is the midpoint of BC. For a point P on the locus for which the triangle PBC is nondegenerate, the median PA of the triangle PBC bisects the angle BPC. Therefore $PB = PC$ (note that $PB : PC = AB : AC = 1 : 1$) and so P must lie on the right bisector of BC. Conversely, any point on the righ bisector save B is on the locus. If the triangle PBC is degenerate, then P must lie on the line BC outside of the closed interval BC (in which case $\angle APB = \angle APC$).

Henceforth, assume that the points ABC are not collinear. The locus does not contain any of the points A, B and C. If P is a point on the open arc BC of the circumcircle of ABC that does not contain A, then $ABPC$ is concyclic and

$$\angle APB = \angle ACB = \angle ABC = \angle APC \ .$$

(More briefly, the angles subtended at P be the equal chords AB and AC are equal. Why are they not supplementary?) If P is any point, except A, on the right bisector of BC, then, by a reflection in this bisector, we see that $\angle APB = \angle APC$. Finally, if P is any point on the BC outside of the closed segment BC, then $\angle APB = \angle APC$ since P, B, C are collinear. Thus, the locus must contain the following three sets: (1) all points on the open arc BC of the circumcircle of ABC that does not contain A; (2) all points on the right bisector of BC except for A; (3) all points on the line BC than do not lie between B and C inclusive.

We show that there are no further points in the locus. Suppose that P lies in the angle formed by AB and AC that includes the right bisector of BC, but that P does not lie on this bisector. Let Q (distinct from P) be the reflection of P in the right bisector. Then $\angle APB = \angle AQC$ and $\angle APC = \angle AQB$. Suppose that $\angle APB = \angle APC$. Then $\angle APC = \angle AQC$ and $\angle APB = \angle AQB$, so that A, B, P, Q, C are concyclic and we must have situation (i). If P lies in the angle exterior to triangle ABC determined by AB and AC produced, then it can be checked that one of the angles APB and APC properly contains the other. The result follows.

Solution 2. If A, B, C are collinear, wolog suppose that P is a point of the locus not on BC for which $PB < PC$ and D is the reflected image of B with respect to PA. Then $AD = BA = AC$ and D lies on the segment PC. Hence $\angle PBA + \angle PCB = \angle PDA + \angle DCA = \angle PDA + \angle ADC = 180°$, so that $\angle BPC = 0°$, a contradiction. Hence P must lie on BC or the right bisector of BC, We can eliminate from the locus all points on the closed segment BC as well as the point A.

Suppose A, B and C are not collinear. Let P be a point on the locus. Consider triangle ABP and ACP. Since $AB = AC$, AP is common and the (noncontained) corresponding angles APB and APC are equal, we have the ambiguous (SSA) case and so either triangles APB and APC are congruent, or else $\angle ABP + \angle ACP = 180°$. If the triangles are conguent, then $PB = PC$ and P lies on the right bisector of BC,

If $\angle ABP + \angle ACP = 180°$, then there are two possibilities. Either B and C lie on the same side of AP, in which case P, B, C are collinear or B and C lie on opposite sides of AP, in which case $ABPC$ is concyclic.

Hence the locus is contained in the union of the right bisector of BC, that part of the line BC not between B and C and the arc BC of the circumcircle of triangle ABC not containing A. Conversely, it is straightforward to verify that every point in this union, except for A, B and C is on the locus.

Comment. Another way in is to apply the law of sines on triangles ABP and ACP and note that

$$\frac{\sin ABP}{|AP|} = \frac{\sin \angle APB}{|AB|} = \frac{\sin \angle APC}{|AC|} = \frac{\sin \angle ACP}{|AP|} \ ,$$

so that $\sin \angle ABP = \sin \angle ACP$. Thus, either $\angle ABP = \angle ACP$ or $\angle ABP + \angle ACP = 180°$.

369. $ABCD$ is a rectangle and APQ is an inscribed equilateral triangle for which P lies on BC and Q lies on CD.

(a) For which rectangles is the configuration possible?

(b) Prove that, when the configuration is possible, then the area of triangle CPQ is equal to the sum of the areas of the triangles ABP and ADQ.

Solution 1. (a) Let the configuration be given and let the length of the side of the equilateral triangle be 1. Suppose that $\angle BAP = \theta$. Then $|AB| = \cos\theta$ and $|AD| = \cos(30° - \theta)$. Observe that $0 \leq \theta \leq 30°$. Then

$$\frac{|AD|}{|AB|} = \frac{\cos 30° \cos\theta + \sin 30° \sin\theta}{\cos\theta} = \frac{\sqrt{3} + \tan\theta}{2} \ .$$

Since $0 \leq \tan\theta \leq 1/\sqrt{3}$, it follows that

$$\frac{\sqrt{3}}{2} \leq \frac{|AD|}{|AB|} \leq \frac{2}{\sqrt{3}} \ .$$

(Alternatively, note that $\cos(30° - \theta)$ increases and $\cos\theta$ decreases with θ, so that

$$\frac{\cos 30°}{\cos 0°} \leq \frac{|AD|}{|AB|} \leq \frac{\cos 0°}{\cos 30°} \ .)$$

Conversely, supposing that this condition is satisfied, we can solve $|AD|/|AB| = \frac{1}{2}(\sqrt{3} + \tan\theta)$ for a value of $\theta \in [0, 30°]$ and determine a configuration for which $\angle BAP = \theta$ and $\angle DAQ = 30° - \theta$. It can be checked that $AP = AQ$ (do this!).

(b) Suppose that the configuration is given, and that L, M and N are the respective midpoints of AP, PQ and QA. Wolog, let the lengths of these three segments be 2. Then BL, CM and DN all have length 1 (why?). Since $\angle BLP = 2\theta$, $\angle CMQ = 60° + 2\theta$ and $\angle DNQ = 60° - 2\theta$, we find that (from the areas of triangles like BLP),

$$\begin{aligned}[CPQ] - [ADQ] - [ABP] &= \sin(60° + 2\theta) - \sin(60° - 2\theta) - \sin 2\theta \\ &= 2\cos 60° \sin 2\theta - \sin 2\theta = 0 \ ,\end{aligned}$$

so that the result holds.

Comment. One can also get the areas of the corner right triangles by taking half the product of their arms. For example,

$$[BAP] = \frac{1}{2}(2\sin\theta)(2\cos\theta) = 2\sin\theta\cos\theta = \sin 2\theta \ .$$

370. A deck of cards has nk cards, n cards of each of the colours C_1, C_2, $\cdots$, C_k. The deck is thoroughly shuffled and dealt into k piles of n cards each, P_1, P_2, $\cdots$, P_k. A game of solitaire proceeds as follows: The top card is drawn from pile P_1. If it has colour C_i, it is discarded and the top card is drawn from pile P_i. If it has colour C_j, it is discarded and the top card is drawn from pile P_j. The game continues in this way, and will terminate when the nth card of colour C_1 is drawn and discarded, as at this point, there are no further cards left in pile P_1. What is the probability that every card is discarded when the game terminates?

Solution. We begin by determining a one-one correspondence between plays of the game and the $(nk)!$ arrangements of the nk cards. For each play of the game, we set the cards aside in the order than they appear. If the game is finishes with the last card, we go through the whole deck and obtain an arrangement in which the last card has colour C_1. If the game finishes early, then we have exhausted the pile P_1, but not all of the remaining pile; all the colours C_1's will have appeared among the first $nk - 1$ cards. We continue the arrangement by dealing out in order all the cards in the pile P_2, then all the cards in the pile P_3 and so on.

Conversely, suppose that we have an arrangement of the nk cards. We reconstruct a game. Look at all the cards up to the last card of colour C_1. Suppose that it contains x_i cards of colour C_i; then that means that there are $n - x_i$ cards that should not be turned over in pile P_i. Take the last $n - x_k$ cards of the arrangement and place them in pile P_k, and then the next last $n - x_{k-1}$ cards and place them in pile P_{k-1}, and so on until we come down to pile P_2. We have backed up in the arrangement to the last card of colour C_1, and its predecessor determines from which pile it was drawn; restore it to that pile. For $2 \leq i \leq x_1 + x_2 + \cdots + x_k$, place the ith card in the arrangement on the pile of the colour of the $(i-1)$th card. Finally, when we get to the first card of the arrangement, all the piles except P_1 have been restored to n cards; place this card on P_1.

Thus, each game determines an arrangement, and each arrangement a game. Therefore, the desired probability is the probability that in an arbitrary arrangement, the last card has colour C_1. As the probability is the same for each of the colours, the desired probability is $1/k$.

371. Let X be a point on the side BC of triangle ABC and Y the point where the line AX meets the circumcircle of triangle ABC. Prove or disprove: if the length of XY is maximum, then AX lies between the median from A and the bisector of angle BAC.

Solution. [F. Barekat] Wolog, suppose that $AB \geq AC$. Let M be the midpoint of BC and N where AM intersect the circumcircle. Let P on BC be the foot of the angle bisector of $\angle BAC$ and let AP intersect the circumcircle at Q. Then BC is partitioned into three segments: BM, MP, PC.

Suppose that X is on the segment BM and that AX meets the circumcircle in Y. Let U on the segment MC satisfy $XM = MU$ and let AU meet the circumcircle in V then

$$AX \cdot XY = BX \cdot XC = CU \cdot UB = AU \cdot UV \ .$$

Now $AX \geq AU$ (one way to see this is to drop a perpendicular from A to XU and use Pythagoras' theorem). Hence $XY \leq UV$, so the maximizing point must lie between M and C.

Now let X lie between P and C. Construct R so that $PXYR$ is a parallelogram. Since Q is the midpoint of the arc BC, the tangent at Q to the circumcircle is parallel to BC and so R lies within the circle, $\angle PQR \leq \angle AQY$ and $\angle QPR = \angle QAY$. Therefore

$$\begin{aligned}\angle PRQ &= 180^\circ - \angle QPR - \angle PQR \geq 180^\circ - \angle QAY - \angle AQY \\ &= \angle AYQ = \angle ACQ = \angle ACB + \angle BCQ \\ &= \angle ACB + \angle BAQ = \angle ACB + \frac{1}{2}\angle BAC \\ &= \frac{1}{2}(2\angle ACB + \angle BAC) \geq \frac{1}{2}(\angle ACB + \angle ABC + \angle BAC) = 90^\circ \ .\end{aligned}$$

Therefore $PQ \geq PR = XY$. It follows that XY assumes its maximum value when X is between M and P, as desired.

372. Let b_n be the number of integers whose digits are all $1, 3, 4$ and whose digits sum to n. Prove that b_n is a perfect square when n is even.

Solution 1. It is readily checked that $b_1 = b_2 = 1$, $b_3 = 2$ and $b_4 = 4$. Consider numbers whose digits sum to $n \geq 5$. There are b_{n-1} of them ending in 1, b_{n-3} of them ending in 3, and b_{n-4} of them ending in 4. We prove by induction, that for each positive integer m,

$$b_{2m} = f_{m+1}^2 \qquad \text{and} \qquad b_{2m-1} = f_{m+1}f_m \ ,$$

where $\{f_n\}$ is the Fibonacci sequences defined by $f_0 = 0, f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$.

The result holds for $m = 1$. Suppose that it holds up to $m = k$. Then

$$\begin{aligned} b_{2k+1} &= b_{2k} + b_{2k-2} + b_{2k-3} = f_{k+1}^2 + f_k^2 + f_k f_{k-1} \\ &= f_{k+1}^2 + f_k(f_k + f_{k-1}) = f_{k+1}^2 + f_k f_{k+1} \\ &= f_{k+1}(f_{k+1} + f_k) = f_{k+1} f_{k+2} \ , \end{aligned}$$

and

$$\begin{aligned} b_{2(k+1)} &= b_{2k+1} + b_{2k-1} + b_{2k-2} = f_{k+2} f_{k+1} + f_{k+1} f_k + f_k^2 \\ &= f_{k+2} f_{k+1} + (f_{k+1} + f_k) f_k \\ &= f_{k+2} f_{k+1} + f_{k+2} f_k = f_{k+2}(f_{k+1} + f_k) = f_{k+2}^2 \ . \end{aligned}$$

The result follows.

Solution 2. As before, $b_n = b_{n-1} + b_{n-3} + b_{n-4}$. From this, we see that

$$b_n = (b_{n-2} + b_{n-4} + b_{n-5}) + (b_{n-2} - b_{n-5} - b_{n-6}) + b_{n-4} = 2b_{n-2} + 2b_{n-4} - b_{n-6} \ ,$$

for $n \geq 7$. Also, for $n \geq 4$,

$$f_n^2 = (f_{n-1} + f_{n-2})^2 = 2f_{n-1}^2 + 2f_{n-2}^2 - (f_{n-1} - f_{n-2})^2 = 2f_{n-1}^2 + 2f_{n-2}^2 - f_{n-3}^2 \ .$$

By induction, it can be shown that $b_{2m} = f_{m+1}^2$ for each positive integer m (do it!).

373. For each positive integer n, define

$$a_n = 1 + 2^2 + 3^3 + \cdots + n^n \ .$$

Prove that there are infinitely many values of n for which a_n is an odd composite number.

Solution. Modulo 3, it can be verified that $n^n \equiv 0$ for $n \equiv 0 \pmod 3$, $n^n \equiv 1$ for $n \equiv 1, 2, 4$ (mod 6), and $n^n \equiv 2$ for $n \equiv 5 \pmod 6$. It follows from this that the sum of any six consecutive values of n^n is congruent to 2 (mod 3), and so the sum of any eighteen consecutive values of n^n is congruent to 0 (mod 3). Since such a sum contains nine odd summands, it must be odd. The sum of any thirty-six consecutive values of n^n contains eighteen odd summands and so is even. It follows that the sum of any thirty-six consecutive values of n^n is a multiple of 6.

It is readily checked that $a_n \equiv 0 \pmod 3$ when $n = 4, 7, 14, 15, 17, 18$. Observe that a_4, a_7, a_{15} are even and a_{14}, a_{17}, a_{18} are odd. Hence a_n is an odd multiple of 3 whenever $n \equiv 14, 17, 18, 22, 25, 33 \pmod{36}$. These numbers are all odd and composite.

Comment. A similar argument can be had for any odd prime p. What is the period of n^n?

374. What is the maximum number of numbers that can be selected from $\{1, 2, 3, \cdots, 2005\}$ such that the difference between any pair of them is not equal to 5?

Solution 1. The maximum number is 1005. For $1 \leq k \leq 5$, let $S_k = \{x : 1 \leq x \leq 2005, x \equiv k (\text{mod } 5)\}$. Each set S_k has 401 numbers, and no number is any of the S_k differs from a number in a different S_k by 5 (or even a multiple of 5). Each S_k can be partitioned into 200 pairs and a singleton:

$$S_k = \{k, 5 + k\} \cup \{10 + k, 15 + k\} \cup \cdots \cup \{1990 + k, 1995 + k\} \cup \{2000 + k\} \ .$$

By the Pigeonhole Principle, each choice of 202 numbers from S_k must contain two numbers in one of the pairs and so which differ by 5. At most 201 numbers can be selected from each S_k with no two differing by 5. For example $\{k, 10 + k, 20 + k, \cdots, 2000 + k\}$ will do. Overall, we can select at most $5 \times 201 = 1005$ numbers, no two differing by 5.

Solution 2. [F. Barekat] The subset $\{1, 2, 3, 4, 5, 11, 12, 13, 14, 15, \cdots, 2001, 2002, 2003, 2004, 2005\}$ contains 1005 numbers, no two differing by 5. Suppose that 1006 numbers are chosen from $\{1, 2, \cdots, 2005\}$. Then, at least 1001 of them must come from the following union of 200 sets:

$$\{1, \cdots, 10\} \cup \{11, \cdots, 20\} \cup \cdots \cup \{1991, \cdots, 2001\} .$$

By the Pigeonhole Principle, at least one of these must contain 6 numbers, two of which must be congruent modulo 5, and so differ by 5. The result follows.

375. Prove or disprove: there is a set of concentric circles in the plane for which both of the following hold:

(i) each point with integer coordinates lies on one of the circles;

(ii) no two points with integer coefficients lie on the same circle.

Solution. There is such a set of concentric circles satisfying (a) and (b), namely the set of all concentric circles centred at $(\frac{1}{3}, \sqrt{2})$. Every point with integer coordinates lies in exactly one of the circles, whose radius is equal to the distance from the point to the common centre. Suppose that the points (a, b), (c, d) with integer coordinates both lie on the same circle. Then

$$(a - (1/3))^2 + (b - \sqrt{2})^2 = (c - (1/3))^2 + (d - \sqrt{2})^2$$

$$\Longleftrightarrow 9a^2 - 6a + 1 + 9b^2 - 18\sqrt{2}b + 18 = 9c^2 - 6c + 1 + 9d^2 - 18\sqrt{2}d + 18$$

$$\Longleftrightarrow 9(a^2 + b^2 - c^2 - d^2) - 6(a - c) = \sqrt{2}(18d - 18b) .$$

The left member of the last equation and the coefficient of $\sqrt{2}$ in the right member are both integers. Since $\sqrt{2}$ is irrational, both must vanish, so that $b = d$ and

$$0 = 3(a^2 - c^2) - 2(a - c) = (a - c)(3(a + c) - 2) .$$

Since a and c are integers, $a + c \neq \frac{2}{3}$, so that $a = c$ and $b = d$. Hence, two points with integer coordinates on the same circle must coincide.

Comment. Since you need a simple example to prove the affirmative, it is cleaner to provide a specific case rather than describe a general case. Some selected the common centre $(\sqrt{2}, \sqrt{3})$, which left them with a more complicated result to prove, that $u + v\sqrt{2} + w\sqrt{3} = 0$ for integers u, v, w implies that $u = v = w = 0$. The argument for this should be provided, since it is possible to determine irrational α, β and nonzero integers p, q, r for which $p + q\alpha + r\beta = 0$ (do it!). An efficient way to do it is to start with

$$u + v\sqrt{2} + w\sqrt{3} = 0 \Longrightarrow u^2 = 2v^2 + 3w^2 + 2\sqrt{6}vw .$$

376. A soldier has to find whether there are mines buried within or on the boundary of a region in the shape of an equilateral triangle. The effective range of his detector is one half of the height of the triangle. If he starts at a vertex, explain how he can select the shortest path for checking that the region is clear of mines.

Solution. Wolog, suppose the equilateral triangle has sides of length 1, so that the range of his detector is $\sqrt{3}/4$. Let the triangle be ABC with A the starting vertex. Since the points B and C must be covered, the soldier must reach the circles of centres B and C and radius $\sqrt{3}/4$. Since $2(\sqrt{3}/4) = \sqrt{3}/2 < 1$, the line of centres is longer than the sum of the two radii and the circles do not intersect. Suppose that the soldier crosses the circumference of the circle with centre B at X and of the circle with centre C at Y. Wolog, let the soldier reach X before Y. Then the total distance travelled by the soldier is not less that

$$|AX| + |XY| \geq |AX| + |XC| - |YC| = |AX| + |XC| - (\sqrt{3}/4) .$$

(Use the triangle inequality.)

Let Z be the midpoint of the arc of the circle with centre B that lies within triangle ABC and W be the point of intersection of the circle with centre C and the segment CZ. The ellipse with foci A and C that passes through Z is tangent to the circle with centre B, so that $|AX| + |XC| \geq |AZ| + |ZC|$. Hence the distance travelled by the soldier is at least

$$|AZ| + |ZC| - (\sqrt{3}/4) = 2(\sqrt{7}/4) - (\sqrt{3}/4) = \frac{2\sqrt{7} - \sqrt{3}}{4} \ .$$

(Use the law of cosines in triangle AZB.) This distance is exactly $(\frac{1}{4})(2\sqrt{7} - \sqrt{3})$ when $X = Z$ and X, Y, C are collinear. We show that this corresponds to a suitable path.

Let the soldier start at A, proceed to Z and thence walk directly towards C, stopping at the point W. From the point Z, the soldier covers the points B and M, the midpoint of AC. Let U be any point on AZ and draw the segment parallel to BM through U joining points on AB and AC. From U, the soldier covers every point on the segment. It follows that the soldier covers every point in the triangle ABM.

Suppose the line through W perpendicular to AC meets AC at P and BC at Q. As in the foregoing paragraph, we see that the soldier covers the trapezoid $MBQP$. Note that the lengths of WP, WC and WQ all do not exceed $\sqrt{3}/4$. It follows that every point of the segments CP and CQ are no further from W than $\sqrt{3}/4$. Hence the soldier covers triangle CPQ. Thus, we have a path of minimum length covering all of triangle ABC.

377. Each side of an equilateral triangle is divided into 7 equal parts. Lines through the division points parallel to the sides divide the triangle into 49 smaller equilateral triangles whose vertices consist of a set of 36 points. These 36 points are assigned numbers satisfying both the following conditions:

(a) the number at the vertices of the original triangle are 9, 36 and 121;

(b) for each rhombus composed of two small adjacent triangles, the sum of the numbers placed on one pair of opposite vertices is equal to the sum of the numbers placed on the other pair of opposite vertices.

Determine the sum of all the numbers. Is such a choice of numbers in fact possible?

Solution 1. The answer is $12(9 + 36 + 121) = 1992$.

More generally, let the equilateral triangle be ABC with the numbers a, b, c at the respective vertices A, B, C. Let the lines of division points parallel to BC, AC and AB be called, respectively, $\alpha-$lines, $\beta-$lines and $\gamma-$lines.

Suppose that u and v are two consecutive entries on, say, an $\alpha-$line and p, q, r are the adjacent entries on the next $\alpha-$line. Then $p + v = u + q$ and $q + v = u + r$, whence $p - q = u - v = q - r$. It follows that any two adjacent points on any $\alpha-$line have the same difference, so that the numbers along any $\alpha-$line are in arithmetic progression. The same applies to $\beta-$ and $\gamma-$lines.

In this way, we can uniquely determine the points along the sides AB, BC and AC, and then along each $\alpha-$line, $\beta-$line and $\gamma-$line. However, we need to check that such an assignment is consistent, *i.e.*, does not yield different results for a given entry gained by working along lines from the three different directions. We do this by describing an assignment, and then showing that it satisfies the condition of the problem.

Let an entry be position i $\alpha-$lines from BC, j $\beta-$lines from AC and k $\gamma-$lines from AB. Thus, any entry on BC corresponds to $i = 0$ and the points A, B, C, respectively, correspond to $(i, j, k) = (7, 0, 0), (0, 7, 0), (0, 0, 7)$. Assign to such a point the value $\frac{1}{7}(ia + jb + kc)$. It can be checked that these satisfy the rhombus condition. For example, the points (i, j, k), $(i, j-1, k+1)$, $(i+1, j-1, k)$ and $(i+1, j-2, k+1)$ are four vertices of a rhombus, and the sum of the numbers assigned to the first and last is equal to the sum of the numbers assigned to the middle two.

We sum the entries componentwise. Along the ith $\alpha-$line, there are $8 - i$ entries whose sum is $\frac{1}{7}[i(8 -$

$i)a + \cdots]$. Hence the sum of all entries is

$$\left[\frac{1}{7}\sum_{i=0}^{7} i(8-i)a\right] + \cdots = \frac{1}{7}[1\cdot 7 + 2\cdot 6 + 3\cdot 5 + 4\cdot 4 + 5\cdot 3 + 6\cdot 2 + 7\cdot 1]a + \cdots = 12a + \cdots \ .$$

Summing along $\beta-$lines and $\gamma-$lines, we find that the sum of all entries is $12(a+b+c)$. In the present situation, this number is 1992.

Solution 2. [F. Barekat] Let a, b, c be the entries at A, B, C. As in Solution 1, we show that the entries along each of AB, BC and CA are in arithmetic progression. The sum of the entries along each of these lines are, respectively, $4(a+b)$, $4(b+c)$, $4(c+a)$ (why?), whence the sum of all the entries along the perimeter of triangle ABC is equal to

$$4(a+b) + 4(b+c) + 4(c+a) - (a+b+c) = 7(a+b+c) \ .$$

Let p, q, r, respectively, on AB, BC, CA be adjacent to A, B, C and u, v, w, respectively, on AC, BA, CB be adjacent to A, B, C. When the perimeter of triangle ABC is removed, there remains a triangle XYZ with sides divided into four equal parts and entries x, y, z, respectively, at vertices X, Y, Z. Since

$$a + b = p + v \ , \quad b + c = q + w \ , \quad c + a = r + u \ ,$$

$$\begin{aligned} x + y + z &= [(p+u) - a] + [(q+v) - b] + [(r+w) - c] \\ &= (p+v) + (q+w) + (r+u) - (a+b+c) = a+b+c \ . \end{aligned}$$

The sum of the entries along the sides of XYZ is equal to

$$\frac{5}{2}(x+y) + \frac{5}{2}(y+z) + \frac{5}{2}(z+x) - (x+y+z) = 4(a+b+c) \ .$$

When the perimeter of triangle XYZ is removed from triangle XYZ, there remains a single small triangle with three vertices. The sum of the entries at these vertices is $x+y+z = a+b+c$. Therefore, the sum of all the entries in the triangular array is $12(a+b+c)$. In the present situation, the answer is 1992.

378. Let $f(x)$ be a nonconstant polynomial that takes only integer values when x is an integer, and let P be the set of all primes that divide $f(m)$ for at least one integer m. Prove that P is an infinite set.

Solution 1. Suppose that $p_k(x)$ is a polynomial of degree k assuming integer values at $x = n, n+1, \cdots, n+k$. Then, there are integers $c_{k,i}$ for which

$$p_k(x) = c_{k,0}\binom{x}{k} + c_{k,1}\binom{x}{k-1} + \cdots + c_{k,k}\binom{x}{0} \ .$$

To see this, first observe that $\binom{x}{k}$, $\binom{x}{k-1}$, $\cdots$, $\binom{x}{0}$ constitute a basis for the vector space of polynomials of degree not exceeding k. So there exist *real* $c_{k,i}$ as specified. We prove by induction on k that the $c_{k,i}$ must in fact be integers. The result is trivial when $k = 0$. Assume its truth for $k \geq 0$. Suppose that

$$p_{k+1}(x) = c_{k+1,0}\binom{x}{k+1} + \cdots + c_{k+1,k+1}$$

takes integer values at $x = n, n+1, \cdots, n+k+1$. Then

$$p_{k+1}(x+1) - p_{k+1}(x) = c_{k+1,0}\binom{x}{k} + \cdots + c_{k+1,k}$$

is a polynomial of degree k which taken integer values at $n, n+1, \cdots, n+k$, and so $c_{k+1,0}, \cdots, c_{k+1,k}$ are all integers. Hence,

$$c_{k+1,k+1} = p_{k+1}(n) - c_{k+1,0}\binom{n}{k+1} - \cdots c_{k+1,k}\binom{n}{1}$$

is also an integer. (This is more than we need; we just need to know that the coefficients of $f(x)$ are all rational.)

Let $f(x)$ be multiplied by a suitable factorial to obtain a polynomial $g(x)$ with integer coefficients. The set of primes dividing values of $g(m)$ at integers m is the union of the set of primes for f and a finite set, so it is enough to obtain the result for g. Note that g assumes the values 0 and 1 only finitely often. Suppose that $g(a) = b \neq 0$ and let $P = \{p_1, p_2, \cdots, p_r\}$ be a finite set of primes. Define

$$h(x) = \frac{g(a + bp_1p_2 \cdots p_r x)}{b} \ .$$

Then $h(x)$ has integer coefficients and $h(x) \equiv 1 \pmod{p_1p_2 \cdots p_r}$. There exists an integer u for which $h(u)$ is divisible by a prime p, and this prime must be distinct from $p_1, p_2, \cdots, p_r$. The result follows.

Solution 2. Let $f(x) = \sum_k^n a_k x_n$. The number $a_0 = f(0)$ is rational. Indeed, each of the numbers $f(0)$, $f(1)$, $\cdots$, $f(n)$ is an integer; writing these conditions out yields a system of $n+1$ linear equations with integer coefficients for the coefficients a_0, a_1, $\cdots$, a_n whose determinant is nonzero. The solution of this equation consists of rational values. Hence all the coefficients of $f(x)$ are rational. Multiply $f(x)$ by the least common multiple of its denominators to get a polynomial $g(x)$ which takes integer values whenever x is an integer. Suppose, if possible, that values of $f(x)$ for integral x are divisible only by primes p from a finite set Q. Then the same is true of $g(x)$ for primes from a finite set P consisting of the primes in Q along with the prime divisors of the least common multiple. For each prime $p \in P$, select a positive integer a_p such that p^{a_p} does not divide $g(0)$. Let $N = \prod\{p^{a_p} : p \in P\}$. Then, for each integer u, $g(Nu) \not\equiv 0 \pmod{N}$. However, for all u, $g(Nu) = \prod p^{b_p}$, where $0 \leq b_p \leq a_p$. Since there are only finitely many numbers of this type, some number must be assumed by g infinitely often, yielding a contradiction. (Alternatively: one could deduce that $g(Nu) \leq N$ for all u and get a contradition of the fact that $|g(Nu)|$ tends to infinity with u.)

Solution 3. [R. Barrington Leigh] Let n be the degree of f. `Lemma.` Let p be a prime and k a positive integer. Then $f(x) \equiv f(x + p^{nk}) \pmod{p^k}$. `Proof by induction on the degree.` The result holds for $n = 0$. Assume that it holds for $n = m - 1$ and $f(x)$ have degree m. Let $g(x) = f(x) - f(x-1)$, so that the degree of $g(x)$ is $m - 1$. Then

$$\begin{aligned} f(x + p^{nk}) - f(x) &= \sum_{i=1}^{p^{nk}} g(x+i) \\ &= \sum_{i=1}^{p^{(n-1)k}} (g(x+i) + g(x+i+p^{(n-1)k}) + \ldots + g(x+i+(p^k-1)p^{(n-1)k}) \\ &\equiv \sum_{i=1}^{p^{(n-1)k}} p^k g(x+i) \equiv 0 \ , \end{aligned}$$

$\pmod{p^k}$. [Note that this does not require the coefficients to be integers.]

Suppose, if possible, that the set P of primes p that divide at least one value of $f(x)$ for integer x is finite, and that, for each $p \in P$, the positive integer a is chosen so that p^a does not divide $f(0)$. Let $q = \prod\{p^a : p \in P\}$. Then p^a does not divide $f(0)$, nor any of the values $f(q^n)$ for positive integer n, as these are all congruent modulo p^a. Since any prime divisor of $f(q^n)$ belongs to P, it must be that $f(q^n)$ is a divisor of q. But this contradicts the fact that $|f(q^n)|$ becomes arbitrarily large with n.

Solution 4. [F. Barekat] Let $f(x) = a_n x^n + \cdots + a_0$ where $n \geq 1$. Substituting $n+1$ integers for x yields a system of $n+1$ linear equations for a_0, a_1, $\cdots$, a_n which has integer coefficients. Such a system has rational solutions, so that the coefficients of the polynomial are rational numbers. (This can also be seen by forming the Lagrange polynomial for the $n+1$ values.) Let $g(x)$ be the product of $f(x)$ and c, a common multiple of all the denominators of the a_i. Then $g(x)$ has integer coefficients and takes integer values when x is an integer.

If $a_0 = 0$, then $n|g(n)$ for each integer n, and there are infinitely many primes among the divisors of the $g(n)$ and therefore among the divisor of the $f(n)$ (since only finitely many primes divide c), when n is integral. Suppose that $a_0 \neq 0$, and, if possible, that $g(n)$ is divisible only by the primes $p_1, p_2, \cdots, p_k$ for integer n. Let $ca_0 = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ and let

$$M = \{p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} : s_i > r_i \forall i\} .$$

The set M has infinitely many elements.

Suppose that $h(x) = (1/ca_0)g(x)$, so that the constant coefficient of $h(x)$ is 1. The polynomial $h(x)$ takes rational values when x is an integer, but only the primes $p_1, p_2, \cdots, p_k$ are involved in the numerator and denominator of these values written in lowest terms. In particular, for $m \in M$, $h(m)$ is an integer congruent to 1 modulo each p_i, so that $h(m) = \pm 1$. However, this would imply that either $h(x) = 1$ or $h(x) = -1$ infinitely often, which cannot occur for an nontrivial polynomail. Hence, there must be infinitely many primes divisors of the values of $g(n)$ for integral n.

Solution 5. [P. Shi] Let t be the largest positive integer for which $f(n)$ is a multiple of t for every positive integer n. Define $g(x) = (1/t)f(x)$. Then $g(n)$ takes integer values for every integer n, the greatest common divisor of all the $g(n)$ (n an integer) is 1, and the set of primes dividing at least one $g(n)$ is a subset of P.

Suppose if possible that $P \equiv \{p_1, p_2, \cdots, p_k\}$ is a finite set. Let $1 \leq i \leq k$. There exists an integer m_i such that $g(m_i)$ is not a multiple of p_i; since $g(m_i + jp_i) \equiv g(m_i) \pmod{p_i}$, $g(n)$ is not a multiple of p_i when $n \equiv m_i \pmod{p_i}$.

By the Chinese Remainder Theorem, there exists infinitely many numbers n for which $n \equiv m_i$ (mod p_i) for each i. For such n, $g(n)$ is not divisible by p_i for any i. At most finitely many such $g(n)$ are equal to ± 1. Each remaining one of the $g(n)$ must have a prime divisor distinct from the p_i, yielding a contradiction. The result follows.

379. Let n be a positive integer exceeding 1. Prove that, if a graph with $2n+1$ vertices has at least $3n+1$ edges, then the graph contains a circuit (*i.e.*, a closed non-self-intersecting chain of edges whose terminal point is its initial point) with an even number of edges. Prove that this statement does not hold if the number of edges is only $3n$.

Solution 1. If there are two vertices joined by two separate edges, then the two edges together constitute a chain with two edges. If there are two vertices joined by three distinct chains of edges, then the number of edges in two of the chains have the same parity, and these two chains together constitute a circuit with evenly many edges. We establish the general result by induction.

When $n = 2$, the graph has 5 vertices at at least 7 edges. Since a graph lacking circuits has fewer edges than vertices, there must be at least one circuit. If there is a circuit of length 5, then any additional edge produces circuit of length 3 and 4. If there is a circuit of length 3, then one of the remaining vertices must be joined to two of the vertices in the cicuit, creating two circuits of length 3 with a common edge. Suppressing this edge gives a circuit of length 4. Accordingly, one can see that there must be a circuit with an even number of edges.

Suppose that the result holds for $2 \leq n \leq m-1$. We may assume that we have a graph G with $2m+1$ edges and at least $3m+1$ vertices that contains no instances where two separate edges join the same pair of vertices and no two vertices are connected by more than two chains. Since $3m+1 > 2m$, the graph is not a tree or union of disjoint trees, and therefore must contain at least one circuit. Consider one of these circuits, L. If it has evenly many edges, the result holds. Suppose that it has oddly many edges, say $2k+1$ with $k \geq 1$. Since any two vertices in the circuit are joined by at most two chains (the two chains that make up the circuit), there are exactly $2k+1$ edges joining pairs of vertices in the circuit. Apart from the circuit, there are $(2m+1)-(2k+1)=2(m-k)$ vertices and $(3m+1)-(2k+1)=3(m-k)+k\geq 3(m-k)+1$ edges.

We now create a new graph G', by coalescing all the vertices and edges of L into a single vertex v and retaining all the other edges and vertices of G. This graph G' contains $2(m-k)+1$ vertices and at least $3(m-k)+1$ edges, and so by the induction hypothesis, it contains a circuit M with an even number of edges. If this circuit does not contain v, then it is a circuit in the original graph G, which thus has a circuit with evenly many edges. If the circuit does contain v, it can be lifted to a chain in G joining two vertices of L by a chain of edges in G'. But these two vertices of L must coincide, for otherwise there would be three chains joining these vertices. Hence we get a circuit, *all* of whose edges lie in G'; this circuit has evenly many edges. The result now follows by induction.

Here is a counterexample with $3n$ edges. Consider $2n+1$ vertices partitioned into a singleton and n pairs. Join each pair with an edge and join the singleton to each of the other vertices with a single edge to obtain a graph with $2n+1$ vertices, $3n$ edges whose only circuits are triangles.

Solution 2. [J. Tsimerman] For any graph H, let $k(H)$ be the number or circuits minus the number of components (two vertices being in the same component if and only if they are connected by a chain of edges). Let G_0 be the graph with $2n+1$ vertices and no edges. Then $k(G_0)=-(2n+1)$. Suppose that edges are added one at a time to obtain a succession G_i of graphs culminating in the graph G with $2n+1$ vertices and at least $3n+1$ edges. Since adding an edge either reduces the number of components (when it connects two vertices of separate components) or increases the number of circuits (when it connects two vertices in the same component), $k(G_{i+1})\geq k(G_i)+1$. Hence $k(G)\geq k(G_{3n+1})\geq -(2n+1)+(3n+1)=n$. Thus, the number of circuits in G is at least equal to the number of components in G plus n, which is at least $n+1$. Thus, G has at least $n+1$ circuits.

If a circuit has two edges, the result is known. If all circuits have at least three edges, then the total number of edges of all circuits is at least $3(n+1)$. Since $3(n+1)>3n+1$, there must be two circuits that share an edge. Let the circuits be A and B and the endpoints of the common edge be u and v. Follow circuit A along from u in the direction away from the adjacent vertex v, and suppose it first meets circuit B and w (which could coincide with v). Then there are three chains connecting u and w, namely the two complementary parts of B and a portion of A. The number of edges of two of these chains have the same parity, and can be used to constitute a circuit with an even number of edges.

A counterexample can be obtained by taking a graph with vertices $a_1, \cdots, a_n, b_0, b_1, \cdots, b_n$, with edges joining the vertex pairs (a_i, b_{i-1}), (a_i, b_i) and (b_{i-1}, b_i) for $1\leq i\leq n$.

380. Factor each of the following polynomials as a product of polynomials of lower degree with integer coefficients:

(a) $(x+y+z)^4-(y+z)^4-(z+x)^4-(x+y)^4+x^4+y^4+z^4$;

(b) $x^2(y^3-z^3)+y^2(z^3-x^3)+z^2(x^3-y^3)$;

(c) $x^4+y^4-z^4-2x^2y^2+4xyz^2$;

(d) $(yz+zx+xy)^3-y^3z^3-z^3x^3-x^3y^3$;

(e) $x^3y^3+y^3z^3+z^3x^3-x^4yz-xy^4z-xyz^4$;

(f) $2(x^4+y^4+z^4+w^4)-(x^2+y^2+z^2+w^2)^2+8xyzw$;

(g) $6(x^5+y^5+z^5)-5(x^2+y^2+z^2)(x^3+y^3+z^3)$.

Solution. (a) Let $P_1(x,y,z)$ be the expression to be factored. Since $P_1(0,y,z)=P_1(x,0,y)=P_1(x,y,0)=0$, three factors of $P_1(x,y,z)$ are x, y and z. Hence, $P_1(x,y,z)=xyzQ_1(x,y,z)$, where $Q_1(x,y,z)$ must be linear and symmetric. Hence $Q_1(x,y,z)=k(x+y+z)$ for some constant k. Since

$3k = P_1(1,1,1) = 81 - 48 + 3 = 36$,

$$P_1(x,y,z) = 12xyz(x+y+z) \ .$$

Comment. The factor $x+y+z$ can be picked up from the Factor Theorem using the substitution $x+y+z=0$ (*i.e.*, $x+y=-z$, $y+z=-x$, $z+x=-y$).

(b)

$$\begin{aligned} x^2(y^3-z^3) &+ y^2(z^3-x^3) + z^2(x^3-y^3) \\ &= x^2(y^3-z^3) + y^2(z^3-x^3) - z^2(z^3-x^3) - z^2(y^3-z^3) \\ &= (x^2-z^2)(y^3-z^3) + (y^2-z^2)(z^3-x^3) \\ &= (x-z)(y-z)[(x+z)(y^2+yz+z^2) - (y+z)(z^2+zx+x^2)] \\ &= (x-z)(y-z)[xy(y-x) + z^2(x-y) + z(y^2-x^2) + z^2(y-x)] \\ &= (x-z)(y-z)(y-x)[xy+z(y+x)] = (x-y)(y-z)(z-x)(xy+yz+zx) \ . \end{aligned}$$

(c)

$$\begin{aligned} x^4+y^4-z^4-2x^2y^2+4xyz^2 &= (x^4+2x^2y^2+y^4) - (z^4+4x^2y^2-4xyz^2) \\ &= (x^2+y^2)^2 - (z^2-2xy)^2 = (x^2+y^2+z^2-2xy)(x^2+y^2-z^2+2xy) \\ &= (x^2+y^2+z^2-2xy)[(x+y)^2-z^2] = (x^2+y^2+z^2-2xy)(x+y+z)(x+y-z) \ . \end{aligned}$$

(d) *Solution 1.*

$$\begin{aligned} (yz+zx+xy)^3 &- y^3z^3 - z^3x^3 - x^3y^3 \\ &= 3(xy^2z^3 + xy^3z^2 + x^2yz^3 + x^2y^3z + x^3yz^2 + x^3y^2z + 2x^2y^2z^2) \\ &= 3xyz(yz^2 + y^2z + xz^2 + xy^2 + x^2z + x^2y + 2xyz) \\ &= 3xyz(x+y)(y+z)(z+x) \ . \end{aligned}$$

Solution 2. Let the polynomial be $P_4(x,y,z)$. Since $P_4(0,y,z) = P_4(x,0,z) = P_4(x,y,0) = P_4(x,-x,0)$ $= P_4(0,y,-y) = P_4(-z,0,z) = 0$, $P_4(x,y,z)$ contains the factors $x, y, z, x+y, y+z, z+x$. Hence

$$P_4(x,y,z) = kxyz(x+y)(y+z)(z+x) \ .$$

Since $8k = P_4(1,1,1) = 24$, $k=3$ and we obtain the factorization.

Solution 3. [D. Rhee]

$$\begin{aligned} P_4(x,y,z) &= [z(x+y)+xy]^3 - x^3y^3 - y^3z^3 - z^3x^3 \\ &= z^3(x+y)^3 + 3z^2(x+y)^2xy + 3z(x+y)(xy)^2 - z^3(x+y)(x^2-xy+y^2) \\ &= (x+y)z[z^2(x+y)^2 + 3z(x+y)xy + 3(xy)^2 - z^2(x+y)^2 + 3z^2(xy)] \\ &= 3(x+y)xyz[z(x+y)+xy+z^2] = 3(x+y)xyz(x+z)(y+z) \ . \end{aligned}$$

(e) Let $P_5(x,y,z)$ be the polynomial to be factored. Since

$$x^3y^3 - x^4yz = x^3y(y^2-xz) = x^2(xy)(y^2-xz) \ ,$$

$$y^3z^3 - xyz^4 = yz^3(y^2-xz) \ ,$$

and

$$z^3x^3 - xy^4z = z^3x^3 - x^2y^2z^2 + x^2y^2z^2 - xy^4z = z^2x^2(zx - y^2) + xy^2z(zx - y^2) \ ,$$

it follows that

$$\begin{aligned} P_5(x, y, z) &= (y^2 - zx)[x^3y + yz^3 - z^2x^2 - xy^2z] \\ &= -(y^2 - zx)(z^2 - xy)(x^2 - yz) = (zx - y^2)(xy - z^2)(yz - x^2) \ , \end{aligned}$$

(f) Let $P_6(x, y, z, w)$ be the polynomial to be factored. Any factorization of $P_6(x, y, z, w)$ will reduce to a factorization of $P_6(x, y, 0, 0)$ when $z = w = 0$, so we begin by factoring this reduced polynomial:

$$P_6(x, y, 0, 0) = 2(x^4 + y^4) - (x^2 + y^2)^2 = (x^2 - y^2)^2 = (x + y)^2(x - y)^2 \ .$$

Similar factorizations occur upon suppressing other pairs of variables. So we look for linear factors that reduce to $x + y$ and $x - y$ when $z = w = 0$, *etc.*. Also the factors must either be symmetrical in x, y, z or come in symmetrical groups. The possibilities, up to sign, are $\{x + y + z + w\}$, $\{x + y + z - w, x + y - z + w, x - y + z + w, -x + y + z + w\}$ and $\{x + y - z - w, x - y + z - w, x - y - z + w\}$. Since $P_6(x, y, z, w)$ has degree 4, there are two possible factorizations:

$$(1) \qquad (x + y + z + w)(x + y - z - w)(x - y + z - w)(x - y - z + w)$$

$$(2) \quad -(x + y + z - w)(x + y - z + w)(x - y + z + w)(-x + y + z + w)$$

Checking (1) yields

$$\begin{aligned} &(x + y + z + w)(x + y - z - w)(x - y + z - w)(x - y - z + w) \\ &= [(x + y)^2 - (z + w)^2][(x - y)^2 - (z - w)^2] \\ &= [(x^2 + y^2 - z^2 - w^2) + 2(xy - zw)][(x^2 + y^2 - z^2 - w^2) - 2(xy - zw)] \\ &= (x^2 + y^2 - z^2 - w^2)^2 - 4(xy - zw)^2 \\ &= x^4 + y^4 + z^4 + w^4 + 2x^2y^2 + 2z^2w^2 - 2x^2z^2 - 2x^2w^2 - 2y^2z^2 - 2y^2w^2 \\ &\qquad - 4x^2y^2 - 4z^2w^2 + 8xyzw \\ &= x^4 + y^4 + z^4 + w^4 - 2(x^2y^2 + x^2z^2 + x^2w^2 + y^2z^2 + y^2w^2 + z^2w^2) + 8xyzw \\ &= 2(x^4 + y^2 + z^4 + w^4) - (x^2 + y^2 + z^2 + w^2)^2 + 8xyzw \ . \end{aligned}$$

Thus, we have found the required factorization. ((2), of course, is not correct.)

(g) Let $P_7(x, y, z)$ be the polynomial to be factored.

Solution 1. Note that

$$\begin{aligned} P_7(x, y, 0) &= 6(x^5 + y^5) - 5(x^2 + y^2)(x^3 + y^3) \\ &= (x + y)[6x^4 - 6x^3y + 6x^2y^2 - 6xy^3 + 6y^4 - 5(x^2 + y^2)(x^2 - xy + y^2) \\ &= (x + y)(x^4 - x^3y - 4x^2y^2 - xy^3 + y^4) \\ &= (x + y)[x^4 - 2x^2y^2 + y^4 - xy(x^2 + 2xy + y^2)] \\ &= (x + y)[(x + y)^2(x - y)^2 - xy(x + y)^2] = (x + y)^3(x^2 - 3xy + y^2) \ . \end{aligned}$$

Similarly, $(y + z)^3$ divides $P_7(0, y, z)$ and $(x + z)^3$ divides $P_y(x, 0, z)$. This suggests that we try the factorization

$$Q_7(x, y, z) \equiv (x + y + z)^3(z^2 + y^2 + z^2 - 3xy - 3yz - 3zx) \ .$$

Since $P_7(1,0,0) = 1 = Q_y(1,0,0)$ and $P_7(1,1,1) = 18 - 45 = -27 \neq Q_7(1,1,1) = 27(-6)$, this does not work. So we need to look at the above factorizations differently:

$$P_7(x,y,0) = (x+y)^2(x^3+y^3-2x^2y-2xy^2) \; ;$$

$$P_7(x,0,z) = (x+z)^2(x^3+z^3-2x^2z-2xz^2) \; ;$$

$$P_7(0,y,z) = (y+z)^2(y^3+z^3-2y^2z-2yz^2) \; .$$

This suggests the trial:

$$R_7(x,y,z) \equiv (x+y+z)^2(x^3+y^3+z^3-2x^2y-2xy^2-2y^2z-2yz^2-2z^2x-2zx^2+kxyz) \; .$$

Now $P_7(1,1,1) = -27$ and $R_7(1,1,1) = 9(-9+k)$, so this will not work unless $k = 6$. Checking, we find that

$$P_7(x,y,z) \equiv (x+y+z)^2(x^3+y^3+z^3-2x^2y-2xy^2-2y^2z-2yz^2-2z^2x-2zx^2+6xyz) \; .$$

Solution 2. [Y. Zhao] For $k = 1,2,3$, let $S_k = x^k + y^k + z^k$; let $\sigma_1 = x+y+z$, $\sigma_2 = xy+yz+zx$ and $\sigma_3 = xyx$. Then $S_1 = \sigma_1$, $S_2 = \sigma_1 S_1 - 2\sigma_2$, $S_3 = \sigma_1 S_2 - \sigma_2 S_1 + 3\sigma_3$, $S_4 = \sigma_1 S_3 - \sigma_2 S_2 + \sigma_3 S_1$ and $S_5 = \sigma_1 S_4 - \sigma_2 S_3 + \sigma_3 S_2$, so that $S_2 = \sigma_1^2 - 2\sigma_2$, $S_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$, $S_4 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 4\sigma_1\sigma_3 + 2\sigma_2^2$ and $S_5 = \sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1^2\sigma_3 + 5\sigma_1\sigma_2^2 - 5\sigma_2\sigma_3$.

$$\begin{aligned} P_7(x,y,z) &= 6S_5 - 5S_2S_3 \\ &= 6(\sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1^2\sigma_3 + 5\sigma_1\sigma_2^2 - 5\sigma_2\sigma_3) - 5(\sigma_1^2 - 2\sigma_2)(\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3) \\ &= \sigma_1^5 - 5\sigma_1^3\sigma_2 + 15\sigma_1^2\sigma_3 = \sigma_1^2(\sigma_1^3 - 5\sigma_1\sigma_2 + 15\sigma_3) \; . \end{aligned}$$

381. Determine all polynomials $f(x)$ such that, for some positive integer k,

$$f(x^k) - x^3 f(x) = 2(x^3 - 1)$$

for all values of x.

Solution. If $f(x)$ is constant, then $f(x) \equiv -2$. Suppose that $f(x)$ is a nonconstant polynomial of positive degree d. Then the degree of the terms of the left side must be greater than 3, so that $f(x^k)$ and $x^3 f(x)$ must have the same leading term. Therefore $\deg f(x^k) = \deg x^3 f(x)$, so that $kd = 3 + d$ or $3 = (k-1)d$. Therefore, $(k,d) = (4,1), (2,3)$.

Suppose that $(k,d) = (4,1)$. Since $f(0) = -2$, we must have $f(x) = ax - 2$ for some constant a. It is readily checked that this solution is valid for all values of a.

Suppose that $(k,d) = (2,3)$. Then $x^3[f(x)+2] = f(x^2) + 2$. From this equation, we see that its two sides must have terms of degree at least 3 and only terms of even degree; thus, its two sides involve terms in x^4 and x^6. It follows that $f(x)$ must have constant term -2, no term in x and x^2 and a term in x^3. Thus, $f(x) = bx^3 - 2$. Again, it can be checked that any function of this form is valid.

Comment. Many solvers forgot to deal with the possibility of a constant function. Also, having gotten the forms $ax - 2$ and $bx^3 - 2$, one should check that they actually work.

382. Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of these intervals. Show that S is a finite union of disjoint intervals of total length not less than 1.

Solution 1. The proof is by induction on the odd number of intervals. The result is obvious if the set contains only one interval. Suppose that it holds when there are $2k-1 \geq 1$ intervals. Let a set of $2k+1$

intervals satisfying the condition be given. Let I and J be the two intervals whose left endpoints are least. Suppose that $I \cap J = K$. Note that the lengths of $I\backslash J$ and $J\backslash I$ are the same.

Suppose that the intervals I and J are removed from the set. Then, by the induction hypothesis, there is a finite union T of disjoint intervals of total length consisting of all points that lie in oddly many intervals apart from I and J. Restore the intervals I and J to form the set S. Outside of the union of I and J, the sets S and T agree. If I is the leftmost interval, then S includes $I\backslash J$ along with $K \cap T$. The only part of T that might not belong to S must lie within the set $J\backslash I$; but this is compensated by the inclusion of $I\backslash J$. The result follows.

Solution 2. [Y. Zhao] That S is a union of disjoint intervals can be established. Let $2n+1$ intervals $I_0, I_1, \cdots, I_{2n}$ of unit length be given in increasing order of left endpoint. Define

$$f_i(x) = \begin{cases} 1, & \text{if } x \in I_i \ ; \\ 0, & \text{if } x \notin I_i \ . \end{cases}$$

for $0 \leq i \leq 2n$. Let

$$F(x) = \sum_{i=0}^{2n} (-1)^i f_i(x) \ .$$

Suppose that x is a real number in $I_0 \cup I_1 \cup \cdots \cup I_n$. Let j be the minimum index and k the maximum index of the intervals that contain x. Then $x \in I_i$ if and only if $j \leq i \leq k$, and so $F(x) = \sum_{i=j}^{k} (-1)^i$. The value of $|F(x)|$ is 0 if and only if there are an even number of summands, *i.e.* $k - j + 1$ is even and 1 if and only if there are an odd number of summands. If x belongs to none of the intervals, then $F(x) = 0$. Hence the length of S is equal to

$$\int_{-\infty}^{\infty} |F(x)| dx \geq \int_{-\infty}^{\infty} F(x) dx = \sum_{i=0}^{2n} (-1)^i \int_{-\infty}^{\infty} f_i(x) dx$$
$$= \sum_{i=0}^{2n} (-1)^i = 1$$

as desired.

383. Place the numbers $1, 2, \cdots, 9$ in a 3×3 unit square so that

(a) the sums of numbers in each of the first two rows are equal;

(b) the sum of the numbers in the third row is as large as possible;

(c) the column sums are equal;

(d) the numbers in the last row are in descending order.

Prove that the solution is unique.

Comment. The problem is not quite correct. The solution is unique up to the order of the first two rows. Most students picked this up.

Solution. The first two rows should contain six numbers whose sum S is as small as possible and is even. This sum is at least $1 + 2 + 3 + 4 + 5 + 6 = 21$, so the sum is at least 22.

If the sum of the first two rows is 22, then the entries must be $1, 2, 3, 4, 5, 7$. The row that contains 1 must contain 3 and 7. The column sums are each 15, so the column that contains 7 cannot contain 8 or 9, so must contain in its third row the number 6. Hence one of the columns consists of $7, 2, 6$. The column that contains 5 cannot contain 8, as the 2 has already been used in another column.

Taking the last row as $(9, 8, 6)$, we obtain the top two rows $(5, 4, 2)$ and $(1, 3, 7)$. This satisfies the conditions. Thus, we have a solution that minimizes the sum of the first two rows and maximizes the sum of the last row.

Comment. The last row sum cannot be more that $9+8+7$, and must be odd (45 minus the sum of the first two rows). So we can start with the last row as $(9,8,6)$ and work from there.

384. Prove that, for each positive integer n,

$$(3-2\sqrt{2})(17+12\sqrt{2})^n+(3+2\sqrt{2})(17-12\sqrt{2})^n-2$$

is the square of an integer.

Solution. Oberve that

$$(1\pm\sqrt{2})^2=3\pm 2\sqrt{2}$$
$$(1\pm\sqrt{2})^4=(3\pm 2\sqrt{2})^2=17\pm 12\sqrt{2}$$

and

$$-1=(1+\sqrt{2})(1-\sqrt{2}) .$$

The given expression is equal to

$$(1+\sqrt{2})^{4n-2}+(1-\sqrt{2})^{4n-2}+2[(1+\sqrt{2})(1-\sqrt{2})]^{2n-1}=[(1+\sqrt{2})^{2n-1}+(1-\sqrt{2})^{2n-1}]^2 .$$

Since

$$(1\pm\sqrt{2})^{2n-1}=\sum_{k=0}^{n-1}\binom{2n-1}{2k}2^k\pm\sqrt{2}\sum_{k=0}^{n-1}\binom{2n-1}{2k+1}2^k ,$$

the quantity in square brackets is the integer

$$\sum_{k=0}^{n-1}\binom{2n-1}{2k}2^{k+1} .$$

The result follows.

385. Determine the minimum value of the product $(a+1)(b+1)(c+1)(d+1)$, given that $a,b,c,d\geq 0$ and

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=1 .$$

Solution 1. By the inequality of the harmonic and geometric means of the four quantities, we have that

$$[(a+1)(b+1)(c+1)(d+1)]^{1/4}\geq\left[\frac{1}{4}\left(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}\right)\right]^{-1}=4 ,$$

whence the product must be at least $4^4=256$. This bound is achieved when $a=b=c=d=3$.

Solution 2. Let $u^4=(a+1)(b+1)$, $v^4=(c+1)(d+1)$. Then, by the Arithmetic-Geometric Means Inequality,

$$\begin{aligned}1&=\frac{a+b+2}{u^4}+\frac{c+d+2}{v^4}\\&\geq\frac{2\sqrt{(a+1)(b+1)}}{u^4}+\frac{2\sqrt{(c+1)(d+1)}}{v^4}\\&=\frac{2}{u^2}+\frac{2}{v^2}=\frac{2(u^2+v^2)}{u^2v^2}\\&\geq\frac{4uv}{u^2v^2}=\frac{4}{uv} ,\end{aligned}$$

so that $uv\geq 4$. The result follows with equality when $a=b=c=d=3$.

386. In a round-robin tournament with at least three players, each player plays one game against each other player. The tournament is said to be *competitive* if it is impossible to partition the players into two sets, such that each player in one set beat each player in the second set. Prove that, if a tournament is not competitive, it can be made so by reversing the result of a single game.

Solution. In the following solutions, let $a > b$ denote that player a beats (wins over) player b. Note that this relation is *not* transitive, *i.e.* $a > b$ and $b > c$ does not necessarily imply that $a > c$.

Solution 1. [A. Wice] Suppose there are n players. We establish two preliminary results:

(1) The players can be labelled so that $a_1 > a_2 > \cdots > a_n$;

(2) If the players can be labelled to form a loop or circuit, thus $a_1 > a_2 > \cdots > a_n > a_1$, then the tournament is competitive.

Statement (1) can be established by induction. (If it holds for all tournaments with fewer than n players, then consider any player z in the tournament with n players. The players that beat z can be formed into a line as can those whom z beats. Now insert the player between the two lines to get a line of players each beating the next.) For statement (2), suppose if possible there are two nonvoid sets A and B partitioning all the players such that each player in A beats each player in B. Any player beaten by a player in B must also lie in B. Suppose we have a loop as in Statement (2). If, say, a_i belongs to B, then so does a_{i+1} and so on all around the loop to a_{i-1}, yielding a contradiction.

Suppose that we have a non-competitive tournament with sets A and B as above. Let $a_1 > a_2 > \cdots > a_k$ and $b_1 > b_2 > \cdots > b_m$ be labellings of the players in A and B as permitted in result (1). We also have $a_k > b_1$ and $a_1 > b_m$. Reverse the game involving a_1 and b_m to make $b_m > a_1$. By result (2), we now have a tournament that is competitive.

Solution 2. Suppose that we are given a noncompetitive tournament T, and that the players are partitioned into two sets A and B for which each player in A beats each player in B. Suppose a is a player in A who loses to the smallest number of competitors in A; let A_1 be the subset of those in A who beat a. Suppose that b is a player in B who wins against the smallest number of players in B; let B_1 be the subset of B who loses to b.

In T, $a > b$. Form a new tournament T' from T by switching the result of the game between a and b, so that $b > a$ and otherwise the results in T and T' are the same. Suppose, if possible, that T' is noncompetitive. Then we can partition the set of players into two subsets U and V, for which each player in U beats each player in V.

Suppose that $a \in V$. Since a beat every player in B besides B, we must have that $U \cap B \subseteq \{b\}$, so that $U \subseteq A \cup \{b\}$. Indeed, $U \subseteq A_1 \cup \{b\}$, so that $A \backslash A_1 \subseteq V$. Consider a player x in U. This player lies in A_1 and must beat every player in $A \backslash A_1$ as well as a, and lose only to other players in A_1, *i.e.*, to fewer players in A than a loses to. But this contradicts the definition of a. Therefore, $a \in U$, so that $b \in U$ as well, since $b > a$ in T'.

Since b is beaten by every player in A in T, $V \cap A \subseteq \{a\}$, so that $V \subseteq B \cup \{a\}$. Indeed, $V \subseteq B_1 \cup \{a\}$, so that $B \backslash B_1 \subseteq U$. Any player in B_1 can win only against other competitors in B_1, ı.e.. to fewer players in B than b beats, giving a contradiction.

Hence $U \cup V = \{a, b\}$, contradicting the fact that the tournament has at least three players.

Comment. Several solvers were too loose in determining the pair that ought to be switched. Not just any pair of players from A and B will do; they have to be carefully delineated. A good thing to do in such a problem is to have an example that you can test your argument against. For example, consider the following tournament with four players a, b, c, d for which $a > c$, $a > d$, $b > a$, $b > c$, $b > d$, $c > d$. This is a noncompetitive tournament for which we can take

$$(A, B) = (\{a, b, c\}, \{d\}) \quad \text{or} \quad (\{a.b\}, \{c, d\}) \quad \text{or} \quad (\{b\}, \{a, c, d\},$$

The only game whose results can be reversed to give a noncompetitive tournament is that between b and d, which will result in the cycle $a > c > d > b > a$. The other reversals result in competitive tournaments: (1) $c > a$, $(A, B) = (\{b, c\}, \{a, d\})$; (2) $a > b$, $(A, B) = (\{a\}, \{b, c, d\})$; (3) $d > a$; $(A, B) = (\{b\}, \{a, c, d\})$; (4) $c > b$; $(A, B) = (\{a, b, c\}, \{d\})$: (5) $d > c$; $(A, B) = (\{a, b, d\}, \{c\})$.

387. Suppose that a, b, u, v are real numbers for which $av - bu = 1$. Prove that

$$a^2 + u^2 + b^2 + v^2 + au + bv \geq \sqrt{3}\ .$$

Give an example to show that equality is possible. (Part marks will be awarded for a result that is proven with a smaller bound on the right side.)

Solution 1. [C. Sun] Let $x = a^2 + b^2$, $y = u^2 + v^2$, $z = au + bv$. Then $xy = z^2 + 1$.

Observe that

$$(t\sqrt{3} + 1)^2 \geq 0 \Longrightarrow 3t^2 + 1 \geq -2t\sqrt{3} \Longrightarrow 4t^2 + 4 \geq (\sqrt{3} - t)^2\ .$$

From this, we find that

$$\begin{aligned}(x + y)^2 \geq 4xy = 4(z^2 + 1) = 4z^2 + 4 \geq (\sqrt{3} - z)^2 \\ \Longrightarrow x + y \geq \sqrt{3} - z \\ \Longrightarrow x + y + z \geq \sqrt{3}\end{aligned}$$

as desired.

Solution 2. [Y. Zhao] Note that

$$a^2 + u^2 + b^2 + v^2 + au + bv = \left(u + \frac{a}{2}\right)^2 + \left(v + \frac{b}{2}\right)^2 + \frac{3}{4}\left(a^2 + b^2\right)\ .$$

For each fixed a and b, the function is minimized when (u, v) is closest to $(\frac{a}{2}, \frac{b}{2})$. But (u, v) lies on the line $bx - ay + 1 = 0$, so the distance between (u, v) and $(-\frac{a}{2}, -\frac{b}{2})$ is at least equal to the distance from $(-\frac{a}{2}, -\frac{b}{2})$ to the line of equation $bx - ay + 1$, namely $(a^2 + b^2)^{-1/2}$. Hence

$$\left(u + \frac{a}{2}\right)^2 + \left(v + \frac{b}{2}\right)^2 + \frac{3}{4}\left(a^2 + b^2\right) \geq \frac{1}{a^2 + b^2} + \frac{3}{4}(a^2 + b^2) \geq \sqrt{3}$$

by the Arithmetic-Geometric Means Inequality. Equality occurs, for example, when

$$(a, b, u, v) = \left(\frac{2^{1/2}}{3^{1/4}}, 0, \frac{-1}{2^{1/2}3^{1/4}}, \frac{3^{1/4}}{2^{1/2}}\right)\ .$$

Solution 3. [G. Ghosn] We use a vector argument, with boldface characters denoting vectors. Let $\mathbf{a} = (a, b)$, $\mathbf{u} = (u, v)$ and $\mathbf{v} = (v, -u)$. It is given that $\mathbf{a} \cdot \mathbf{v} = 1$. Since $\mathbf{u}$ and $\mathbf{v}$ form a basis for which $\mathbf{u} \cdot \mathbf{v} = 0$, $\{\mathbf{u}, \mathbf{v}\}$ is an orthogonal basis for two-dimensional Euclidean space. Hence there are scalars α and β for which $\mathbf{a} = \alpha\mathbf{u} + \beta\mathbf{v}$. Taking the inner (dot) product of this equation with $\mathbf{v}$ yields $\beta = |\mathbf{v}|^{-2} = u^2 + v^2$.

We have that $a^2 + b^2 = \mathbf{a} \cdot \mathbf{a} = (\alpha^2 + \beta^2)(u^2 + v^2)$ and $\alpha u + \beta v = \mathbf{a} \cdot \mathbf{u} = \alpha(u^2 + v^2)$. Hence

$$\begin{aligned}a^2 + u^2 + b^2 + v^2 + au + bv &= (\alpha^2 + \beta^2 + 1 + \alpha)(u^2 + v^2) \\ &= \frac{1}{4}[(2\alpha + 1)^2 + 3 + 4\beta^2](u^2 + v^2) \\ &\geq \frac{3}{4}(u^2 + v^2) + \frac{1}{u^2 + v^2} \geq 2\left(\frac{\sqrt{3}}{2}\right)(u^2 + v^2)\left(\frac{1}{u^2 + v^2}\right) = \sqrt{3}\ ,\end{aligned}$$

by the Arithmetic-Geometric Means Inequality, with equality if and only if $\alpha = -1/2$ and $u^2 + v^2 = 2/\sqrt{3}$. We can achieve equality with

$$(a, b, u, v) = \left(\frac{3^{1/4}}{2^{1/2}}, \frac{-1}{2^{1/2}3^{1/4}}, 0, \frac{2^{1/2}}{3^{1/4}} \right) .$$

Solution 4. [A. Wice] Let $\mathbf{a} = (a, b)$ and $\mathbf{u} = (u, v)$, and let θ be the angle between the vectors $\mathbf{a}$ and $\mathbf{u}$. The area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{u}$ is equal to

$$|\mathbf{a} \times \mathbf{u}| = |\mathbf{a}||\mathbf{u}| \sin\theta = |av - bu| = 1 .$$

Observe that $0 < \theta < 180°$. We have that

$$\begin{aligned} a^2 + u^2 + b^2 + v^2 + au + bv &= |\mathbf{a}|^2 + |\mathbf{u}|^2 + \mathbf{a} \cdot \mathbf{u} \\ &= |\mathbf{a}|^2 + |\mathbf{u}|^2 + |\mathbf{a}||\mathbf{u}| \cos\theta \\ &\geq |\mathbf{a}||\mathbf{u}|(2 + \cos\theta) = \frac{2 + \cos\theta}{\sin\theta} \ , \end{aligned}$$

by the Arithmetic-Geometric Means Inequality.

Now

$$\begin{aligned} 1 &\geq -\cos(\theta + 60°) = -\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta \\ &\Longrightarrow 2 + \cos\theta \geq \sqrt{3}\sin\theta \\ &\Longrightarrow \frac{2 + \cos\theta}{\sin\theta} \geq \sqrt{3} \ , \end{aligned}$$

with equality if and only if $\theta = 120°$. Hence

$$a^2 + u^2 + b^2 + v^2 + au + bv \geq \sqrt{3}$$

with equality if and only if $|\mathbf{a}| = |\mathbf{u}| = 2^{1/2}/3^{1/4}$.

Hence we select

$$(a, b, u, v) = (k\cos\alpha, k\sin\alpha, k\cos(\alpha + 120°), k\sin(\alpha + 120°)$$

with $k = 2^{1/2}/3^{1/4}$ and some angle α. Taking $\alpha = 30°$ yields the example

$$(a, b, u, v) = \left(\frac{3^{1/4}}{2^{1/2}}, \frac{1}{3^{1/4} \cdot 2^{1/2}}, \frac{-3^{1/4}}{2^{1/2}}, \frac{1}{3^{1/4} \cdot 2^{1/2}} \right) .$$

Solution 5. Let $a = p\cos\phi$, $b = p\sin\phi$, $u = q\cos\theta$ and $v = q\sin\theta$, where p and q are positive reals. Then $1 = av - bu = pq\sin(\theta - \phi)$, from which we deduce that $pq \geq 1$ and that $\cos^2(\theta - \phi) = (p^2q^2 - 1)/(p^2q^2)$. Therefore

$$\begin{aligned} a^2 + u^2 + b^2 + v^2 + au + bv &= p^2 + q^2 + pq\cos(\theta - \phi) \\ &\geq p^2 + q^2 - pq\sqrt{(p^2q^2 - 1)/(p^2q^2)} = p^2 + q^2 - \sqrt{p^2q^2 - 1} \\ &\geq 2pq - \sqrt{p^2q^2 - 1} \ , \end{aligned}$$

by the Arithmetic-Geometric Means Inequality.

We need to show that $2t - (t^2 - 1)^{1/2} \geq 3^{1/2}$ for $t \geq 1$, or equivalently,

$$4t^2 + t^2 - 1 - 4t\sqrt{t^2 - 1} \geq 3 \Longleftrightarrow 5t^2 - 4 \geq 4t\sqrt{t^2 - 1} .$$

This in turn is equivalent to $25t^4 - 40t^2 + 16 \geq 16t^4 - 16t^2$ which reduces to the true inequality $(3t^2 - 4)^2 \geq 0$. The minimum of the left member of the inequality occurs when $t = 2/\sqrt{3}$ and $\cos^2(\theta - \phi) = (t^2 - 1)/t^2 = 1/4$.

Taking $p = q = 2^{1/2}3^{-1/4}$. $\phi = 150°$ and $\theta = 30°$ yields the example in Solution 4.

Solution 6. [C. Bao] This solution uses Lagrange Multipliers. Let

$$F(a,b,u,v,\lambda) = a^2 + u^2 + b^2 + v^2 + au + bv - \lambda(av - bu - 1) \ .$$

Then, the Lagrange conditions become

$$0 = \frac{\partial F}{\partial a} = 2a + u - \lambda v$$

$$0 = \frac{\partial F}{\partial b} = 2b + v + \lambda u$$

$$0 = \frac{\partial F}{\partial u} = 2u + a + \lambda b$$

$$0 = \frac{\partial F}{\partial v} = 2v + b - \lambda a$$

from which we obtain that

$$3(a+u) + \lambda(b-v) = 0 = (b-v) + \lambda(a+u) \ .$$

Therefore $3(a+u) = \lambda^2(a+u)$, so that, either $\lambda = \pm\pm\sqrt{3}$ or $a + u = b - v = 0$ at a critical point.

Suppose, first, that $\lambda^2 = 3$. Then

$$2a^2 + au + au + 2u^2 = \lambda(av - bu) \Longrightarrow \lambda = 2(a^2 + u^2 + au) \ ,$$

and

$$2v^2 + bv + vb + 2b^2 = \lambda(av - bu) \Longrightarrow \lambda = 2(b^2 + v^2 + bv) \ .$$

Therefore, at a critical point,

$$a^2 + u^2 + b^2 + v^2 + au + bv = \lambda \ .$$

Since, double the left side is equal to $(a+u)^2 + (b+v)^2 + a^2 + u^2 + b^2 + v^2$, we must have that $\lambda = \sqrt{3}$.

At this point in the argument, a technical difficulty arises, as it must be argued somehow that the critical point is a minimum, rather than a maximum or a saddle point. One way to do this is to establish that the objective function becomes infinite when we move towards infinity on the constraint surface, that it must attain a minimum value on the constraint surface (which requires a compactness argument) and use the fact that a single value of λ is turned up for a critical point.

The second possibility is that $a + u = b - v = 0$. Since $av - bu = 1$, this leads to $uv = -1/2$ and

$$a^2 + u^2 + b^2 + v^2 + au + bv = u^2 + 3v^2 \geq 2\sqrt{3}|uv| = \sqrt{3}$$

at the critical points.

Comment. This was not an easy problem and I garnered a larger collection of nice solutions than I expected. The lower bound of 2 is easily obtained by noting that

$$\begin{aligned} &2[a^2 + u^2 + b^2 + v^2 + au + bv] \\ &\qquad = [a^2 + u^2 + b^2 + v^2 + 2au + 2bv] + [a^2 + u^2 + b^2 + v^2 + 2bu - 2av] + 2 \\ &\qquad = (a+u)^2 + (b+v)^2 + (b+u)^2 + (a-v)^2 + 2 \geq 2 \ . \end{aligned}$$

Equality would require that $a = v = -u$ and $b = -u = -v$, which cannot be realized simultaneously.

388. A class with at least 35 students goes on a cruise. Seven small boats are hired, each capable of carrying 300 kilograms. The combined weight of the class is 1800 kilograms. It is determined that any group of 35 students can fit into the boats without exceeding the capacity of any one of them. Prove that it is unnecessary to leave any student off the cruise.

Solution. We prove the result by induction on the number of students. By hypothesis, if there are only 35 students, then all can be accommodated. Suppose that there are n students, where $n \geq 35$. Let the weights of these students be $w_1, w_2, \cdots, w_n$ kg in decreasing order.

Suppose that we have accommodated the k heaviest of these students for some $k \geq 35$. If $k < n$, then the weight w_{k+1} of the $(k+1)$th heaviest student satisfies

$$36w_{k+1} \leq w_1 + \cdots + w_{35} + w_{36} + \cdots + w_{k+1} \leq 1800 \ ,$$

whence $w_{k+1} \leq 50$. The amount of capacity available in the boats is

$$\begin{aligned} 2100 - (w_1 + \cdots + w_k) &\geq 2100 - (1800 - w_{k+1}) = 300 + w_{k+1} \\ &\geq 6w_{k+1} + w_{k+1} = 7w_{k+1} \ . \end{aligned}$$

Since all seven boats can accommodate at least $7w_{k+1}$ kg, at least one of them can accomodate at least w_{k+1} kg, and so the $(k+1)$th heaviest student can get into this boat. We can continue on in this way until all students are loaded.

389. Let each of m distinct points on the positive part of the $x-$axis be joined by line segments to n distinct points on the positive part of the $y-$axis. Obtain a formula for the number of intersections of these segments (exclusive of endpoints), assuming that no three of the segments are concurrent.

Solution 1. An intersection is determined by a choice of two points on each axis, and to each such choice there is exactly one intersection in which the segments are formed by taking the outer point on one axis and joining it to the inner point on the other. Thus there are $\binom{m}{2}\binom{n}{2}$ intersections.

Solution 2. [J. Park] Let the points on the $x-$axis be in order $X_1, \cdots, X_m$ and the points on the $y-$axis be in order $Y_1, \cdots, Y_n$. We draw the segments one at a time, starting with the segments $[X_i, Y_n]$ $(1 \leq i \leq m)$. This produces no interesection points. Now, for $1 \leq i \leq m$, draw $[X_i, Y_{n-1}]$, which produces $i-1$ intersection points with $[X_j, Y_n]$ when $1 \leq j \leq i-1$. All these segments ending in Y_{n-1} produce $1+2+\cdots+(m-1) = \binom{m}{2}$ intersections with segments ending in Y_n.

The segments ending in Y_{n-2} produce $\binom{m}{2}$ intersections with segments ending in Y_{n-1} and $\binom{m}{2}$ intersections with segments ending in Y_n. Continuing on, we find that the segments ending in Y_{n-j} make $j\binom{m}{2}$ intersections altogether with segments ending in Y_k for $k > n-j$ for $1 \leq j \leq n-1$. Hence the total number of intersections is

$$(1 + 2 + \cdots + \overline{n-1})\binom{m}{2} = \binom{n}{2}\binom{m}{2} \ .$$

390. Suppose that $n \geq 2$ and that $x_1, x_2, \cdots, x_n$ are positive integers for which $x_1 + x_2 + \cdots + x_n = 2(n+1)$. Show that there exists an index r with $0 \leq r \leq n-1$ for which the following $n-1$ inequalities hold:

$$x_{r+1} \leq 3$$

$$x_{r+1} + x_{r+2} \leq 5$$

$$\cdots$$

$$x_{r+1} + x_{r+2} + \cdots + r_{r+i} \leq 2i + 1$$

$$\cdots$$

$$x_{r+1} + x_{r+2} + \cdots + x_n \leq 2(n-r) + 1$$

$$\cdots$$

$$x_{r+1} + \cdots + x_n + x_1 + \cdots + x_j \leq 2(n+j-r) + 1$$

$$\cdots$$

$$x_{r+1} + x_{r+2} + \cdots + x_n + x_1 + \cdots + x_{r-1} \leq 2n - 1$$

where $1 \leq i \leq n-r$ and $1 \leq j \leq r-1$. Prove that, if all the inequalities are strict, then r is unique, and that, otherwise, there are exactly two such r.

Solution 1. First, consider the case that $n = 2$. Then $x_1 + x_2 = 6$. When $x_1 = 1, 2$, take $r = 0$; when $x_1 = 4, 5$, take $r = 1$, and when $x_1 = x_3 = 3$, take $r = 0$ or $r = 1$. Thus the result holds.

We prove the result by induction. Suppose that the result holds for $n = m \geq 2$. Let $x_1 + x_2 + x_3 + \cdots + x_{m+1} = 2(m+2)$. Observe that at least one x_i does not exceeds 2, since $3(m+1) > 2(m+2)$.

Case (i): Suppose that $x_k = 2$ for some k, which we may suppose exceeds 1. (Note that the conditions are cyclic in the indices.) Consider the set $\{x_1, x_2, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{m+1}\}$. This satisfies the hypotheses for the case $n = m$. We may suppose that $r = 0$, as the argument is essentially the same for any index. Then

$$\sum_{i=1}^{s} x_i \leq 2s + 1 \qquad (1 \leq s \leq k-1)$$

$$\sum_{i=1}^{k-1} x_i + \sum_{i=k+1}^{s} x_i \leq 2(s-1) + 1 \qquad (k+1 \leq s \leq m) \ .$$

Restoring x_k, we find that $\sum_{i=1}^{s} x_i \leq 2s+1$ for $1 \leq s \leq m$. Thus, a solution for $n = m$ yields a solution for $n = m+1$. Conversely, any solution for $n = m+1$ yields a solution for $n = m$. If $r \neq k$, the same value of r works in both cases. If $r = k$, then taking out x_k will give a solution for the $n = m$ case with $r = k+1$. Equality will occur in the solution for $n = m$ if and only if it occurs in the solutions for $n = m+1$.

Case (ii): Suppose $x_k = 1$ and $x_{k+1} \geq 2$ (when $k = m+1$, interpret this as $x_1 \geq 2$). The set $\{x_1, x_2, \cdots, x_{k-1}, x_{k+1} - 1, \cdots, x_{m+1}\}$ satisfies the hypothesis for the $n = m$ case. We may suppose that $r = 0$. When $k \geq 2$,

$$\sum_{i=1}^{s} x_i \leq 2s + 1 \qquad (1 \leq s \leq k-1)$$

$$\sum_{i=1}^{k-1} x_i + \left(\sum_{i=k+1}^{s} x_i \right) - 1 \leq 2(s-1) + 1 \qquad (k+1 \leq s \leq m+1) \ .$$

Restoring x_k, we find that $\sum_{i=1} x_i \leq 2s+1$. Thus $r = 0$ continues to work for the $n = m+1$ case. However, when $k = 1$ and $r = 0$, we have that, when $2 \leq s \leq m-1$, $(\sum_{i=2}^{s} x_i) - 1 \leq 2s - 1$ so that $\sum_{i=2}^{s} x_i \leq 2s$. Hence $\sum_{i=1}^{s} x_i \leq 2s+1$ for $1 \leq i \leq m-1$. Equality will occur in the solution for $n = m$ if and only if it occurs in the solutions for $n = m+1$. The results follows.

Solution 2. [B.H. Deng] We establish the existence of the requisite value of r. If the inequalities hold for $r = 1$, Suppose that at least one inequality fails for $r = 1$. Then we can select $m \geq 2$ so that

(1) $x_2 + \cdots + x_m = 2(m-1) + w$, where $w \geq 2$;

(2) for $2 \leq i < m$, $x_2 + \cdots + x_i = 2(i-1) + u$ and $u < w$;

(3) for $m < j \leq n$, $x_2 + \cdots + x_j = 2(j-1) + v$ and $v \leq w$.

Thus, m is the first index where the discrepancy between $x_2 + \cdots + x_m$ and $2(m-1)$ achieves its maximum value.

We show that $r = m$. If $m < j \leq n$, then

$$x_{m+1} + \cdots + x_j = [2(j-1) + v] - [2(m-1) + w] = 2(j-m) + (v-w)$$
$$\leq 2(j-m) < 2(j-m) + 1 \ .$$

In particular,

$$x_{m+1} + \cdots + x_n = (x_1 + \cdots + x_n) - (x_1 + \cdots + x_m)$$
$$= 2(n+1) - x_1 - [2(m-1) + w] = 2(n-m) + 4 - x_1 - w$$

from which

$$x_{m+1} + \cdots + x_n + x_1 = 2(n-m+1) + (2-w) \leq 2(n-m+1) \ .$$

If $2 \leq i < m$, then

$$x_{m+1} + \cdots + x_n + x_1 + \cdots + x_i = 2(n-m) + 4 - w + 2(i-1) + u$$
$$= 2(n-m+i) + 2 + (u-w) < 2(n-m+i) + 2 \ ,$$

from which

$$x_{m+1} + \cdots + x_n + x_1 + \cdots + x_i \leq 2(n-m+i) + 1 \ .$$

We deal with the question of uniqueness. Wolog, we may suppose that the inequalities hold when $r = 1$. Suppose that $k \geq 2$ is such that $x_2 + \cdots + x_i \leq 2(i-1)$ for $2 \leq i < k$ and $x_2 + \cdots + x_k = 2(k-1) + 1$. For $k < j \leq n$, we have that

$$x_{k+1} + \cdots + x_j = (x_2 + \cdots + x_j) - (x_2 + \cdots + x_k)$$
$$\leq 2(j-1) + 1 - 2(k-1) - 1 = 2(j-k).$$

Also,

$$x_{k+1} + \cdots + x_n = 2(n+1) - x_1 - 2(k-1) - 1 = 2(n-k+1) + 1 - x_1$$

so that

$$x_{k+1} + \cdots + x_n + x_1 \leq 2(n-k+1) + 1 \ .$$

For $2 \leq i \leq k-1$,

$$x_{k+1} + \cdots + x_n + x_1 + \cdots + x_i \leq 2(n-k+1) + 1 + 2(i-1) = 2(n-k+i) + 1 \ ,$$

so that the inequalities hold for $r = k$.

Comment. When $n > 2$ and all the x_i exceed 1, then there is a direct argument. Since $3n > 2(n+1)$ and $x_1 + \cdots + x_n = 2(n+1)$, we must have $x_i \leq 2$ for at least one index i. Since $x_i \geq 2$ for each i, there are two possibilities. Either, all the x_i but one are equal to 2 and the remaining one is equal to 4; or, all the x_i but two are equal to 2 and the remaining two are equal to 3. If $x_k = 4$, then we must take $r = k$ and we get strict inequality throughout.

In the second case, suppose, wolog, that $x_1 = 3$. If $x_2 = 3$ as well, then we can take $r = 1$ or $r = 2$; in the first case, $x_{r+1} = 3$, while in the second, $x_3 + \cdots + x_n + x_1 = 2(n-1) + 1 = 2n - 1$. If $x_k = 3$ for $3 \leq k \leq n-1$, then we can take $r = 1$ (in which case, for example, $x_2 + x_3 + \cdots + x_k = 2(k-1) + 1$) or $r = k$ (in which case, for example, $x_{k+1} + \cdots + x_1 = 2(n+1-k) + 1$.

391. Show that there are infinitely many nonsimilar ways that a square with integer side lengths can be partitioned into three nonoverlapping polygons with integer side lengths which are similar, but no two of which are congruent.

Comment. Unfortunately, there was an error in the formulation of the problem, and it is not known whether nonsimilar partitions can be made with integer side lengths as requested. However, full credit was given to any students who achieved infinitely many nonsimilar partitions, without satisfying the numerical condition.

Let $ABCD$ be the square. Let w be any integer and suppose that $u = 1 + w$ and $v = 1 + w + w^2$. Thus, $u(1 + w + w^2) = v(1 + w)$, and this will be the side length of the square. Select points E on AB, F on CD, G on DA and H on EF so that $|AE| = u$, $|EB| = uw(1 + w)$, $|CF| = u(1 + w)$, $|FD| = uw^2$, $|DG| = vw$, $|GA| = v$. Then the three trapezoids $AEHG$, $GHFD$ and $CFEB$ are similar and $|GH| = uw$. Different values of w give nonsimilar partitions, and the sides of the trapezoids parallel to the sides of the square will have integer lengths. For the slant sides to have integer lengths, it is necessary to make

$$u^2(w-1)^2 + v^2 = (w+1)^2(w-1)^2 + (w^2+w+1)^2 = 2w^4 + 2w^3 + w^2 + 2w + 2$$

a perfect square. The only possibility discovered so far is the case $w = 1$. We might achieve others if we can find values of m and n for which $w^2 - 1 = 2mn$ and $w^2 + w + 1 = m^2 - n^2$. It is unlikely that there are infinitely many possibilities. Can these by found using some other strategy?

392. Determine necessary and sufficient conditions on the real parameter a, b, c that

$$\frac{b}{cx+a} + \frac{c}{ax+b} + \frac{a}{bx+c} = 0$$

has exactly one real solution.

Solution 1. We look at a number of cases.

`Case 1.` $abc = 0$. If at least two of a, b, c vanish, then the equation is undefined. If, say, $a = 0$, $bc \neq 0$, then the equation becomes

$$\frac{b}{cx} + \frac{c}{b} = 0 \ ,$$

which, for $x \neq 0$ is equivalent to $c^2x + b^2 = 0$; this has the unique real solution $x = -b^2/c^2$.

`Case 2.` $abc \neq 0$, $(a^2 - bc)(b^2 - ca)(c^2 - ab) = 0$. Suppose, first, say, $a^2 - bc = b^2 - ca = 0$. Then, also, $c^2 - ab = 0$. It follows that $b/a = a/c = c/b = k$ for some nonzero value of k. Wew have that $k = c/b = (c/a)(a/b) = 1/k^2$, whence $k^3 = 1$ and $k = 1$. The equation becomes $0 = 3/(x + 1)$ with no real solution.

Suppose, on the other hand, say, that $a^2 - bc = 0$ and $(b^2 - ca)(c^2 - ab) \neq 0$. Then $b/a = a/c = k$ for some real $k \neq 1$. Then $b/c = k^2$ and the equation becomes

$$0 = \left(\frac{b}{c} + \frac{c}{a}\right)\left(\frac{1}{x+k}\right) + \left(\frac{a}{c}\right)\left(\frac{1}{k^2x+1}\right) = \frac{g(x)}{k(x+k)(k^2x+1)}$$

where $g(x) = (k^5 + 2k^2)x + (2k^3 + 1)$. Now $g(-k) = -(k^3 - 1)(k^3 + 1)$ and $g(-1/k^2) = k^3 - 1$. If $k \neq -1$, then $g(x)$ is divisible by neither $(x + k)$ nor $(k^2x + 1)$ and the equation has a single real solution. If $k = -1$, then $b = c = -a$ and the equation reduces to $1/(x + 1) = 0$ with no real solution.

`Case 3.` $abc(a^2 - bc)(b^2 - ca)(c^2 - ab)(ab^2 + bc^2 + ca^2) \neq 0$. For x such that $(cx + a)(ax + b)(bx + c) \neq 0$, the equation is equivalent to the quadratic equation $f(x) = 0$, where

$$\begin{aligned} f(x) &= b(ax+b)(bx+c) + c(cx+a)(bx+c) + a(cx+a)(ax+b) \\ &= (ab^2 + bc^2 + ca^2)x^2 + (a^3 + b^3 + c^3 + 3abc)x + (b^2c + c^2a + a^2b) \ . \end{aligned}$$

The discriminant of this quadratic is given by

$$\begin{aligned} D &\equiv (a^3 + b^3 + c^3 + 3abc)^2 - 4(ab^2 + bc^2 + ca^2)(b^2c + c^2a + a^2b) \\ &= a^6 + b^6 + c^6 - 3a^2b^2c^2 - 2(a^3b^3 + b^3c^3 + c^3a^3) + 2abc(a^3 + b^3 + c^3) \ . \end{aligned}$$

Since $f(-a/c) = bc^{-2}(a^2 - bc)(b^2 - ac) \neq 0$, $f(-b/a) \neq 0$ and $f(-c/b) \neq 0$, the equation $f(x) = 0$ has exactly the same solutions as the given equation. When $D = 0$, the equation has exactly one real solution. When $D \neq 0$, it has either no real solutions or exactly two real solutions.

`Case 4.` $abc(a^2 - bc)(b^2 - ca)(c^2 - ab)(a^3 + b^3 + c^3 + 3abc) \neq 0$; $ab^2 + bc^2 + ca^2 = 0$. In this case,

$$f(x) = (a^3 + b^3 + c^3 + 3abc)x + (b^2c + c^2a + a^2b)$$

and $f(x) = 0$ has exactly one real solution in the domain of the given equation.

`Case 5.` $abc(a^2 - bc)(b^2 - ca)(c^2 - ab) \neq 0$; $a^3 + b^3 + c^3 + 3abc = ab^2 + bc^2 + ca^2 = 0$. In this case, there is no real solution.

To sum up, we see that there is exactly one real solution if and only if one of the following conditions holds:

(a) exactly one of a, b, c vanishes;
(b) exactly two of b/a, a/c, c/b are equal to a real number distinct from -1 and $+1$;
(c) none of a, b, c, $a^2 - bc$, $b^2 - ac$, $c^2 - ab$, $ab^2 + bc^2 + ca^2$ vanishes and $D = 0$;
(d) none of a, b, c, $a^2 - bc$, $b^2 - ac$, $c^2 - ab$, $a^3 + b^3 + c^3 3abc$ vanishes and $ab^2 + bc^2 + ca^2 = 0$.

Comment. The difficulty of this problems lies in sorting out the different possibilities in order to make a comprehensive analysis. Most solvers simply put the left side of the equation over a common denominator and analyzed the quadratic equation with realizing what might happen with the coefficients. For example, one should be wary of the possibility that $a/b = b/c$, which means that $ax + b$ is a constant multiple of $bx + c$. In this case, putting the left side over a common cubic denominator introduces a spurious factor proportional to $ax + b$; the least common denominator is in fact no more than a quadratic.

Another take on the problem is to look at the graph of the left side. If the three linear denominators are not proportional, then there will be three vertical asymptotes, and one can analyze what the graph does between these asymptotes. This is a nice exercise for you to work on and show that the results are consistent with those obtained in the foregoing solution.

393. Determine three positive rational numbers x, y, z whose sum s is rational and for which $x - s^3$, $y - s^3$, $z - s^3$ are all cubes of rational numbers.

Solution 1. [A. Kong] Let $x = a/k$, $y = b/k$, $z = c/k$ where a, b, c, k are all integers. Suppose that $b = ma$ and $c = na$. Then

$$x - s^3 = \frac{1}{k^3}[ak^2 - (a + b + c)^3] = \frac{a}{k^3}[k^2 - a^2(1 + m + n)^3]$$

$$y - s^3 = \frac{1}{k^3}[bk^2 - (a + b + c)^3] = \frac{a}{k^3}[mk^2 - a^2(1 + m + n)^3]$$

$$z - s^3 = \frac{1}{k^3}[ck^2 - (a + b + c)^3] = \frac{a}{k^3}[nk^2 - a^2(1 + m + n)^3] \ .$$

We try to make $x = s^3$, so that $k^2 = a^2(1 + m + n)^3$; for example, let $a = 1$ and $1 + m + n$ be a square. When $(a, k, m, n) = (1, 8, 1, 2)$, we obtain the successful example $(x, y, z) = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4})$.

Solution 2. [C. Sun] Let $x - s^3 = a^3$, $y - s^3 = b^3$ and $z - s^3 = c^3$, so that $s - 3s^3 = a^3 + b^3 + c^3$. We try to make $a + b + c = x + y + z = s$; then we would have

$$\begin{aligned} s(2s - 1)(2s + 1) &= 4s^3 - s = s^3 - (s - 3s^3) \\ &= (a + b + c)^3 - (a^3 + b^3 + c^3) = 3(a + b)(b + c)(c + a) \ . \end{aligned}$$

Try $a+b = s$, $3(b+c) = 2s+1$ and $c+a = 2s-1$. Putting this with $a+b+c = s$ yields $c = (a+b+c)-(a+b) = 0$, $a = (c + a) - c = 2s - 1$ and $b = \frac{1}{3}(2s + 1)$. Then

$$s = a + b + c = \frac{8s - 2}{3} \Longrightarrow s = \frac{2}{5} \ .$$

This yields $(a,b,c) = (-\frac{1}{5}, \frac{3}{5}, 0)$ whence

$$(x,y,z) = (a^3+s^3, b^3+s^3, c^3+s^3) = \left(\frac{7}{125}, \frac{35}{125}, \frac{8}{125}\right) .$$

It can be checked that this works.

Solution 3. [F. Barekat] An example is

$$(x,y,z) = \left(\frac{49}{256}, \frac{49}{256}, \frac{62}{256}\right) .$$

To get this, let $x = p/u$, $y = q/v$ and $z = r/w$ and write s in the form m/uvw. Then

$$m^3 - pu^2v^3w^3 = \alpha^3 \ , \quad m^3 - qv^2u^3w^3 = \beta^3 \ , \quad m^3 - rw^2u^3v^3 = \gamma^3$$

which leads to $m(3m^2 - (uvw)^2) = \alpha^3 + \beta^3 + \gamma^3$. Playing around, we arrive at the possibility $m = 5$, $uvw = 8$ and $\alpha = \beta = 3$, $\gamma = 1$.

Solution 4. We try to make

$$x - s^3 = \frac{u^3s^3}{t^3} \ , \quad y - s^3 = \frac{v^3s^3}{t^3} \ , \quad z - s^3 = \frac{w^3s^3}{t^3} \ ,$$

which leads to

$$s - 3s^3 = s^3\left(\frac{u^3+v^3+w^3}{t^3}\right)$$

and

$$1 = s^2\left(\frac{u^3+v^3+w^3}{t^3} + 3\right) .$$

We select u, v, w to make the quantity in parentheses square (for example, we can try $u^3 + v^3 + w^3 = t^3$).

For example, $(u,v,w,t,s) = (3,4,5,6,1/2)$ yields

$$(8x, 8y, 8z) = \left(\frac{243}{216}, \frac{280}{216}, \frac{341}{216}\right)$$

and $(u,v,w,t,s) = (3,5,6,2,1/7)$ yields

$$(343x, 343y, 343z) = \left(\frac{35}{8}.\frac{133}{8}.\frac{224}{8}\right) .$$

These check out.

394. The average age of the students in Ms. Ruler's class is 17.3 years, while the average age of the boys is 17.5 years. Give a cogent argument to prove that the average age of the girls cannot also exceed 17.3 years.

Solution. Suppose that there are b boys and g girls and that the average age of the girls is a. Then the sum of all the ages of the students in the class is

$$17.3(b+g) = 17.5b + ag$$

whence $(17.3 - a)g = 0.2b$. Since the right side as well as g are positive, $17.3 - a$ must also be positive. Hence $a < 17.3$.

395. None of the nine participants at a meeting speaks more than three languages. Two of any three speakers speak a common language. Show that there is a language spoken by at least three participants.

Solution 1. `Case (i)`. *Each pair has a common language.* There are $\binom{9}{2} = 36$ pairs with at most $3 \times 9 = 27$ languages. By the Pigeonhole Principle, there exists a language spoken by at least two different pairs, which includes either three or four participants.

`Case (ii)`. *There exists a pair consisting of, say, A and B, who have no language in common.* These two know k languages between them, where $2 \leq k \leq 6$. Let the other participants be $P_1, \cdots, P_7$. Consider each triplet $\{A, B, P_i\}$, we see that each participant P_i knows at least one of the k languages. By the Pigeonhole Principle, one of the k languages is known by at least two of the P_i along with either A or B. The result follows.

Solution 2. [F. Barekat] If every speaker speaks a language shared by two other participants, then the result holds. On the other hand, suppose that there is a participant P each of whose languages is shared by at most one other person. Then at most three people speak one of P's languages and there are five participants, A, B, C, D, E, who have no language in common with P. Considering each pair of these five with P, we see that each pair of the five participants speaks a common language. Thus, A shares a language with each of B, C, D, E, and, speaking at most three languages, must share the same language with two of them. The result follows.

Solution 3. [A. Kong] We prove the result by contradiction. Suppose if possible that no language is spoken by more than two participants. Form a graph with 9 vertices corresponding to the participants and connect two vertices by an edge if and only if the corresponding participants have a common language. The number of edges does not exceed the number of languages. There are $\binom{9}{3} = 84$ triplets of vertices, each of which involves at least one edge. Any edge can be involved in at most 7 triplets. As two edges can be involved in 14 distinct triplets only if the edges have no common vertex, we can find at most four edges that can be involved in seven triplets each with no two triplets having an edge in common. Since each vertex can be an endpoint of at most three edges and each edge involves two vertices, the number of edges is at most $\lfloor \frac{1}{2}(3 \times 9) \rfloor = 13 = 7 + 6$. Hence, the number of triplets that are involved with these edges is at most $4 \times 7 + 9 \times 6 = 82$, a contradiction.

Comment. The hypothesis allows for the possibility that some participant may know *fewer* that three languages, so you should not base your argument on everyone knowing exactly three languages. This is a situation where a contradiction argument can be avoided, and you should try to do so.

396. Place 32 white and 32 black checkers on a 8×8 square chessboard. Two checkers of different colours form a *related pair* if they are placed in either the same row or the same column. Determine the maximum and the minimum number of related pairs over all possible arrangements of the 64 checkers.

Solution. The maximum number of related pairs is 256, achieved by putting 4×4 blocks of black checkers in diagonally opposite corners of the board and 4×4 blocks of white checkers in the other diagonally opposite corner, or else by alternating the colours as on a standard chessboard. The minimum number of related pairs is 128, achieved by filling four columns entirely with black checkers and the other four columns with white checkers. We now prove that these bounds hold.

Suppose that in a given row or column, there are x black and $8 - x$ white checkers. Then the number of related pairs is

$$x(8 - x) = 16 - (x - 4)^2 \leq 16$$

with equality if and only if $x = 4$. Hence the total number of related pairs cannot exceed $16 \times 16 = 256$.

The number of related pairs is independent of the order in which the rows or columns of checkers appear, so, wolog, we may suppose that the number of white checkers in the columns decreases as we go from left to right. Suppose that there is a white checker to the right of a black checker in some row, so that the white checker appears in a column with r white checkers and the black checker appears in a column with s white checkers, where $r \leq s$. Suppose now that we interchange the positions of just these two checkers. The number of related pairs in the rows remains unchanged, while the number of related pairs in the columns gets reduced by

$$[r(8-r)+s(8-s)]-[(r-1)(9-r)+(s+1)(7-s)]=2(s+1-r)>0\ .$$

We can continue this sort of exchange, reducing the number of related pairs each time, until every white checker is to the left of all the black checkers in the same row. Thus, we need consider only configurations in which no black checker is to the left of or above white checker.

Suppose that the ith row has x_i white checkers and the jth column has y_j white checkers, where $8 \geq x_1 \geq x_2 \geq \cdots \geq x_8 \geq 0$ and $8 \geq y_1 \geq y_2 \geq \cdots \geq y_8 \geq 0$. We have that $a_1+\cdots+x_8 = y_1+\cdots+y_8 = 32$. Setting $x_9 = 0$, we see that for each i, $x_i - x_{i+1}$ is equal to the number of indices j for which $y_j = i$.

The total number of related pairs is

$$\begin{aligned}\sum_{i=1}^{8} x_i(8-x_i)+\sum_{j=1}^{8} y_j(8-y_j) &= 8\times\sum x_i + 8\times\sum y_j - \sum x_i^2 - \sum y_j^2 \\ &= 8\times 32 + 8\times 32 - (x_1^2+\cdots+x_8^2) - \sum_{i=1}^{8}(x_i-x_{i+1})i^2 \\ &= 512 - [(x_1^2+\cdots+x_8^2)+(x_1-x_2)+4(x_2-x_3)+9(x_3-x_4)+\cdots] \\ &= 512 - [(x_1+x_2^2+2x_2+x_3^2+4x_3+\cdots)+(x_1+\cdots+x_8)] \\ &= 512 - [x_1^2+(x_2+1)^2+\cdots+(x_8+7)^2-140+32] \\ &= 620 - [x_1^2+(x_2+1)^2+\cdots+(x_8+7)^2]\ .\end{aligned}$$

Thus, we require the maximum of $x_1^2+(x_2+1)^2+\cdots+(x_8+7)^2$ when $x_1+\cdots+x_8 = 32$ and $8 \geq x_1 \geq \cdots \geq x_8 \geq 0$.

At this point, the argument becomes tedious and a simpler one is sought. Let $z_i = x_i + (i-1)$ with $1 \leq i \leq 8$. It is straightforward to check that $x_5 \leq 6$, $x_6 \leq 5$, $x_7 \leq 4$ and $x_8 \leq 4$, so that $z_1 \leq 8$, $z_2 \leq 9$, $z_3 \leq 10$, $z_4 \leq 11$, $z_5 \leq 10$, $z_6 \leq 10$, $z_7 \leq 10$ and $z_8 \leq 11$. The value 11 is possible for z_i only when $i = 4$ and we must have $(x_1,\cdots,x_8) = (8,8,8,8,0,0,0,0)$ or $i = 8$ and we must have $(x_1,\cdots,x_8) = (4,4,4,4,4,4,4,4)$. In both cases, the square sum is 492. In a similar way, we find that $x_1 \geq 4$, $x_2 \geq 4$, $x_3 \geq 3$, $x_4 \geq 2$ and so $z_1 \geq 4$, $z_2 \geq 5$, $z_3 \geq 5$, $z_4 \geq 5$, $z_5 \geq 4$, $z_6 \geq 5$, $z_7 \geq 6$, $z_8 \geq 7$. The value 4 is possible for z_i only when $i = 1$ or $i = 5$ and we have $(x_1,\cdots,x_8) = (8,8,8,8,0,0,0,0)$ or $i = 8$ and we must have $(x_1,\cdots,x_8) = (4,4,4,4,4,4,4,4)$. Otherwise, we must have $5 \leq z_i \leq 10$ for each i, and checking out the possibilities leads to square sums less than 492.

397. The altitude from A of triangle ABC intersects BC in D. A circle touches BC at D, intersects AB at M and N, and intersects AC at P and Q. Prove that

$$(AM+AN):AC = (AP+AQ):AB\ .$$

Solution 1. Let the circle intersect AD again at E.

$$\begin{aligned}(AE+AD)\cdot AD &= AE\cdot AD + AD^2 = AM\cdot AN + AB^2 - BD^2 \\ &= AM\cdot AN + AB^2 - BN\cdot BM = AM\cdot AN + (AN+NB)\cdot(AM+MB) - BN\cdot BM \\ &= AM\cdot AN + AN\cdot AB + NB\cdot AM = AM\cdot(AN+NB) + AN\cdot AB \\ &= (AM+AN)\cdot AB\ .\end{aligned}$$

Similarly, $(AE+AD)\cdot AD = (AP+AQ)\cdot AC$. The result follows.

Solution 2. [F. Barekat] Let O be the centre of the circle, and let S and T be the respective midpoints of MN and PQ. Then $OS \perp AB$, $OT \perp AC$,

$$AM + AN = 2AS = 2AO \cos \angle BAD = 2AO \sin \angle ABC$$

and

$$AP + AQ = 2AT = 2AO \cos \angle CAD = 2AO \sin \angle ACB \ .$$

Hence

$$AB : AC = \sin \angle ACB : \sin \angle ABC = (AP + AQ) : (AM + AN)$$

as desired.

398. Given three disjoint circles in the plane, construct a point in the plane so that all three circles subtend the same angle at that point.

Solution. If two circles of radii r and R are given with respective centres O and P, and if Q is a point at which both circles subtend equal angles, then $OQ : OP = r : R$. To prove this, draw tangents from Q to meet the circles of centres O and P at A and B respectively, so that $\angle AQO$ and $\angle BQP$ are half the subtended angles. Then the proportion is a consequence of the similarity of the triangles QAO and QBP.

Suppose first that $r < R$. The locus of Q turns out to be a circle (a circle of Apollonius). One way to see this is to introduce coordinates with O at the origin, P at $(p, 0)$ and Q at (x, y). The equation of the locus is $\sqrt{x^2 + y^2} = (r/R)\sqrt{(x-p)^2 + y^2}$. This simplifies to $(R^2 - r^2)(x^2 + y^2) + 2pr^2x - p^2r^2 = 0$, the equation of a circle. Let one pair of common tangents to the circles intersect at the point V on the same side of both circles and the other pair at W between the two circles. V is the centre of a dilation with factor r/R that takes the larger circle to the smaller, and W is the centre of a dilation with factor $-r/R$ that takes the larger circle to the smaller. V and W both lie on the locus and form a line of symmetry for the locus; hence it is a diameter of the locus circle. So to construct the locus, it suffices to determine the points V and W. This can be done for example by drawing parallel diameters to the two circles and noting that the line joining pairs of their endpoints must pass through either V or W (since the dilations takes one diameter to the other).

To solve the problem, for each of two pairs of the three circles, determine circle at which the two circles subtend equal angles. If these circles intersect, then the intersection points will be points at which all three circles subtend equal angles.

It remains to consider the case where at least two of the circles have the same radius. In this case, a reflection about the right bisector of the line of centres takes one circle to the other, and this right bisector is the locus desired. So the desired point is in this case, either the intersection of a line and a circle or of two lines.

399. Let n and k be positive integers for which $k < n$. Determine the number of ways of choosing k numbers from $\{1, 2, \cdots, n\}$ so that no three consecutive numbers appear in any choice.

Solution 1. An admissible choice of k numbers corresponds to a sequence of k 1's and $n - k$ 0's in a sequence of n terms where 1 appears in the ith position if and only if i is selected as one of the k numbers and three 1's do not appear consecutively. We count the number of such sequences.

Suppose that an admissible sequence has a occurrences of 11 and b occurrences of 1, separated by 0's, so that $k = 2a+b$. The patterns of two 1's can be interpolated among the patterns of one 1 in $\binom{a+b}{a}$ ways. $a+b-1$ zeros must be placed in that many slots between adjacent patterns of 11 and 1. (If $a + b - 1 > n - k$, then a sequence of the type specified cannot occur, but this will automatically come out in the final expression.)

If $a + b - 1 \le n - k$, then $(n - k) - (a + b - 1)$ 0's remain to be allocated, either at the beginning or the end of the sequence or in the $a + b - 1$ slots that already contain one 0. Recall that u identical objects can be distributed among v distinguishable boxes in $\binom{u+v-1}{v-1}$ ways. (To see this, place $u + v - 1$ identical objects in a line; select $v-1$ gaps between adjacent pairs to determine a partition into v boxes each with at least one object; now, remove the superfluous v objects, one from each box.) Applying this to the present situation, we deduce that the spare $(n-k)-(a+b-1)$ 0's can be distributed in

$$\binom{n-k-(a+b-1)+(a+b-1)-1}{a+b-1} = \binom{n-k-1}{k-a}$$

ways.

Thus, the total number of ways of selecting an admissible set of k numbers form $\{1,2,\cdots,n\}$ is

$$\sum\{\binom{a+b}{b}\binom{n-k+1}{a+b} : 2a+b=k, a\geq 0, b\geq 0\} = \sum_{a\geq 0}\binom{k-a}{a}\binom{n-k+1}{k-a} .$$

Solution 2. [K. Kim] Reformulate the problem as selecting a sequence of n integers with k ones and $n-k$ zeros, with 1 being selected if and only if i is selected from among $\{1,2,\cdots,n\}$. At most two ones can appear side by side. Begin with the $n-k$ zeros; there are $n-k+1$ "slots" separated by the zeros into which we may insert 0, 1 or 2 ones. Suppose that we have i pairs of ones in the slots, where $0\leq 2i\leq k$. We can pick the slots for these in $\binom{n-k+1}{i}$ ways. There are $n-k+1-i$ slots left over and we can fit the remaining singelton ones in them in $\binom{n-k+1-i}{k-2i}$ ways. Hence the total number of ways is

$$\sum\left\{\binom{n-k+1}{i}\binom{n-k+1-i}{k-2i} : 0\leq 2i\leq k\right\} .$$

Solution 3. [F. Barekat] Recall that

$$(1-x)^{-t} = \sum_{i=0}^{\infty}\binom{t+i-1}{t-1}x^i .$$

We transform the given problem into an equivalent problem. Each choice $(b_1,\cdots,b_k)$ of k distinct numbers from $\{1,2,\cdots,m\}$ given in increasing order corresponds to a choice $(c_1,\cdots,c_k)$ of k not necessarily distinct numbers from $\{1,2,\cdots,n-k+1\}$ given in increasing order with $c_i=b_i-(i-1)$ $(1\leq i\leq k)$.

Three of the numbers b_i are consecutive if and only if the corresponding three numbers c_i are equal. Hence the answer to the problem is equal to the number of choices of k numbers from $\{1,2,\cdots,n-k+1\}$ for which at most two numbers are equal to any value.

For each i with $1\leq i\leq n-k+1$, construct the quadratic $1+x+x^2$ and define the generating function

$$\begin{aligned} f(x) &= (1+x+x^2)^{n-k+1} = (1-x^3)^{n-k+1}(1-x)^{-(n-k+1)} \\ &= \sum_{j=0}^{\infty}(-1)^j\binom{n-k+1}{j}x^{3j}\sum_{i=0}^{\infty}\binom{n+i-k}{n-k}x^i \\ &= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{\infty}(-1)^j\binom{n-k+1}{j}\binom{n-3j}{n-k}\right]x^k . \end{aligned}$$

For each k, the coefficient counts the number of ways we can form x^k in the expression $f(x)$ by selecting $1, x$ or x^2 from the ith factor. This is the answer to the problem.

Comment. A similar argument to that of the third solution gives $\sum_{j=0}^{\infty}(-1)^j\binom{n-k+1}{j}\binom{n-rj}{n-k}$ for the number of choices that avoid r consecutive integers.

If $f(n,k)$ represents the number of admissible selections, then, for $n\geq 1$, $f(n,1)=n$ and $f(n,2)=\binom{n}{2}$ and for $n\geq 3$, $f(n,3)=\binom{n}{3}-(n-2)=\frac{1}{6}(n-2)(n-3)(n+2)$.

When $n \geq 5$. $k \geq 4$, we can develop some recursion relations. An admissible set of k numbers can be selected from $\{1, 2, \cdots, n\}$ not including n in $f(n-1, k)$ ways. Or we can select n and the remaining $k-1$ numbers from $\{1, 2, \cdots, n-1\}$ provided that both $n-2$ and $n-1$ are not selected in $f(n-1, k-1) - f(n-4, k-3)$ ways. Thus,

$$f(n, k) = f(n-1, k) + f(n-1, k-1) - f(n-4, k-3) \ .$$

Alternatively, we can look at the three cases where n is not chosen, where n is chosen but $n-1$ is not and where both n and $n-1$ are chosen but $n-2$ is not. This yields that

$$f(n, k) = f(n-1, k) + f(n-2, k-1) + f(n-3, k-2) \ .$$

In particular, we have that $f(n, 4) = \binom{n}{4} - (n-3)^2 = \frac{1}{24}(n-3)(n-4)(n^2+n-18)$ and $f(n, 5) = \binom{n}{5} - \frac{1}{2}(n-3)(n-4)^2 = \frac{1}{120}(n-3)(n-4)(n-5)(n-6)(n+8)$.

We have the following table of values

n ↓	$k \rightarrow$	1	2	3	4	5	6	7
1		1						
2		2	1					
3		3	3	0				
4		4	6	2	0			
5		5	10	7	1	0		
6		6	15	16	6	0	0	
7		7	21	30	19	3	0	0
8		8	28	50	45	16	1	0
9		9	36	77	90	51	10	0

400. Let a_r and b_r $(1 \leq r \leq n)$ be real numbers for which $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ and

$$b_1 \geq a_1 \ , \quad b_1 b_2 \geq a_1 a_2 \ , \quad b_1 b_2 b_3 \geq a_1 a_2 a_3 \ , \quad \cdots \quad , b_1 b_2 \cdots b_n \geq a_1 a_2 \cdots a_n \ .$$

Show that

$$b_1 + b_2 + \cdots + b_n \geq a_1 + a_2 + \cdots + a_n \ .$$

Solution. Since $b_1 \cdots b_s > 0$ for all s, each b_i is positive. Let $c_0 = 1$ and define

$$c_1 = \frac{b_1}{a_1} \ , \quad c_2 = \frac{b_1 b_2}{a_1 a_2} \ , \quad \cdots, \quad c_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \ .$$

Then $c_i \geq 1$ and

$$b_i = \frac{c_i}{c_{i-1}} a_i$$

for $1 \leq i \leq n$. We have that

$$\begin{aligned}
&(b_1 + \cdots + b_n) - (a_1 + \cdots + a_n) \\
&\quad = \left(\frac{c_1}{c_0} - 1\right) a_1 + \left(\frac{c_2}{c_1} - 1\right) a_2 + \cdots + \left(\frac{c_n}{c_{n-1}} - 1\right) a_n \\
&\quad = (c_1 - 1)(a_1 - a_2) + \left(c_1 + \frac{c_2}{c_1} - 2\right)(a_2 - a_3) + \left(c_1 + \frac{c_2}{c_1} + \frac{c_3}{c_2} - 3\right)(a_3 - a_4) + \cdots \\
&\qquad + \left(c_1 + \frac{c_2}{c_1} + \cdots + \frac{c_i}{c_{i-1}} - i\right)(a_i - a_{i-1}) + \cdots + \left(c_1 + \frac{c_2}{c_1} + \cdots + \frac{c_n}{c_{n-1}} - n\right) a_n \ .
\end{aligned}$$

By the Arithmetic-Geometric Means Inequality,

$$\frac{1}{i}\left[c_1+\frac{c_2}{c_1}+\cdots+\frac{c_i}{c_{i-1}}\right]\geq\left[c_1\left(\frac{c_2}{c_1}\right)\cdots\left(\frac{c_i}{c_{i-1}}\right)\right]^{1/i}=c_i^{1/i}\geq 1\ ,$$

so that $c_1+(c_2/c_1)+\cdots+(c_i/c_{i-1})\geq i$ and the result follows.

Comments. The transformation of the series $\sum((c_i/c_{i-1}-1)a_i$ is a standard way of dealing with series, known as *summation by parts* and analogous to the calculus technique of integration by parts. It is used as a means of incorporating the hypothesis $a_1\geq a_2\geq\cdots\geq a_n>0$.

An interesting argument comes from F. Barekat who claims a stronger result. The proof almost works, but there is a small fly in the ointment. It is not clear to me that this can be worked around or whether the claimed result is false and a counterexample can be found. We suppose that a_n and b_n are defined for all n such that $b_1b_2\cdots b_n\geq a_1a_2\cdots a_n>0$. It is claimed that, for all n,

$$(b_1+b_2+\cdots+b_n)-(a_1+a_2+\cdots+a_n)\geq a_n\left(1-\frac{a_1a_2\cdots a_n}{b_1b_2\cdots b_n}\right)\ .$$

When $n=1$, we have that

$$b_1-a_1=b_1\left(1-\frac{a_1}{b_1}\right)\geq a_1\left(1-\frac{a_1}{b_1}\right)\ .$$

Suppose that the result holds for $n=m$. Then

$$\begin{aligned}
&(b_1+\cdots+b_m+b_{m+1})-(a_1+\cdots+a_m+a_{m+1})-a_{m+1}\left(1-\frac{a_1\cdots a_ma_{m+1}}{b_1\cdots b_mb_{m+1}}\right)\\
&\geq a_m\left(1-\frac{a_1\cdots a_m}{b_1\cdots b_m}\right)+b_{m+1}+\frac{a_{m+1}a_1\cdots a_{m+1}}{b_1\cdots b_{m+1}}-2a_{m+1}\\
&=\frac{1}{b_1\cdots b_{m+1}}[a_mb_1\cdots b_mb_{m+1}-a_mb_{m+1}a_1\cdots a_m+b_{m+1}b_1\cdots b_{m+1}+a_{m+1}a_1\cdots a_{m+1}\\
&\qquad\qquad -2a_{m+1}b_1\cdots b_{m+1}]\\
&=\frac{1}{b_1\cdots b_{m+1}}[b_1\cdots b_m(a_mb_{m+1}+b_{m+1}^2-2a_{m+1}b_{m+1})-(a_1\cdots a_m)(a_mb_{m+1}-a_{m+1}^2)]\\
&=\frac{1}{b_1\cdots b_{m+1}}[(b_1\cdots b_m-a_1\cdots a_m)(a_mb_{m+1}+b_{m+1}^2-2a_{m+1}b_{m+1})\\
&\qquad\qquad +(a_1\cdots a_m)(b_{m+1}^2-2a_{m+1}b_{m+1}+a_{m+1}^2)]\\
&=\frac{1}{b_1\cdots b_{m+1}}[b_{m+1}(b_1\cdots b_m-a_1\cdots a_m)(a_m+b_{m+1}-2a_{m+1})+(a_1\cdots a_m)(b_{m+1}-a_{m+1})^2]\\
&=\frac{1}{b_1\cdots b_{m+1}}[b_{m+1}(b_1\cdots b_m-a_1\cdots a_m)((a_m-a_{m+1})\\
&\qquad\qquad +(b_{m+1}-a_{m+1}))+(a_1\cdots a_m)(b_{m+1}-a_{m+1})^2]\ .
\end{aligned}$$

The argument can be completed if $b_{m+1}\geq a_{m+1}$, but seems to be in trouble otherwise.

401. Five integers are arranged in a circle. The sum of the five integers is positive, but at least one of them is negative. The configuration is changed by the following moves: at any stage, a negative integer is selected and its sign is changed; this negative integer is added to each of its (immediate) neighbours (*i.e.*, its absolute value is subtracted from each of its neighbours).

Prove that, regardless of the negative number selected for each move, the process will eventually terminate with all integers nonnegative in exactly the same number of moves with exactly the same configuration.

Solution. We associate with each arrangement of numbers a doubly infinite sequence, and analyze how the corresponding sequence alters with each move. Suppose the numbers, given clockwise, are a, b, c, d, e, and that their positive sum is s. Pick one number, say a, as a starting point and construct the sequence of running totals as we proceed closkwise: $a, a+b, a+b+c, a+b+c+d, s, s+a, \cdots$. This produces a sequence of blocks of five numbers for which each block of five is obtained from a previous one by adding s to its entries, Now extend the sequence backwards, preserving this "quasi-periodic" property.

We make some observations. The (doubly-infinite) sequence is increasing if and only if all entries in the circle are nonnegative. If the circle has a negative entry, then the sequence decreases at this particular entry. Given any term in the sequence, there are at most finitely many terms following it that do not exceed it.

Suppose, without loss of generality, that the number c in the circle is negative and that the sum in the sequence up to a is t. Then we have the consecutive terms: t, $t+b$, $t+b+c$, $t+b+c+d$, $\cdots$; note that $t+b+c < t+b$. Now perform the operation on the five numbers, selecting c as the relevant negative number. Then, the numbers in the circle become a, $b+c$, $-c$, $d+c$, e and the corresponding terms in the sequence are t, $t+b+c$, $t+b$, $t+b+c+d$, $\cdots$. In other words, the effect on the sequence is that, for every pair of entries corresponding to b and c, the terms get interchanged, and a decreasing pair become increasing.

We define the following isomorphism (mathematically equivalent situation). Each configuration of five integers of five integers in a circle with a designated starting entry (a) for the sequence corresponds to a doubly infinite sequence that has the quasiperiodicity defined above, and every such sequence gives rise to a circle of five integers. The operation of the problem corresponds to the switching of periodic sets of decreasing pairs to increasing pairs. Note that the entries of the doubly infinite sequence stay the same; they just get rearranged.

Suppose that we focus on five consecutive positions in the original doubly infinite sequence. Each of these has a finite number, say p, of terms following it in the sequence that are smaller, and a finite number, say q, of terms preceding it that are bigger. Each switching operation on two entries will decrease p for one and decrease q for the other. Eventually, each of the five terms in the five original positions will end up $p-q$ positions to the right and we will have an increasing sequence. The result follows.

Comment. We can go part way, showing that the sequence of moves will terminate, by associating with each configuration a positive quantity that decreases. With the integers a, b, c, d, e with $c < 0$ and sum $s > 0$, we form the quantity $(a-c)^2+(b-d)^2+(c-e)^2+(d-a)^2+(e-b)^2$. If we make a move, pivoting on c, to get $a, b+c, -c, d+c, e$, the corresponding quantity is $(a+c)^2+(b-d)^2+(c+e)^2+(d+c-a)^(e-b-c)^2$. This is smaller than the preceding quantity by the positive amount $(-2c)s$. This difference depends on the size of $|c|$, and so we cannot get a fix on how long it will take to achieve a circle whose integers are all nonnegative.

402. Let the sequences $\{x_n\}$ and $\{y_n\}$ be defined, for $n \geq 1$, by $x_1 = x_2 = 10$, $x_{n+2} = x_{n+1}(x_n+1)+1$ $(n \geq 1)$ and $y_1 = y_2 = -10$, $y_{n+2} = y_{n+1}(y_n+1)+1$ $(n \geq 1)$. Prove that there is no number that is a term of both sequences.

Solution 1. We prove by induction that $x_n \equiv 10$ and $y_n \equiv -10 \equiv 91$ modulo 101 for all positive integers n. The proof is by induction.

The congruences are true for $n = 1, 2$. Suppose that they are true for all natural numbers less than $k+2$, and in particular that $x_k \equiv x_{k+1} \equiv 10$. Then

$$x_{k+2} \equiv 10(10+1)+1 = 111 \equiv 10$$

modulo 101. Similarly, $y_k \equiv y_{k+1} \equiv -10$ implies

$$y_{k+2} \equiv (-10)(-10+1)+1 = 91 \equiv -10$$

modulo 101. Since x_n and y_n are in different congrence classes modulo 101, the desired result follows.

Comment. How would one hit on this idea? One might get at it by working out the first few terms and seeing that the congruence modulo 101 remains stable. Of one might ask if there is a modulus for which the terms remain in the same modular class as the first two terms. This would require that the modulus m would satisfy $10 \equiv 10(10+1)+1 = 111$ modulo m, whence m should be 101.

Solution 2. [Y. Zhao] Working out the first few terms yields

$$\{x_n\} = \{10, 10, 111, 1222, 136865, 167385896, \cdots\}$$

and

$$\{y_n\} = \{-10, -10, 91, -818, -75255, 61483338, \cdots\} .$$

We prove by induction that, for all $n > 2$, (1) $x_n > |y_n|$ and, for all $n > 1$, (2) $|y_{n+1}| > x_n + 2$. In the proof, we repeatedly appeal to $|xy| = |x||y|$ and $|x+1| \geq |x| - 1$ for all real x and y. (Establish these.)

Ad (1): This holds for $n = 3$ and $n = 4$. Suppose (1) is true for $n < k$. Then

$$\begin{aligned} x_k &= x_{k-1}(x_{k-2}+1)+1 > |y_{k-1}|(|y_{k-2}|+1)+1 \\ &\geq |y_{k-1}| \cdot |y_{k-2}+1| + 1 = |y_{k-1}(y_{k-2}+1)| + 1 \\ &\geq |y_{k-1}(y_{k-2}+1)+1| = |y_k| . \end{aligned}$$

Thus, (1) follows by an induction argument.

Ad (2): This holds for $n = 2$ and $n = 3$. Suppose (2) is true for $n < k$. Then

$$\begin{aligned} |y_{k+1}| &= |y_k(y_{k-1}+1)+1| \geq |y_k(y_{k-1}+1)| - 1 \\ &= |y_k||y_{k-1}+1| - 1 \geq |y_k| \cdot (|y_{k-1}| - 1) - 1 > (x_{k-1}+2)(x_{k-2}+2-1) - 1 \\ &= x_{k-1} \cdot x_{k-2} + 2x_{k-2} + x_{k-1} + 1 > x_{k-1}(x_{k-2}+1) + 1 + 2 = x_k + 2 . \end{aligned}$$

Thus, (2) follows by an induction argument.

Now, we have that $x_2 < |y_3| < x_3 < |y_4| < x_4 < |y_5| < \cdots$, so that no numbers apart from 10 and -10 can appear twice among the terms of the two sequences. The result follows.

403. Let $f(x) = |1-2x| - 3|x+1|$ for real values of x.

(a) Determine all values of the real parameter a for which the equation $f(x) = a$ has two different roots u and v that satisfy $2 \leq |u - v| \leq 10$.

(b) Solve the equation $f(x) = \lfloor x/2 \rfloor$.

Solution 1. [D. Rhee] (a) We have that

$$f(x) = \begin{cases} x+4, & \text{if } x \leq -1; \\ -5x-2, & \text{if } -1 < x < \frac{1}{2}; \\ -x-4, & \text{if } x \geq \frac{1}{2}. \end{cases}$$

The graph of $f(x)$ consists of three line segments with nodes at $(-1, 3)$ and $(1/2, -9/2)$. The equation $f(x) = a$ has

• no solutions if $a > 3$;

• two solutions determined by the intersection points of the line $y = a$ and the lines $y = x + 4$ and $y = -5x - 2$ when $-9/2 \leq a \leq 3$;

• two solutions determined by the intersection points of the line $y = a$ and the lines $y = x + 4$ and $y = -x - 4$ when $a < -9/2$.

When $-9/2 \leq a \leq 3$, the intersection points in question are $(a-4, a)$ and $(-\frac{1}{5}(a+2), a)$. The inequality $2 \leq |u - v| \leq 10$ is equivalent to

$$2 \leq -\frac{1}{5}(a+2) - (a-4) = \frac{1}{5}(18 - 6a) \leq 10$$

(subject to the constraint on a). The values of a that satisfy the requirements of the problem are $-9/2 \leq a \leq 4/3$.

When $a < -9/2$, the intersection points of the graphs are $(a-4, a)$ and $(-a-4, a)$ and the inequality $2 \leq |u-v| \leq 10$ is equivalent to $2 \leq -2a \leq 10$. The values of a that satisfy the requirements of the problem are given by $-5 \leq a \leq -9/2$.

To sum up, the equation $f(x) = a$ has two different solutions u, v that satisfy $2 \leq u - v \leq 10$ if and only if $-5 \leq a \leq 4/3$.

(b) Consider the graphs of the functions $f(x)$ and $g(x) \equiv \lfloor x/2 \rfloor$, exploring the possible intersection points of the second graph with the three "branches" of the first graph.

Where do the graphs of $y = g(x)$ and $y = x + 4$ intersect? When $x < -10$, $\lfloor x/2 \rfloor > x/2 - 1 = x/2 - 5 + 1 \geq x/2 + x/2 + 4 = x + 4$, while, when $x > -8$, $\lfloor x/2 \rfloor < x/2 = x/2 + 4 - 4 \leq x/2 + 4 + x/2 = x + 4$. Thus, the solution must satisfy $-10 \leq x \leq 8$, when $\lfloor x/2 \rfloor$ takes one of the two values -5 and -4, We find that $x = -8$ or $x = -9$.

As for the intersection of the graphs of $y = g(x)$ and $y = -5x - 2$, since we are considering only those values of x for which $-1 \leq x \leq 1/2$ and $g(x)$ is equal to -1 or 0, we find that $x = -1/5$. Finally, when $x \geq 1/2$, $g(x)$ is positive while $-x - 4$ is negative.

Therefore, the only solutions of $f(x) = \lfloor x/2 \rfloor$ are $x = -9, -8, -1/5$.

Solution 2. [Y. Zhao] (a) Similar to the foregoing.

(b) Let $x/2 = n + r$, where n is the integer $\lfloor x/2 \rfloor$ and $0 \leq r < 1$. When $x < -1$, we find that $r = -(n+4)/2$. Since $0 \leq r < 1$, $n = -4$ or $n = -5$ so that $x = -8$ or $x = -9$. These check out.

When $-1 \leq x \leq 1/2$, $r = (-11n - 2)/10$ which leads to $-2/11 \geq n \geq -12/11$ and so $n = -1$ and $r = 9/10$. Thus, $x = -1/5$, which checks out.

When $x > 1/2$, there are no solutions (as in the first solution). The only solutions are $x = -9, -8, -1/5$.

Solution 3. [V. Krakovna] (a) Similar to the foregoing.

(b) This solution is based on a lot of information and relationships from the graphs of $f(x)$ and $g(x)$ and illustrates the geometric approach to solving equations. The equation $f(x) = \lfloor x/2 \rfloor$ has solutions when the line $y = x/2$ comes close (within vertical distance 1) of the graph of $y = f(x)$. For any solution, $f(x)$ must take integer values, as $\lfloor x/2 \rfloor$ always does.

Since $f(x)$ is negative and $x/2$ positive when $x \geq 0$, there are no solutions in this domain. When $-1 \leq x \leq 0$, $x/2$ is negative and the only negative integer values assumed by $f(x)$ are -1 and -2, Only the former yields a solution: $x = -1/5$. When $x < -1$, $x/2 - 1 \leq f(x) \leq x/2$ only when $-10 \leq x \leq -8$, and so $f(x)$ must assume one of the values $-6, -5, -4$. Checking out, we get the solutions $x = -9, -8$.

404. Several points in the plane are said to be *in general position* if no three are collinear.

(a) Prove that, given 5 points in general position, there are always four of them that are vertices of a convex quadrilateral.

(b) Prove that, given 400 points in general position, there are at least 80 nonintersecting convex quadrilaterals, whose vertices are chosen from the given points. (Two quadrilaterals are nonintersecting if they do not have a common point, either in the interior or on the perimeter.)

(c) Prove that, given 20 points in general position, there are at least 969 convex quadrilaterals whose vertices are chosen from these points. (**Bonus:** Derive a formula for the number of these quadrilaterals given n points in general position.)

Solution. [V. Krakovna] (a) Among the five points, there are three X, Y, Z that form a triangle with the other two points M, N outside of the triangle. To see this, select one of the $\binom{5}{3}$ possible triangles that has the smallest area. We note that a convex quadrilateral is characterized by the fact that any two adjacent vertices lie in the same halfplane determined by the side that does not contain these vertices.

Case (i): One of the points M lies outside the triangle on the opposite side of YZ to X but within the angular sector formed by, say, XY and XZ produced. Then $MYXZ$ is a convex quadrilateral.

Case (ii): Both points M and N lie on the opposite side of YZ to X, with M lying in the angle formed by XY and ZY produced and N lying in the angle formed by XZ and YZ produced. Then $MNZY$ is a convex quadrilateral.

Case (iii): Both points M and N lie on the opposite side of YZ to X and in the angle formed by XY and ZY produced. If N and X are on the same side of the line MY, then $XNMY$ is a convex quadrilateral; otherwise, N and Z are on the same side of MY and $MNZY$ is a convex quadrilateral.

(b) Partition the 400 points into groups of five as follows. Choose a line that is not parallel to any of the lines determined by two of the points. Imagine this line passing over the 400 points; it passes over them one at a time, so we can use it to separate off points five at a time. Thus, we have a succession of sets of five points, each included between a pair of lines parallel to the chosen line. From (a), for each group of five, we can select four that are vertices of a convex quadrilateral, so that there are at least 80 convex quadrilaterals determined.

(c) For n points in general position, there are at least $\binom{n}{5}/(n-4)$ convex quadrilaterals. There are $\binom{n}{5}$ possible sets of five points, each of which has a subset of 4 points yielding a convex quadrilateral. However, for each quadrilateral, there are $n-4$ posibilities for the fifth point of the group to which it might belong, so that each convex quadrilateral could be counted up to $n-4$ times. Hence, the number of distinct convex quadrilaterals is at least $\binom{n}{5}/(n-4)$. When $n = 20$, this number is 969.

405. Suppose that a permutation of the numbers from 1 to 100, inclusive, is given. Consider the sums of all triples of consecutive numbers in the permutation. At most how many of these sums can be odd?

Solution. [V. Krakovna] In the sequence of 100 numbers, there are 98 triples of consecutive numbers. We will show that it is impossible for all of them to have odd sums.

Let e and o denote an even and an odd number, respectively, in the sequence. A triple has an odd sum if and only if it is (o, o, o) or (e, e, o) in any order. We can get a succession of odd sums by extended sequences of the form $\cdots eeoeeoeeo \cdots$ or $\cdots oooooo \cdots$. However, because there are 50 even numbers, we cannot maintain either of these patterns for the whole sequence. So there must be some transition point between the two types; at such a point, there must be a double e followed or preceded by a string of os, and at least one even triple sum must occur.

However, we can arrange the numbers so that 97 of the 98 triples have odd sums:

$$(2, 4, 1, 6, 8, 3, 10, 12, 5, \cdots, 98, 100, 49, 51, 53, 55, \cdots, 97, 99) ,$$

the only even sum being $100 + 49 + 51$.

406. Let a, b. c be natural numbers such that the expression

$$\frac{a+1}{b} + \frac{b+1}{c} + \frac{c+1}{a}$$

is also equal to a natural number. Prove that the greatest common divisor of a, b and c, $\gcd(a, b, c)$, does not exceed $\sqrt[3]{ab + bc + ca}$, *i.e.*,

$$\gcd(a, b, c) \le \sqrt[3]{ab + bc + ca} .$$

Solution. [A. Guo] Let k be the sum in the problem and let d be the greatest common divisor of a, b, c.

Then $a = du$, $b = dv$, $c = dw$ for some positive integers u, v, w and

$$\begin{aligned} k &= \frac{a+1}{b} + \frac{b+1}{c} + \frac{c+1}{a} \\ &= \frac{a^2c + ac + b^2a + ab + bc + c^2b}{abc} \end{aligned}$$

if and only if

$$ab + bc + ca = kabc - (a^2c + b^2a + c^2b)$$

if and only if

$$d^2(uv + vw + wu) = d^3(kuv - (u^2w + v^2u + w^2v)) \ .$$

Hence d must divide $uv + vw + wu$, and so d^3 divides, and thus does not exceed, $ab + bc + ca$. The result follows.

407. Is there a pair of natural numbers, x and y, for which

(a) $x^3 + y^4 = 2^{2003}$?

(b) $x^3 + y^4 = 2^{2005}$?

Provide reasoning for your answers to (a) and (b).

Solution. (a) The answer is "no". Consider the equation modulo 13. By Fermat's Little Theorem, $2^{12} \equiv 1$, whence $2^{2003} \equiv 2^{11} = 2048 \equiv 7$ modulo 13. The only possibilities for the congruence of x^3 modulo 13 are 0, 1, 5, 8, 12, and the only possibilities for the congrence of y^4 modulo 13 are 0, 1, 3, 9. We cannot achieve a sum of 7 modulo 13.

(b) The answer is "yes". Observe that

$$2^{2005} = 2^{2004} + 2^{2004} = (2^{668})^3 + (2^{501})^4$$

so that one possible choice is $(x, y) = (2^{668}, 2^{501})$.

408. Prove that a number of the form $a000 \cdots 0009$ (with $n + 2$ digits for which the first digit a is followed by n zeros and the units digit is 9) cannot be the square of another integer.

Solution. (D. Rhee) It is easy to check that $n \neq 0$. Suppose, if possible, that $n \geq 1$ and that $a \cdot 10^{n+1} + 9 = b^2$. Then $a \cdot 10^{n+1} = (b-3)(b+3)$. The right side is even and the product of two numbers differing by 6; these two numbers have different remainders when divided by 5. Since the left side is a multiple of 5^{n+1}, exactly one of $b - 3$ and $b + 3$ is divisible by 5^{n+1}. This implies that $b + 3$. the greater of the two factors is at least equal to $2 \cdot 5^{n+1}$, while the smaller one is at most equal to $a \cdot 2^n$. Therefore,

$$6 = (b+3) - (b-3) \geq 2 \cdot 5^{n+1} - a \cdot 2^n \geq 2 \cdot 5^{n+1} - 9 \cdot 2^n = 2^n(5^{n+1}/2^{n-1} - 9) \geq 2(25 - 9) = 32 > 6 \ ,$$

which is false. Hence, our assumption was wrong and there is no integer b for which the given number is equal to its square.

409. Find the number of ways of dealing n cards to two persons ($n \geq 2$), where the persons may receive unequal (positive) numbers of cards. Disregard the order in which the cards are received.

Solution. If we allow hands with no cards, there are 2^n ways in which they may be dealt (each card may go to one of two people). There are two cases in which a person gets no cards. Subtracting these gives the result: $2^n - 2$.

410. Prove that $\log n \geq k \log 2$, where n is a natural number and k the number of distinct primes that divide n.

Solution. Let n be a natural number greater than 1 and $p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ its prime factorization. Since $p_i \geq 2$ and $a_i \geq 1$ for all i,

$$n \geq 2^{a_1+a_2+\cdots+a_k} \geq 2^k \ .$$

This is also true for $n = 1$, for in this case, $k = 0$ and $n = 2^0$. Thus, for any base b exceeding 1,

$$\log_b n \geq \log_b 2^k = k \log_b 2 \ .$$

411. Let b be a positive integer. How many integers are there, each of which, when expressed to base b, is equal to the sum of the squares of its digits?

Solution. A simple calculation shows that 0 and 1 are the only single-digit solutions. We show that there are no solutions with three or more digits. Suppose that $n = a_0 + a_1 b + \cdots + a_m b^m$ where $m \geq 2$, $1 \leq a_m \leq b-1$ and $0 \leq a_i \leq b-1$ for $0 \leq i \leq m-1$. Then

$$\begin{aligned}(a_0 + a_1 b + \cdots + a_m b^m) &- (a_0^2 + a_1^2 + \cdots + a_m)^2 \\ &= a_1(b-a_1) + a_2(b^2-a_2) + \cdots + a_m(b^k - a_m) - a_0(a_0-1) \\ &\geq a_m(b^m - a_m) - a_0(a_0-1) \geq 1\cdot(b^2-(b-1)) - (b-1)(b-2) \\ &= 2b - 1 \geq 0 \ .\end{aligned}$$

Thus, there are at most two digits for any example.

Let $N(b)$ denote the total number of solutions, and $N_2(b)$ the number of two digit solutions. Thus, $N(n) = N_2(b) + 2$.

Thus, $N_2(n)$ is the number of pairs (a_0, a_1) satisfying

$$a_0 + a_1 b = a_0^2 + a_1^2 \ , \qquad 0 \leq a_0 < b, 1 \leq a_1 < b \ . \tag{1}$$

The transformation given by $2a_0 = p+1$, $2a_1 = b+q$ establishes a one-one correspondence between the pairs (a_0, a_1) satisfying (1) and the pairs (p, q) satisfying

$$p^2 + q^2 = 1 + b^2 \ , \qquad p \text{ odd } , 3 \leq p \leq b, 1 \leq q \leq b \ . \tag{3}$$

Now we can express the number of solutions of (2) in terms of the number $r(k)$ of solutions to

$$c^2 + d^2 = k \ . \tag{3}$$

Suppose that b is even. Then $1 + b^2$ is odd, so that exactly one of p or q is odd. Thus, given a solution (p, q) to (2) we can generate three others that solve (3) via $(c, d) = (-p, q), (q, p), (q-p)$. We also add the eight remaining solutions $(\pm 1, \pm b)$ and $(\pm b, \pm 1)$. This shows that $r(1+b^2) = 4N_2(b) + 8 = 4N(b)$.

Suppose that b is odd. Then $1 + b^2 \equiv 2 \pmod 4$; hence, both p and q must be odd. Thus, from any solution (p, q) to (2) we can generate another solution to (3) via $(c, d) = (-p, q)$. We also add the remaining four uncounted solutions, $(\pm 1, \pm b)$. This shows that $r(1+b^2) = 2N_2(b) + 4 = 2N(b)$.

The quantity $r(k)$ can be computed from a formula given, for example, in the book *Introduction to the Theory of Numbers* by Hardy and Wright. Using the fact that no prime of the form $4j+3$ can divide $1+b^2$, we find that

$$r(1+b^2) = \begin{cases} 4\tau(1+b^2) \ , & \text{if } b \text{ is even,} \\ 2\tau(1+b^2) \ , & \text{if } b \text{ is odd,} \end{cases}$$

where $\tau(n)$ is the number of positive integer divisors of n. Thus $N(b) = \tau(1+b^2)$.

412. Let A and B be the midpoints of the sides, EF and ED, of an equilateral triangle DEF. Extend AB to meet the circumcircle of triangle DEF at C. Show that B divides AC according to the golden section. (That is, show that $BC : AB = AB : AC$.)

Solution. Consider the chords ED and CC'. The angles EBC' and CBD are equal, since they are vertically opposite, while angles $C'ED$ and DCC' are equal since they are subtended by the same chord $C'D$. Thus triangles $C'EB$ and DCB are similar. Therefore $EB : C'B = BC : BD$.

Since $EB = BD = AB$,

$$BC : AB = BC : BD = EC : C'B = AB : AC \ .$$

413. Let I be the incentre of triangle ABC. Let A', B' and C' denote the intersections of AI, BI and CI, respectively, with the incircle of triangle ABC. Continue the process by defining I' (the incentre of triangle $A'B'C'$), then $A''B''C''$, *etc.*. Prove that the angles of triangle $A^{(n)}B^{(n)}C^{(n)}$ get closer and closer to $\pi/3$ as n increases.

Solution. From triangle IAC we have that $\angle AIC = \pi - \frac{A}{2} - \frac{C}{2} = \frac{\pi+B}{2}$, so that $B' = \angle A'B'C' = \frac{1}{2}\angle A'IC' = \frac{1}{2}\angle AIC = \frac{\pi+B}{4}$. Similar relations hold for A' and C'. Assuming, wolog, $A \leq B \leq C$, then $A' = \frac{1}{4}(\pi + A) \leq B' \leq \frac{1}{4}(\pi + B) \leq C' = \frac{1}{4}(\pi + C)$, and $C' - A' = \frac{1}{4}(C - A)$, so that triangle $A'B'C'$ is "four times closer" to equilateral than triangle ABC is. The result follows.

414. Let $f(n)$ be the greatest common divisor of the set of numbers of the form $k^n - k$, where $2 \leq k$, for $n \geq 2$. Evaluate $f(n)$. In particular, show that $f(2n) = 2$ for each integer n.

Solution. For any prime p, $f(n)$ cannot contain a factor p^2 because $p^2 \not| k(k^{n-1} - 1)$ for $k = p$. For any n, $2|f(n)$.

If p is an odd prime and if a is a primitive root modulo p, then $p|a(a^{n-1} - 1)$ only if $(p-1)|(n-1)$. On the other hand, if $(p-1)|(n-1)$, then $p|(k^n - k)$ for every k. Thus, if P_n is the product of the distinct odd primes p for which $(p-1)|(n-1)$, then $f(n) = 2P_n$. (In particular, $6|f(n)$ for every odd n.)

As $p - 1$ is not a divisor of $2n - 1$ for any odd prime p, it follows that $f(n) - 2$.

Comments. The symbol | means "divides" or "is a divisor of". For every prime p, there is a number a (called the *primitive root* modulo p such that $p - 1$ is the smallest values of k for which $a^k \equiv 1$ modulo p.

415. Prove that

$$\cos\frac{\pi}{7} = \frac{1}{6} + \frac{\sqrt{7}}{6}\left(\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right) + \sqrt{3}\sin\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right)\right) .$$

Solution. The identity

$$\cos 7\theta = (\cos\theta + 1)(8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1)^2 - 1$$

(derive this using de Moivre's theorem, or otherwise) implies that the three roots of $f(x) = 8x^3 - 4x^2 - 4x + 1$ are $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$ and $\cos\frac{5\pi}{7}$. Observe that $\cos\frac{\pi}{7} > \cos\frac{3\pi}{7} > 0 > \cos\frac{5\pi}{7}$. Thus, $\cos\frac{\pi}{7}$ is the only root of the cubic polynomial $f(x)$ greater than $\cos\frac{3\pi}{7}$.

Let

$$a = \cos\left(\frac{1}{3}\arccos\frac{-1}{2\sqrt{7}}\right) ,$$

and let

$$c = \frac{1}{6} + \frac{\sqrt{7}}{6}\left(\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right) + \sqrt{3}\cos\left(\frac{1}{3}\arccos\frac{1}{2\sqrt{7}}\right)\right)$$
$$= \frac{1}{6} + \frac{\sqrt{7}}{3}\cos\left(\frac{1}{3}\left(\pi - \arccos\frac{1}{2\sqrt{7}}\right)\right)$$
$$= \frac{1}{6} + \frac{\sqrt{7}}{3}a \ .$$

The function $g(x) = \cos(\frac{1}{3}\arccos x)$ is increasing for $-1 \le x \le 1$, so that $a > \cos(\frac{1}{3}\arccos(-1)) = \frac{1}{2}$. Therefore

$$x > \frac{1+\sqrt{7}}{6} > \frac{1}{2} > \cos\frac{3\pi}{7} \ .$$

Since $6c - 1 = 2\sqrt{7}a$, the identity $4\cos^3\theta - 3\cos\theta = \cos 3\theta$ gives

$$\frac{1}{14\sqrt{7}}(6c-1)^3 - \frac{3}{2\sqrt{7}}(6c-1) = \frac{-1}{2\sqrt{7}} \ .$$

Hence

$$f(c) = \frac{14\sqrt{7}}{27}\left(\frac{1}{14\sqrt{7}}(6c-1)^3 - \frac{3}{2\sqrt{7}}(6c-1) + \frac{1}{2\sqrt{7}}\right) = 0 \ ,$$

and so $c = \cos\frac{\pi}{7}$.

416. Let P be a point in the plane.

(a) Prove that there are three points A, B, C for which $AB = BC$, $\angle ABC = 90°$, $|PA| = 1$, $|PB| = 2$ and $|PC| = 3$.

(b) Determine $|AB|$ for the configuration in (a).

(c) A rotation of $90°$ about B takes C to A and P to Q. Determine $\angle APQ$.

Solution 1. (a) We first show that a figure similar to the desired figure is possible and then get the lengths correct by a dilatation. Place a triangle in the cartesian plane with A at $(0,1)$, B at $(0,0)$ and C at $(1,0)$. Let P be at (x,y). The condition that $PA : PB = 1 : 2$ yields that

$$x^2 + y^2 = 4[x^2 + (y-1)^2] \Longleftrightarrow 0 = 3x^2 + 3y^2 - 8y + 4 \ .$$

The condition that $PB : PC = 2 : 3$ yields that

$$9[x^2 + y^2] = 4[(x-1)^2 + y^2] \Longleftrightarrow 0 = 5x^2 + 5y^2 + 8x - 4 \ .$$

Hence $3x + 5y = 4$, so that $9x^2 = 16 - 40y + 25y^2$ and

$$0 = 2(17y^2 - 32y + 14) \ .$$

Solving these equations yields

$$(x,y) = \left(\frac{5\sqrt{2}-4}{17}, \frac{16-3\sqrt{2}}{17}\right) \ ,$$

a point that lies within the positive quadrant, and

$$(x,y) = \left(\frac{-5\sqrt{2}-4}{17}, \frac{16+3\sqrt{2}}{17}\right) \ ,$$

a point that lies within the second quadrant..

(b) In the first situation,

$$|PB|^2 = \frac{20 - 8\sqrt{2}}{17} \; .$$

Rescaling the figure so that $|PB| = 2$, we find that the rescaled square has side length equal to the **square root** of

$$(17)/(5 - 2\sqrt{2}) = 5 + 2\sqrt{2} \; .$$

In the second situation,

$$|PB|^2 = \frac{20 + 8\sqrt{2}}{17} \; .$$

Rescaling the figure so that $|PB| = 2$, we find that the rescaled square has side length equal to the **square root** of

$$(17)/(5 + 2\sqrt{2}) = 5 - 2\sqrt{2} \; .$$

(c) Since triangle BPQ is right isosceles, $|PQ| = 2\sqrt{2}$. Since also $|AQ| = |CP| = 3$ and $|AP| = 1$, $\angle APQ = 90°$, by the converse of Pythagoras' theorem.

Comment. *Ad* (a), P is on the intersection of two Apollonius circles with diameter joining $(-2, 0)$ and $(2/5, 0)$ passing through the points $(0, 2/\sqrt{5})$ on the $y-$axis and with diameter joining $(0, 2)$ and $(0, 2/3)$. These intersect within the triangle and outside of the triangle.

Solution 2. Suppose that the square has side length x. Let the perpendiculat distance from P to AB be a and from P to BC be b, both distances measured within the right angle. Then we have the three equations: (1) $a^2 + b^2 = 4$; (2) $a^2 + (x - b)^2 = 1$ or $x^2 = 2bx - 3$; (3) $b^2 + (x - a)^2 = 9$ or $x^2 = 2ax + 5$. Hence $2x(b - a) = 8$, so that $x = 4(b - a)^{-1}$. Also $4x = x^2(b - a) = 5b + 3a$, which along with $b - a = 4/x$ yields

$$2a = x - \frac{5}{x} \qquad \text{and} \qquad 2b = \frac{3}{x} + x \; .$$

Thus

$$16 = \left(x - \frac{5}{x}\right)^2 + \left(\frac{3}{x} + x\right)^2 = 2x^2 + \frac{34}{x^2} - 4$$
$$\Longrightarrow x^4 - 10x^2 + 17 = 0$$
$$\Longrightarrow x^2 = 5 \pm 2\sqrt{2} \; .$$

For $2a$ to be positive, we require that $x^2 > 5$ and so $x = \sqrt{5 + 2\sqrt{2}}$ and P is inside triangle ABC. Since

$$(5 - 2\sqrt{2})^2 < \left(5 - \frac{2 \times 7}{5}\right)^2 = \left(\frac{11}{5}\right)^2 < 5 \; ,$$

the second value of x yields negative a and the point lies on the opposite side of AB to C.

For (c), we consider two cases:

(1) $|AB| = \sqrt{5 + 2\sqrt{2}}$ and P lies inside the triangle ABC. Applying the law of cosines to triangle APB yields $\cos \angle APB = -1/\sqrt{2}$ and $\angle APB = 135°$. Hence $\angle APQ = \angle APB - \angle QPB = 135° - 45° = 90°$.

(2) $|AB| = \sqrt{5 - 2\sqrt{2}}$ and P lies outside the triangle ABC. Then the law of cosines applied to triangle APB yields $\cos \angle APB = 1/\sqrt{2}$ and $\angle APB = 45°$. Hence $\angle APQ = \angle APB + \angle BPQ = 45° + 45° = 90°$.

Solution 3. [D. Dziabenko] We can juxtapose two right triangles of sides $(2, 2, 2\sqrt{2})$ and $(1, 2\sqrt{2}, 3)$ to obtain a quadrilateral with $|XY| = |XW| = 2$, $|YZ| = 1$, $|ZW| = 3$ and $|YW| = 2\sqrt{2}$. Since $\angle XYZ = 135°$, we can use the law of cosines to find that $|XZ| = \sqrt{5 + 2\sqrt{2}}$.

A rotation of $90°$ about X takes W to Y and Z to T, so that $|YZ| = 1$, $|XY| = 2$, $|YT| = |WZ| = 3$ and $|XZ| = |XT| = \sqrt{5+2\sqrt{2}}$. Relabel Y as P, X as B, Z as A and T as C to get the desired configuration. For (b), we have that $|AB| = |XZ| = \sqrt{5+2\sqrt{2}}$, and, for (c), that $Q = W$ and $\angle APQ = \angle XYW = 90°$.

Solution 4. [J. Kileel] Let $P \sim (0,0)$, $B \sim (0,2)$, $A \sim (a,b)$, $C \sim (c,d)$. The conditions to be satisfied are: (1) $a^2 + b^2 = 1$; (2) $c^2 + d^2 = 9$; (3) $a^2 + (b-2)^2 = c^2 + (d-2)^2 \Longrightarrow d = b+2$;

$$(4) \qquad \frac{b-2}{a} = \frac{c}{2-d} = \frac{c}{-b} \Longrightarrow -b^2 + 2b = ac \Longrightarrow b^4 - 4b^3 + 4b^2 = (1-b^2)(5-b^2-4b) \ .$$

Hence

$$0 = 8b^3 - 10b^2 - 4b + 5 = (4b-5)(2b^2-1) \ .$$

Since $b = -5/4$ is extraneous (why?), either $b = 1/\sqrt{2}$ or $b = -1\sqrt{2}$.

Lat $b = 1/\sqrt{2}$. From symmetry, it suffices to take $a = 1/\sqrt{2}$ and we get

$$(a,b,c,d) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{2\sqrt{2}-1}{\sqrt{2}}, \frac{2\sqrt{2}+1}{\sqrt{2}}\right) \ ,$$

and

$$|AB|^2 = |BC|^2 = 5 - 2\sqrt{2} \ .$$

Lat $b = -1/\sqrt{2}$. Again we take $a = 1/\sqrt{2}$ and we get

$$(a,b,c,d) = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-2\sqrt{2}-1}{\sqrt{2}}, \frac{2\sqrt{2}-1}{\sqrt{2}}\right) \ ,$$

and

$$|AB|^2 = |BC|^2 = 5 + 2\sqrt{2} \ .$$

Thus the configuration is possible and we have the length of $|AB|$. In the first case, the rotation about B that takes C to A is clockwise and carries P to $Q \sim (-2,2)$. It is straightforward to check that $\angle APQ = 90°$. In the second case, the rotation about B that takes C to A is counterclockwise and carries P to $Q \sim (2,2)$. Again, $\angle APQ = 90°$.

Solution 5. Place B at $(0,0)$, A at $(0,a)$, C at $(a,0)$ and P at (b,c). Then we have to satisfy the three equations: (1) $(a-c)^2 + b^2 = 1$; (2)$b^2 + c^2 = 4$; (3) $(b-a)^2 + c^2 = 9$. Taking the differences of the first two and of the last two lead to the equations

$$2c = a + \frac{3}{a} \qquad\qquad 2b = a - \frac{5}{a}$$

from which, through substitution in (2), we get that $a^4 - 10a^2 + 17 = 0$. This leads to the possibilities that $a^2 = 5 \pm 2\sqrt{2}$, and we can complete the argument as in the foregoing solutions.

417. Show that for each positive integer n, at least one of the five numbers 17^n, 17^{n+1}, 17^{n+2}, 17^{n+3}, 17^{n+4} begins with 1 (at the left) when written to base 10.

Solution 1. It is equivalent to show that, for each natural number n, one of 1.7^{n+k} $(0 \leq k \leq 4)$ begins with the digit 1. We begin with this observation: if for some positive integers u and r, $1.7^u < 10^r \leq 1.7^{u+1}$, then

$$1.7^{u+1} = (1.7)(1.7)^u < (1.7)10^r < 2 \cdot 10^r$$

and the first digit of 1.7^{u+1} is 1.

We obtain the desired result by induction. $1.7^1 = 1.7$ begins with 1, so one of the first five powers of 1.7 begins with 1. Suppose that for some positive integer n exceeding 4, one, at least, of every five consecutive powers of 1.7 up to 1.7^n begins with 1. Let $m \leq n$ be the largest positive integer for which $10^v < 1.7^m < 2 \cdot 10^v$ for some integer v. Then $1.7^m < 10^{v+1}$ and

$$1.7^{m+5} = (1.7)^m (1.7)^5 = (1.7)^m (14.19857) > 10^{v+1}$$

with the result that, for u equal to one of the numbers m, $m+1$, $m+2$, $m+3$, $m+4$, $1.7^u < 10^{v+1} \leq 1.7^{u+1}$. Hence, one of the numbers 1.7^{m+k} $(1 \leq k \leq 5)$ begins with the digit 1. If it is 1.7^{m+k}, then $m+k > n$ and we have established the result up to $m+k$.

Solution 2. For $n = 1$, 17^n begins with 1. Suppose that, for some positive integer k, 17^k begins with 1. Then, either

$$10^a < 17^k < \frac{10^5}{17^4} 10^a$$

or

$$\frac{10^5}{17^4} 10^a < 17^k < 2 \times 10^a$$

for some positive integer a. In the former case,

$$10^{a+5} < 17^5 \times 10^a < 17^{k+5} < 17 \times 10^{a+5}$$

so that 17^{k+5} begins with 1. In the latter case,

$$10^{a+5} < 17^{k+4} < 2 \times 10^a \times 17^4 < 2 \times 10^a \times 300^2 = 1.8 \times 10^{a+5}$$

so that 17^{k+4} begins with 1. The result follows.

Solution 3. Let $17^n = a \cdot 10^m + b$ where $0 \leq b < 10^m$. Then

$$a \times 10^m < 17^n < (a+1)10^m$$

so that

$$(1.7a)10^{m+1} < 17^{n+1} < (1.7)(a+1)10^{m+1} \ .$$

Let $6 \leq a \leq 9$. Then

$$10^{m+2} < (1.7)6 \times 10^{m+1} < 17^{n+1} < 1.7 \times 10^{m+2}$$

and 17^{n+1} begins with 1. Let $4 \leq a \leq 5$. Then

$$6 \times 10^{m+1} < 4(17)10^m \leq (17a)10^m < 17^{n+1} < (1.7)6 \times 10^{m+1} < (1.02)10^{m+1}$$

so that, either 17^{n+1} begins with 1, or 17^{n+1} begins with 6, 7, 8 or 9 and 17^{n+2} begins with 1. When $a = 3$, $5 \times 10^{m+1} < 17^{n+1} < 7 \times 10^{m+1}$ and either 17^{n+2} or 17^{n+3} begins with 1. When $a = 2$, then $3 \times 10^{m+1} < 17^{n+1} < 6 \times 10^{m+1}$ and one of 17^{n+2}, 17^{n+3}, 17^{n+4} begins with 1. Finally, if $a = 1$, one can similarly show that one of 17^{n+k} $(1 \leq k \leq 5)$ begins with 1. The argument now can be completed by induction.

Solution 4. [D. Dziabenko] 17^n beginning with 1 is equivalent to $10^m < 17^n < 2 \times 10^m$ for some positive integer m, which in turn is equivalent to

$$m < n \log 17 < m + \log 2$$

or

$$p < n \log 1.7 < p + \log 2$$

for some positive integer $p(= m - n)$.

Suppose that 17^n begins with 1. We observe that $\log 1.7 < \log 2 = (1/3)\log 8 < 1/3$ and that $2^{10} > 10^3$, whereupon $\log 2 > 3/10$ and

$$\log 1.7 = (\log 17) - 1 > (\log 16) - 1 = (4\log 2) - 1 > \frac{6}{5} - 1 = \frac{1}{5}$$

$$\Longrightarrow 1 < 5\log 1.7 < 5\log 2 < 5/3$$

and so the integer part of $(n+5)\log 17$ is exactly one more than the integer part of $n\log 17$.

From the foregoing, each interval of length $\log 2$ must contain a multiple of $\log 1.7$ and in particular the interval

$$\{x : p+1 < x < p+1+\log 2\}$$

must contain at least one of $(n+k) \leq 1.7$ $(1 \leq k \leq 5)$. We can now complete the argument for the result by induction.

418. (a) Show that, for each pair m,n of positive integers, the minimum of $m^{1/n}$ and $n^{1/m}$ does not exceed $3^{1/2}$.

(b) Show that, for each positive integer n,

$$\left(1 + \frac{1}{\sqrt{n}}\right)^2 \geq n^{1/n} \geq 1 \ .$$

(c) Determine an integer N for which

$$n^{1/n} \leq 1.00002005$$

whenever $n \geq N$. Justify your answer.

Solution. (a) Wolog, we may assume that $m \leq n$, so that $m^{1/n} \leq n^{1/n}$. It suffices to show that, for each positive integer n, $n^{1/n} \leq 3^{1/3} (< 3^{1/2})$ or that $n \leq 3^{n/3}$. Since $3 > 64/27$, it follows that $3^{1/3} - 1 > (4/3) - 1 = 1/3 > 0$ and the result holds for $n = 1$. Suppose as an induction hypothesis, that it holds for n. Then, since $3^{n/3} \geq n$,

$$\begin{aligned} 3^{(n+1)/3} &\geq (3 + \overline{n-3})3^{1/3} > 3^{4/3} + n - 3 \\ &= n + 3(3^{1/3} - 1) > n + 1 \ . \end{aligned}$$

(b) Note that

$$\left(1 + \frac{1}{\sqrt{n}}\right)^n \geq 1 + n\left(\frac{1}{\sqrt{n}}\right) = 1 + \sqrt{n} > \sqrt{n} \ .$$

Alternatively, we can note that, by taking a term out of the binomial expansion,

$$\begin{aligned} (\sqrt{n} + 1)^{2n} &> \binom{2n}{2}(\sqrt{n})^{2n-2} = \frac{2n(2n-1)}{2} n^{n-1} \\ &= (2n-1)n^n \geq n^{n+1} \ , \end{aligned}$$

from which

$$\left(1 + \frac{1}{\sqrt{n}}\right)^{2n} = \frac{(\sqrt{n}+1)^{2n}}{n^n} > n \ .$$

(c) By (b), it suffices to make sure that $(1 + n^{-1/2})^2 \leq 1.00002005$. Let $N = 10^{10}$. Then, for $n \geq N$, we have that $\sqrt{n} \geq 10^5$, so that

$$(1 + n^{-1/2})^2 \leq (1.00001)^2 = 1.0000200001 < 1.00002005 \ .$$

419. Solve the system of equations

$$x+\frac{1}{y}=y+\frac{1}{z}=z+\frac{1}{x}=t$$

for x,y,z not all equal. Determine xyz.

Solution 1. Taking pairs of the three equations, we obtain that

$$x-y=\frac{y-z}{yz}, \qquad y-z=\frac{x-z}{xz}, \qquad z-x=\frac{x-y}{xy} .$$

Since equality of any two of x,y,z implies equality of all three, x,y,z must be distinct. Multiplying these three equations together we find that $(xyz)^2=1$.

When $xyz=1$, then $z=1/xy$ and we find that solutions are given by

$$(x,y,z)=\left(x,-\frac{1}{x+1},-\frac{x+1}{x}\right)$$

as long as $x\neq 0,-1$. When $xyz=-1$, then we obtain the solutions

$$(x,y,z)=\left(x,\frac{1}{1-x},\frac{x-1}{x}\right) .$$

Thus, $xyz=1$ or $xyz=-1$.

Solution 2. We have that $xy+1=yt$ and $yz+1=zt$, so that $xyz+z=yzt=zt^2-t$, whence $z(t^2-1)=xyz+t$. Similarly, $y(t^2-1)=x(t^2-1)=xyz+t$. If $x\neq y$, since $(x-y)(t^2-1)=0$, we must have that $t=\pm 1$. We find that $(x,y,z,t)=((1-z)^{-1},z^{-1}(z-1),z,1)$ and $xyz=-1$ or $(x,y,z,t)=(-(z+1)^{-1},-z^{-1}(z+1),z,-1)$ and $xyz=1$. Thus xyz is equal to 1 or -1.

Solution 3. We have that $y=1/(t-x)$ and $z=t-(1/x)=(xt-1)/x$. This leads to

$$\frac{1}{t-x}+\frac{x}{xt-1}=t\Longrightarrow 0=xt^3-(1+x^2)t^2-xt+(1+x^2)=(t^2-1)[xt-(1+x^2)]=0 .$$

Similarly,

$$0=(t^2-1)[yt-(1+y^2)]=(t^2-1)[zt-(1+z^2)] .$$

Either $t^2=1$ or x,y,z are the roots of the quadratic equation $\lambda^2-t\lambda+1=0$. Since a quadratic has at most two roots, two of x and y must be equal, say $x=y$. But then $y=z$ contrary to hypothesis. Hence $t^2=1$.

Multiplying the three equations together yields that

$$t^3=xyz+3t+\frac{1}{xyz}$$

from which

$$0=(xyz)^2+(3t-t^3)(xyz)+1=(xyz)^2+(3t-t)(xyz)+t^2=(xyz+t)^2 .$$

Hence $xyz=t$. As in the previous solutions, we check that $t=1$ and $t=-1$ are both possible.

420. Two circle intersect at A and B. Let P be a point on one of the circles. Suppose that PA meets the second circle again at C and PB meets the second circle again at D. For what position of P is the length of the segment CD maximum?

Solution 1. The segment CD always has the same length. The strategy is to show that the angle subtended by CD on its circle is equal to the sum of the angles subtended by AB on the two circles, and so is constant. There re a number of configurations possible. Note that (i) and (ii) do not occur with the same pair of circle. The strategy is to show that the angle subtended by CD on its circle is equal to the sum or difference of the angles subtended by AB on its two circles, and so CD is constant.

(i) A is between P and C; B is between P and D;

(ii) C is between P and A; D is between P and B;

(iii) A is between P and C; D is between P and B;

(iv) C is between P and A; B is between P and D;

(v) P is between A and C and also between B and D.

Ad (i), $\angle CBD = \angle PCB + \angle BPC = \angle ACB + \angle APB$. Ad (ii), $\angle CBD = \angle ACB - \angle APB$. Ad (iii), by the angle sum of a triangle, $\angle CAD = 180° - \angle CBD = \angle BCA + \angle BPA$. Since $ADBC$ is concyclic, $\angle CBD = \angle PAD = 180° - \angle APD - \angle ADP = \angle ADB - \angle APB$. Case (iv) is similar to (iii). Ad (v), $\angle DBC = \angle DPC - \angle PCB = \angle APB - \angle ACB$. The angle subtended by CD on the arc opposite P is $180° - \angle DBC = \angle ACB + (180° - \angle APB)$. Also, $\angle DBC = \angle APB - \angle ADB = (180° - \angle ADB) - (180° - \angle APB)$.

Solution 2. We have the same set of cases as in the first solution. Let U be the centre of the circle PAB and V the centre of the circle $ABDC$. Let UV and AB intersect in O; note that $UV \perp AB$. It is straightforward to show that triangles PAB and PDC are similar, whence $CD : AB = PC : PB$ and that triangles PBC and UBV are similar, whence $PC : PB = UV : UB$. Therefore, $CD : AB = UV : UB$ and the results follows.

421. Let $ABCD$ be a tetrahedron. Prove that

$$|AB| \cdot |CD| + |AC| \cdot |BD| \geq |AD| \cdot |BC| \ .$$

Solution 1. First, we establish a small proposition. Let $\mathbf{u}$ and $\mathbf{v}$ be any unit vectors in space and p and q any scalars. Then

$$|p\mathbf{u} + q\mathbf{v}| = |p\mathbf{v} + q\mathbf{u}| \ .$$

This is intuitively obvious, but can be formally established as follows:

$$\begin{aligned} |p\mathbf{u} + q\mathbf{v}|^2 &= (p\mathbf{u} + q\mathbf{v}) \cdot (p\mathbf{u} + q\mathbf{v}) = p^2 + q^2 + 2pq\mathbf{u} \cdot \mathbf{v} \\ &= (p\mathbf{v} + q\mathbf{u}) \cdot (p\mathbf{v} + q\mathbf{u}) = |p\mathbf{v} + q\mathbf{u}|^2 \ . \end{aligned}$$

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be unit vectors and b, c, d be positive scalars for which $\overrightarrow{AB} = b\mathbf{u}$, $\overrightarrow{AC} = c\mathbf{v}$ and $\overrightarrow{AD} = d\mathbf{w}$. Thus $\overrightarrow{BC} = c\mathbf{v} - b\mathbf{u}$, $\overrightarrow{CD} = d\mathbf{w} - c\mathbf{v}$ and $\overrightarrow{BD} = d\mathbf{w} - b\mathbf{u}$.

Then

$$\begin{aligned} |AB||CD| + |AC||BD| &= b|d\mathbf{w} - c\mathbf{v}| + c|d\mathbf{w} - b\mathbf{u}| \\ &= b|d\mathbf{v} - c\mathbf{w}| + c|b\mathbf{w} - d\mathbf{u}| = |bd\mathbf{v} - bc\mathbf{w}| + |cb\mathbf{w} - cd\mathbf{u}| \\ &\geq |bd\mathbf{v} - cd\mathbf{u}| = d|b\mathbf{v} - c\mathbf{u}| = d|c\mathbf{v} - b\mathbf{u}| = |AD||BC| \ , \end{aligned}$$

as required.

Solution 2. Consider the planes of ABC and DBC as being hinged along BC. If we flatten the tetrahedron by spreading the planes apart to a dihedral angle of $180°$, then D moves to a position D' relative to A and $|AD'| \geq |AD|$. The other distances between pairs of points remain the same. It is, thus, enough to establish the result when A, B, C, D are coplanar. Suppose this to be the case.

Let a, b, c, d be complex numbers representing respectively the four points A, B, C, D. Then

$$\begin{aligned} |AB||CD| + |AC||BD| &= |(a-b)(c-d)| + |(c-a)(b-d)| \\ &\geq |(a-b)(c-d) + (c-a)(b-d)| = |(a-d)(c-b)| = |AD||BC| \ . \end{aligned}$$

(The result in the plane is known as Ptolemy's Inequality.)

Solution 3. [Q. Ho Phu] On the ray AC determine C' so that $|AC||AC'| = |AB|^2$; on the ray AD determine D' so that $|AD||AD'| = |AB|^2$. Since $AB : AC = AC' : AB$ and angle A is common, triangles ABC and $AC'B$ are similar, whence $BC' : BC = AB : AC$ and

$$|BC'| = \frac{|BC||AB|}{|AC|} = \frac{|BC||AD||AB|}{|AC||AD|} \ .$$

Similarly,

$$|BD'| = \frac{|BD||AB|}{|AD|} = \frac{|BD||AC||AB|}{|AD||AC|} \ ,$$

and

$$|C'D'| = \frac{|CD||AD'|}{|AC|} = \frac{|CD||AB|^2}{|AD||AC|} \ .$$

In the triangle $BC'D'$, we have that $|BD'| + |C'D'| > |BC'|$, whence

$$|BD||AC| + |CD||AB| > |AD||BC|$$

as desired.

422. Determine the smallest two positive integers n for which the numbers in the set $\{1, 2, \cdots, 3n-1, 3n\}$ can be partitioned into n disjoint triples $\{x, y, z\}$ for which $x + y = 3z$.

Solution. Suppose that the partition consists of the triples $[x_k, y_k, z_k]$ $(1 \leq k \leq n)$. Then

$$\sum_{i=1}^{3n} i = \sum_{k=1}^{n} (x_k + y_k + z_k) = 4 \sum_{k=1}^{n} z_k$$

so that 4 must divide $\frac{1}{2}3n(3n+1)$, or that $3n(3n+1)$ is a multiple of 8. Thus, either $n \equiv 0$ or $n \equiv 5$ (mod 8).

$n = 5$ is possible. Here are some examples:

$$[1, 11, 4], [2, 13, 5], [3, 15, 6], [9, 12, 7], [10, 14, 8]$$
$$[1, 14, 5], [2, 10, 4], [3, 15, 6], [9, 12, 7], [11, 13, 8]$$
$$[1, 8, 3], [2, 13, 5], [12, 15, 9], [4, 14, 6], [10, 11, 7]$$
$$[1, 11, 4], [2, 7, 3], [5, 13, 6], [10, 14, 8], [12, 15, 9]$$
$$[1, 8, 3], [2, 13, 5], [4, 14, 6], [10, 11, 7], [12, 15, 9]$$

Adjoining to any of these solutions the eight triples

$$[19, 29, 16], [21, 30, 17], [26, 28, 18], [27, 33, 20], [31, 35, 22], [32, 37, 23], [34, 38, 24], [36, 39, 25]$$

yields a possibility for $n = 13$.

For $n = 8$, we have

$$[1, 5, 2], [3, 9, 4], [6, 18, 8], [7, 23, 10], [14, 19, 11], [16, 20, 12], [17, 22, 13], [21, 24, 15]$$

There are many other possibilities.