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Notes. A real-valued function on the reals is *increasing* if and only if $f(u) \leq f(v)$ whenever $u < v$. It is *strictly increasing* if and only if $f(u) < f(v)$ whenever $u < v$.

The inverse tangent function is denoted by $\tan^{-1} x$ or $\arctan x$. It is defined by the relation $y = \tan^{-1} x$ if and only if $\pi/2 < y < \pi/2$ and $x = \tan y/$

199. Let A and B be two points on a parabola with vertex V such that VA is perpendicular to VB and θ is the angle between the chord VA and the axis of the parabola. Prove that

$$\frac{|VA|}{|VB|} = \cot^3 \theta \ .$$

200. Let n be a positive integer exceeding 1. Determine the number of permutations $(a_1, a_2, \cdots, a_n)$ of $(1, 2, \cdots, n)$ for which there exists exactly one index i with $1 \leq i \leq n$ and $a_i > a_{i+1}$.

201. Let $(a_1, a_2, \cdots, a_n)$ be an arithmetic progression and $(b_1, b_2, \cdots, b_n)$ be a geometric progression, each of n positive real numbers, for which $a_1 = b_1$ and $a_n = b_n$. Prove that

$$a_1 + a_2 + \cdots + a_n \geq b_1 + b_2 + \cdots + b_n \ .$$

202. For each positive integer k, let $a_k = 1 + (1/2) + (1/3) + \cdots + (1/k)$. Prove that, for each positive integer n,

$$3a_1 + 5a_2 + 7a_3 + \cdots + (2n+1)a_n = (n+1)^2 a_n - \frac{1}{2}n(n+1) \ .$$

203. Every midpoint of an edge of a tetrahedron is contained in a plane that is perpendicular to the opposite edge. Prove that these six planes intersect in a point that is symmetric to the centre of the circumsphere of the tetrahedron with respect to its centroid.

204. Each of $n \geq 2$ people in a certain village has at least one of eight different names. No two people have exactly the same set of names. For an arbitrary set of k names (with $1 \leq k \leq 7$), the number of people containing at least one of the $k (\geq 1)$ names among his/her set of names is even. Determine the value of n.

205. Let $f(x)$ be a convex realvalued function defined on the reals, $n \geq 2$ and $x_1 < x_2 < \cdots < x_n$. Prove that

$$x_1 f(x_2) + x_2 f(x_3) + \cdots + x_n f(x_1) \geq x_2 f(x_1) + x_3 f(x_2) + \cdots + x_1 f(x_n) \ .$$

206. In a group consisting of five people, among any three people, there are two who know each other and two neither of whom knows the other. Prove that it is possible to seat the group around a circular table so that each adjacent pair knows each other.

207. Let n be a positive integer exceeding 1. Suppose that $A = (a_1, a_2, \cdots, a_m)$ is an ordered set of $m = 2^n$ numbers, each of which is equal to either 1 or -1. Let

$$S(A) = (a_1a_2, a_2a_3, \cdots, a_{m-1}a_m, a_ma_1) .$$

Define, $S^0(A) = A$, $S^1(A) = S(A)$, and for $k \geq 1$, $S^{k+1} = S(S^k(A))$. Is it always possible to find a positive integer r for which $S^r(A)$ consists entirely of 1s?

208. Determine all positive integers n for which $n = a^2 + b^2 + c^2 + d^2$, where $a < b < c < d$ and a, b, c, d are the four smallest positive divisors of n.

209. Determine all positive integers n for which $2^n - 1$ is a multiple of 3 and $(2^n - 1)/3$ has a multiple of the form $4m^2 + 1$ for some integer m.

210. ABC and DAC are two isosceles triangles for which B and D are on opposite sides of AC, $AB = AC$, $DA = DC$, $\angle BAC = 20°$ and $\angle ADC = 100°$. Prove that $AB = BC + CD$.

211. Let ABC be a triangle and let M be an interior point. Prove that

$$\min\{MA, MB, MC\} + MA + MB + MC < AB + BC + CA .$$

212. A set S of points in space has at least three elements and satisfies the condition that, for any two distinct points A and B in S, the right bisecting plane of the segment AB is a plane of symmetry for S. Determine all possible finite sets S that satisfy the condition.

213. Suppose that each side and each diagonal of a regular hexagon $A_1A_2A_3A_4A_5A_6$ is coloured either red or blue, and that no triangle $A_iA_jA_k$ has all of its sides coloured blue. For each $k = 1, 2, \cdots, 6$, let r_k be the number of segments A_kA_j $(j \neq k)$ coloured red. Prove that

$$\sum_{k=1}^{6}(2r_k - 7)^2 \leq 54 .$$

214. Let S be a circle with centre O and radius 1, and let P_i $(1 \leq i \leq n)$ be points chosen on the (circumference of the) circle for which $\sum_{i=1}^{n} \overrightarrow{OP_i} = \mathbf{0}$. Prove that, for each point X in the plane, $\sum |XP_i| \geq n$.

215. Find all values of the parameter a for which the equation $16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$ has exactly four real solutions which are in geometric progression.

216. Let x be positive and let $0 < a \leq 1$. Prove that

$$(1 - x^a)(1 - x)^{-1} \leq (1 + x)^{a-1} .$$

217. Let the three side lengths of a scalene triangle be given. There are two possible ways of orienting the triangle with these side lengths, one obtainable from the other by turning the triangle over, or by reflecting in a mirror. Prove that it is possible to slice the triangle in one of its orientations into finitely many pieces that can be rearranged using rotations and translations in the plane (but not reflections and rotations out of the plane) to form the other.

218. Let ABC be a triangle. Suppose that D is a point on BA produced and E a point on the side BC, and that DE intersects the side AC at F. Let $BE + EF = BA + AF$. Prove that $BC + CF = BD + DF$.

219. There are two definitions of an ellipse.

(1) An ellipse is the locus of points P such that the sum of its distances from two fixed points F_1 and F_2 (called *foci*) is constant.

(2) An ellipse is the locus of points P such that, for some real number e (called the *eccentricity*) with $0 < e < 1$, the distance from P to a fixed point F (called a *focus*) is equal to e times its perpendicular distance to a fixed straight line (called the *directrix*).

Prove that the two definitions are compatible.

220. Prove or disprove: A quadrilateral with one pair of opposite sides and one pair of opposite angles equal is a parallelogram.

221. A *cycloid* is the locus of a point P fixed on a circle that rolls without slipping upon a line u. It consists of a sequence of arches, each arch extending from that position on the locus at which the point P rests on the line u, through a curve that rises to a position whose distance from u is equal to the diameter of the generating circle and then falls to a subsequent position at which P rests on the line u. Let v be the straight line parallel to u that is tangent to the cycloid at the point furthest from the line u.

(a) Consider a position of the generating circle, and let P be on this circle and on the cycloid. Let PQ be the chord on this circle that is parallel to u (and to v). Show that the locus of Q is a similar cycloid formed by a circle of the same radius rolling (upside down) along the line v.

(b) The region between the two cycloids consists of a number of "beads". Argue that the area of one of these beads is equal to the area of the generating circle.

(c) Use the considerations of (a) and (b) to find the area between u and one arch of the cycloid using a method that does not make use of calculus.

222. Evaluate

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2}{n^2}\right) .$$

223. Let a, b, c be positive real numbers for which $a + b + c = abc$. Prove that

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2} .$$

224. For $x > 0$, $y > 0$, let $g(x, y)$ denote the minimum of the three quantities, x, $y + 1/x$ and $1/y$. Determine the maximum value of $g(x, y)$ and where this maximum is assumed.

225. A set of n lighbulbs, each with an *on-off* switch, numbered $1, 2, \cdots, n$ are arranged in a line. All are initially off. Switch 1 can be operated at any time to turn its bulb on of off. Switch 2 can turn bulb 2 on or off if and only if bulb 1 is off; otherwise, it does not function. For $k \geq 3$, switch k can turn bulb k on or off if and only if bulb $k-1$ is off and bulbs $1, 2, \cdots, k-2$ are all on; otherwise it does not function.

(a) Prove that there is an algorithm that will turn all of the bulbs on.

(b) If x_n is the length of the shortest algorithm that will turn on all n bulbs when they are initially off, determine the largest prime divisor of $3x_n + 1$ when n is odd.

226. Suppose that the polynomial $f(x)$ of degree $n \geq 1$ has all real roots and that $\lambda > 0$. Prove that the set $\{x \in \mathbf{R} : |f(x)| \leq \lambda |f'(x)|\}$ is a finite union of closed intervals whose total length is equal to $2n\lambda$.

227. Let n be an integer exceeding 2 and let $a_0, a_1, a_2, \cdots, a_n, a_{n+1}$ be positive real numbers for which $a_0 = a_n$, $a_1 = a_{n+1}$ and

$$a_{i-1} + a_{i+1} = k_i a_i$$

for some positive integers k_i, where $1 \leq i \leq n$.

Prove that

$$2n \leq k_1 + k_2 + \cdots + k_n \leq 3n .$$

228. Prove that, if $1 < a < b < c$, then

$$\log_a(\log_a b) + \log_b(\log_b c) + \log_c(\log_c a) > 0 \ .$$

229. Suppose that n is a positive integer and that $0 < i < j < n$. Prove that the greatest common divisor of $\binom{n}{i}$ and $\binom{n}{j}$ exceeds 1.

230. Let f be a strictly increasing function on the closed interval $[0, 1]$ for which $f(0) = 0$ and $f(1) = 1$. Let g be its inverse. Prove that

$$\sum_{k=1}^{9} \left(f\left(\frac{k}{10}\right) + g\left(\frac{k}{10}\right)\right) \leq 9.9 \ .$$

231. For $n \geq 10$, let $g(n)$ be defined as follows: n is mapped by g to the sum of the number formed by taking all but the last three digits of its square and adding it to the number formed by the last three digits of its square. For example, $g(54) = 918$ since $54^2 = 2916$ and $2 + 916 = 918$. Is it possible to start with 527 and, through repeated applications of g, arrive at 605?

232. (a) Prove that, for positive integers n and positive values of x,

$$(1 + x^{n+1})^n \leq (1 + x^n)^{n+1} \leq 2(1 + x^{n+1})^n \ .$$

(b) Let $h(x)$ be the function defined by

$$h(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ x, & \text{if } x > 1. \end{cases}$$

Determine a value N for which

$$|h(x) - (1 + x^n)^{\frac{1}{n}}| < 10^{-6}$$

whenever $0 \leq x \leq 10$ and $n \geq N$.

233. Let $p(x)$ be a polynomial of degree 4 with rational coefficients for which the equation $p(x) = 0$ has *exactly one* real solution. Prove that this solution is rational.

234. A square of side length 100 is divided into 10000 smaller unit squares. Two squares sharing a common side are called *neighbours*.

(a) Is it possible to colour an even number of squares so that each coloured square has an even number of coloured neighbours?

(b) Is it possible to colour an odd number of squares so that each coloured square has an odd number of coloured neighbours?

235. Find all positive integers, N, for which:

(i) N has exactly sixteen positive divisors: $1 = d_1 < d_2 < \cdots < d_{16} = N$;

(ii) the divisor with the *index* d_5 (namely, d_{d_5}) is equal to $(d_2 + d_4) \times d_6$ (the product of the two).

236. For any positive real numbers a, b, c, prove that

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2} \ .$$

237. The sequence $\{a_n : n = 1, 2, \cdots\}$ is defined by the recursion

$$a_1 = 20 \qquad\qquad a_2 = 30$$

$$a_{n+2} = 3a_{n+1} - a_n \qquad \text{for } n \geq 1 \ .$$

Find all natural numbers n for which $1 + 5a_n a_{n+1}$ is a perfect square.

238. Let ABC be an acute-angled triangle, and let M be a point on the side AC and N a point on the side BC. The circumcircles of triangles CAN and BCM intersect at the two points C and D. Prove that the line CD passes through the circumcentre of triangle ABC if and only if the right bisector of AB passes through the midpoint of MN.

239. Find all natural numbers n for which the diophantine equation

$$(x + y + z)^2 = nxyz$$

has positive integer solutions x, y, z.

240. In a competition, 8 judges rate each contestant "yes" or "no". After the competition, it turned out, that for any two contestants, two judges marked the first one by "yes" and the second one also by "yes"; two judges have marked the first one by "yes" and the second one by "no"; two judges have marked the first one by "no" and the second one by "yes"; and, finally, two judges have marked the first one by "no" and the second one by "no". What is the greatest number of contestants?

241. Determine $\sec 40° + \sec 80° + \sec 160°$.

242. Let ABC be a triangle with sides of length a, b, c oppposite respective angles A, B, C. What is the radius of the circle that passes through the points A, B and the incentre of triangle ABC when angle C is equal to (a) 90°; (b) 120°; (c) 60°. (With thanks to Jean Turgeon, Université de Montréal.)

243. The inscribed circle, with centre I, of the triangle ABC touches the sides BC, CA and AB at the respective points D, E and F. The line through A parallel to BC meets DE and DF produced at the respective points M and N. The modpoints of DM and DN are P and Q respectively. Prove that A, E, F, I, P, Q lie on a common circle.

244. Let $x_0 = 4$, $x_1 = x_2 = 0$, $x_3 = 3$, and, for $n \geq 4$, $x_{n+4} = x_{n+1} + x_n$. Prove that, for each prime p, x_p is a multiple of p.

245. Determine all pairs (m, n) of positive integers with $m \leq n$ for which an $m \times n$ rectangle can be tiled with congruent pieces formed by removing a 1×1 square from a 2×2 square.

246. Let $p(n)$ be the number of partitions of the positive integer n, and let $q(n)$ denote the number of finite sets $\{u_1, u_2, u_3, \cdots, u_k\}$ of positive integers that satisfy $u_1 > u_2 > u_3 > \cdots > u_k$ such that $n = u_1 + u_3 + u_5 + \cdots$ (the sum of the ones with odd indices). Prove that $p(n) = q(n)$ for each positive integer n.

For example, $q(6)$ counts the sets $\{6\}$, $\{6, 5\}$, $\{6, 4\}$, $\{6, 3\}$, $\{6, 2\}$, $\{6, 1\}$, $\{5, 4, 1\}$, $\{5, 3, 1\}$, $\{5, 2, 1\}$, $\{4, 3, 2\}$, $\{4, 3, 2, 1\}$.

247. Let $ABCD$ be a convex quadrilateral with no pairs of parallel sides. Associate to side AB a point T as follows. Draw lines through A and B parallel to the opposite side CD. Let these lines meet CB produced at B' and DA produced at A', and let T be the intersection of AB and $B'A'$. Let U, V, W be points similarly constructed with respect to sides BC, CD, DA, respectively. Prove that $TUVW$ is a parallelogram.

248. Find all real solutions to the equation

$$\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1 \ .$$

249. The non-isosceles right triangle ABC has $\angle CAB = 90°$. Its inscribed circle with centre T touches the sides AB and AC at U and V respectively. The tangent through A of the circumscribed circle of triangle ABC meets UV in S. Prove that:

(a) $ST \| BC$;

(b) $|d_1 - d_2| = r$, where r is the radius of the inscribed circle, and d_1 and d_2 are the respective distances from S to AC and AB.

250. In a convex polygon $\mathfrak{P}$, some diagonals have been drawn so that no two have an intersection in the interior of $\mathfrak{P}$. Show that there exists at least two vertices of $\mathfrak{P}$, neither of which is an enpoint of any of these diagonals.

251. Prove that there are infinitely many positive integers n for which the numbers $\{1, 2, 3, \cdots, 3n\}$ can be arranged in a rectangular array with three rows and n columns for which (a) each row has the same sum, a multiple of 6, and (b) each column has the same sum, a multiple of 6.

252. Suppose that a and b are the roots of the quadratic $x^2 + px + 1$ and that c and d are the roots of the quadratic $x^2 + qx + 1$. Determine $(a - c)(b - c)(a + d)(b + d)$ as a function of p and q.

253. Let n be a positive integer and let $\theta = \pi/(2n + 1)$. Prove that $\cot^2 \theta$, $\cot^2 2\theta$, $\cdots$, $\cot^2 n\theta$ are the solutions of the equation

$$\binom{2n+1}{1} x^n - \binom{2n+1}{3} x^{n-1} + \binom{2n+1}{5} x^{n-2} - \cdots = 0 \ .$$

254. Determine the set of all triples (x, y, z) of integers with $1 \le x, y, z \le 1000$ for which $x^2 + y^2 + z^2$ is a multiple of xyz.

255. Prove that there is no positive integer that, when written to base 10, is equal to its kth multiple when its initial digit (on the left) is transferred to the right (units end), where $2 \le k \le 9$ and $k \ne 3$.

256. Find the condition that must be satisfied by y_1, y_2, y_3, y_4 in order that the following set of six simultaneous equations in x_1, x_2, x_3, x_4 is solvable. Where possible, find the solution.

$$x_1 + x_2 = y_1 y_2 \qquad x_1 + x_3 = y_1 y_3 \qquad x_1 + x_4 = y_1 y_4$$

$$x_2 + x_3 = y_2 y_3 \qquad x_2 + x_4 = y_2 y_4 \qquad x_3 + x_4 = y_3 y_4 \ .$$

257. Let n be a positive integer exceeding 1. Discuss the solution of the system of equations:

$$ax_1 + x_2 + \cdots + x_n = 1$$

$$x_1 + ax_2 + \cdots + x_n = a$$

$$\cdots$$

$$x_1 + x_2 + \cdots + ax_i + \cdots + x_n = a^{i-1}$$

$$\cdots$$

$$x_1 + x_2 + \cdots + x_i + \cdots + ax_n = a^{n-1} \ .$$

258. The infinite sequence $\{a_n; n = 0, 1, 2, \cdots\}$ satisfies the recursion

$$a_{n+1} = a_n^2 + (a_n - 1)^2$$

for $n \ge 0$. Find all rational numbers a_0 such that there are four distinct indices p, q, r, s for which $a_p - a_q = a_r - a_s$.

259. Let ABC be a given triangle and let $A'BC$, $AB'C$, ABC' be equilateral triangles erected outwards on the sides of triangle ABC. Let Ω be the circumcircle of $A'B'C'$ and let A'', B'', C'' be the respective intersections of Ω with the lines AA', BB', CC'.

Prove that AA', BB', CC' are concurrent and that

$$AA'' + BB'' + CC'' = AA' = BB' = CC' \ .$$

260. $TABC$ is a tetrahedron with volume 1, G is the centroid of triangle ABC and O is the midpoint of TG. Reflect $TABC$ in O to get $T'A'B'C'$. Find the volume of the intersection of $TABC$ and $T'A'B'C'$.

261. Let $x, y, z > 0$. Prove that

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(x+y)(y+z)}} + \frac{z}{z + \sqrt{(x+z)(y+z)}} \leq 1 \ .$$

as above to get a linear polynomial with root r.

262. Let ABC be an acute triangle. Suppose that P and U are points on the side BC so that P lies between B and U, that Q and V are points on the side CA so that Q lies between C and V, and that R and W are points on the side AB so that R lies between A and W. Suppose also that

$$\angle APU = \angle AUP = \angle BQV = \angle BVQ = \angle CRW = \angle CWR \ .$$

The lines AP, BQ and CR bound a triangle T_1 and the lines AU, BV and CW bound a triangle T_2. Prove that all six vertices of the triangles T_1 and T_2 lie on a common circle.

263. The ten digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are each used exactly once altogether to form three positive integers for which the largest is the sum of the other two. What are the largest and the smallest possible values of the sum?

264. For the real parameter a, solve for real x the equation

$$x = \sqrt{a + \sqrt{a + x}} \ .$$

A complete answer will discuss the circumstances under which a solution is feasible.

265. Note that $959^2 = 919681$, $919 + 681 = 40^2$; $960^2 = 921600$, $921 + 600 = 39^2$; and $961^2 = 923521$, $923 + 521 = 38^2$. Establish a general result of which these are special instances.

266. Prove that, for any positive integer n, $\binom{2n}{n}$ divides the least common multiple of the numbers $1, 2, 3, \cdots, 2n-1, 2n$.

267. A non-orthogonal reflection in an axis a takes each point on a to itself, and each point P not on a to a point P' on the other side of a in such a way that a intersects PP' at its midpoint and PP' always makes a fixed angle θ with a. Does this transformation preserves lines? preserve angles? Discuss the image of a circle under such a transformation.

268. Determine all continuous real functions f of a real variable for which

$$f(x + 2f(y)) = f(x) + y + f(y)$$

for all real x and y.

269. Prove that the number

$$N = 2 \times 4 \times 6 \times \cdots \times 2000 \times 2002 + 1 \times 3 \times 5 \times \cdots \times 1999 \times 2001$$

is divisible by 2003.

270. A straight line cuts an acute triangle into two parts (not necessarily triangles). In the same way, two other lines cut each of these two parts into two parts. These steps repeat until all the parts are triangles. Is it possible for all the resulting triangle to be obtuse? (Provide reasoning to support your answer.)

271. Let x, y, z be natural numbers, such that the number

$$\frac{x - y\sqrt{2003}}{y - z\sqrt{2003}}$$

is rational. Prove that

(a) $xz = y^2$;

(b) when $y \neq 1$, the numbers $x^2 + y^2 + z^2$ and $x^2 + 4z^2$ are composite.

272. Let $ABCD$ be a parallelogram whose area is 2003 sq. cm. Several points are chosen on the sides of the parallelogram.

(a) If there are 1000 points in addition to A, B, C, D, prove that there always exist three points among these 1004 points that are vertices of a triangle whose area is less that 2 sq. cm.

(b) If there are 2000 points in addition to A, B, C, D, is it true that there always exist three points among these 2004 points that are vertices of a triangle whose area is less than 1 sq. cm?

273. Solve the logarithmic inequality

$$\log_4(9^x - 3^x - 1) \geq \log_2 \sqrt{5} \ .$$

274. The inscribed circle of an isosceles triangle ABC is tangent to the side AB at the point T and bisects the segment CT. If $CT = 6\sqrt{2}$, find the sides of the triangle.

275. Find all solutions of the trigonometric equation

$$\sin x - \sin 3x + \sin 5x = \cos x - \cos 3x + \cos 5x \ .$$

276. Let a, b, c be the lengths of the sides of a triangle and let $s = \frac{1}{2}(a + b + c)$ be its semi-perimeter and r be the radius of the inscribed circle. Prove that

$$(s - a)^{-2} + (s - b)^{-2} + (s - c)^{-2} \geq r^{-2}$$

and indicate when equality holds.

277. Let m and n be positive integers for which $m < n$. Suppose that an arbitrary set of n integers is given and the following operation is performed: select any m of them and add 1 to each. For which pairs (m, n) is it always possible to modify the given set by performing the operation finitely often to obtain a set for which all the integers are equal?

278. (a) Show that $4mn - m - n$ can be an integer square for infinitely many pairs (m, n) of integers. Is it possible for either m or n to be positive?

(b) Show that there are infinitely many pairs (m, n) of positive integers for which $4mn - m - n$ is one less than a perfect square.

279. (a) For which values of n is it possible to construct a sequence of abutting segments in the plane to form a polygon whose side lengths are $1, 2, \cdots, n$ exactly in this order, where two neighbouring segments are perpendicular?

(b) For which values of n is it possible to construct a sequence of abutting segments in space to form a polygon whose side lengths are $1, 2, \cdots, n$ exactly in this order, where any two of three sucessive segments are perpendicular?

280. Consider all finite sequences of positive integers whose sum is n. Determine $T(n,k)$, the number of times that the positive integer k occurs in all of these sequences taken together.

281. Let a be the result of tossing a black die (a number cube whose sides are numbers from 1 to 6 inclusive), and b the result of tossing a white die. What is the probability that there exist real numbers x, y, z for which $x + y + z = a$ and $xy + yz + zx = b$?

282. Suppose that at the vertices of a pentagon five integers are specified in such a way that the sum of the integers is positive. If not all the integers are non-negative, we can perform the following operation: suppose that x, y, z are three consecutive integers for which $y < 0$; we replace them respectively by the integers $x + y, -y, z + y$. In the event that there is more than one negative integer, there is a choice of how this operation may be performed. Given any choice of integers, and any sequence of operations, must we arrive at a set of nonnegative integers after a finite number of steps?

For example, if we start with the numbers $(2, -3, 3, -6, 7)$ around the pentagon, we can produce $(1, 3, 0, -6, 7)$ or $(2, -3, -3, 6, 1)$.

Solutions

199. Let A and B be two points on a parabola with vertex V such that VA is perpendicular to VB and θ is the angle between the chord VA and the axis of the parabola. Prove that

$$\frac{|VA|}{|VB|} = \cot^3 \theta \ .$$

Comment. A lot of students worked harder on this problem than was necessary. It should be noted that all parabolas are similar (as indeed all circles are similar); this means that you can establish a general result about parabolas by dealing with a convenient one. Let us see why this is so. One definition of a parabola is that it is the locus of points that are equidistant from a given point (called the *focus*) and a given line (called the *directrix*) that does not contain the point. Any point-line pair can be used, and each such pair can be transformed into another by a similarity transformation. (Translate one point on to the other, make a rotation to make the two lines parallel and perform a dilation about the point that makes the two lines coincide.) The same transformation will take the prabola defined by one pair to the parabola defined by the other. You should point out in your solution that there is no loss of generality in taking the particular case of a parabola whose equation in the plane is $y = ax^2$. But you do not have to be even that general; it is enough to assume that the parabola has the equation $y = x^2$ or $x = y^2$. (*Exercise:* Determine the focus and the directrix for these parabolas.) Some of the solvers did not appear to be aware that parabolas need not have vertical or horizontal axes; the axis of a parabola can point in any direction.

Solution. Wolog, suppose that the parabola is given by $y^2 = x$, so that its vertex is the origin and its axis is the $x-$axis. Suppose $A \sim (u, v)$ is a point on the parabola whose radius vector makes an angle θ with the axis; then $v/u = \tan\theta$. Hence $1/u = v^2/u^2 = \tan^2\theta$, so that $A \sim (\cot^2\theta, \cot\theta)$. Similarly, it can be shown that $B \sim (\tan^2\theta, -\tan\theta)$. Hence

$$\frac{|VA|^2}{|VB|^2} = \frac{\cot^2\theta(\cot^2\theta + 1)}{\tan^2\theta(\tan^2\theta + 1)} = \cot^6\theta \ ,$$

and the result follows.

200. Let n be a positive integer exceeding 1. Determine the number of permutations $(a_1, a_2, \cdots, a_n)$ of $(1, 2, \cdots, n)$ for which there exists exactly one index i with $1 \le i \le n-1$ and $a_i > a_{i+1}$.

Comment. Some solvers found it difficult to appreciate what was going on in this problem. It is often a good beginning strategy to actually write out the appropriate permutations for low values of n. This does two things for you. First, it gives you a sense of what goes into constructing the right permutations and so how your argument can be framed. Secondly, it gives you some data against which you can check your final answer.

Solution 1. For $n \geq 1$, let p_n be the number of permutations of the first n natural numbers that satisfy the condition. Suppose that $a_i = n$ for some i with $1 \leq i \leq n-1$. Then $(a_1, a_2, \cdots, a_{i-1})$ and $(a_{i+1}, \cdots, a_n)$ must both be in increasing order, so that the appropriate permutation is determined uniquely once its first $i-1$ entries are found. There are $\binom{n-1}{i-1}$ ways of choosing these entries. If $a_n = n$, then there are p_{n-1} ways of ordering the first $n-1$ numbers to give an appropriate permutation. Hence

$$p_n = \left[\sum_{i=1}^{n-1} \binom{n-1}{i-1}\right] + p_{n-1} = 2^{n-1} - 1 + p_{n-1} \ .$$

Thus, substituting for each p_i in turn, we have that

$$p_n = (2^{n-1} - 1) + (2^{n-2} - 1) + \cdots + (2^2 - 1) + (2 - 1) + (1 - 1) = 2^n - 1 - n = 2^n - (n+1) \ .$$

Solution 2. [H. Li; M. Zaharia] For $n \geq 2$, let p_n be the number of acceptable permutations. We have that $p_2 = 1$. Consider first the placing of the numbers $1, 2, \cdots, n-1$ in some order. If they appear in their natural order, then we can slip in n before any one of them to get an acceptable permutation; there are $n-1$ ways of doing this. If there exists a single consecutive pair (r, s) of numbers for which $r < s$ and r follows s, then we can slip n between s and r or at the end to get an acceptable permutation. There are $2p_{n-1}$ possibilities. If there is more than one pair (r, s) of consecutive pairs with $r < s$ and r following s, then no placement of n will yield an acceptable permutation. Hence

$$p_n = 2p_{n-1} + (n-1)$$

so that

$$\begin{aligned} p_n + n + 1 &= 2(p_{n-1} + n) = 2^2(p_{n-2} + n - 1) \\ &= \cdots = 2^{n-2}(p_2 + 3) = 2^{n-2} \cdot 4 = 2^n \ , \end{aligned}$$

whence $p_n = 2^n - (n+1)$.

Solution 3. [R. Barrington Leigh] Let $1 \leq k \leq n-1$ and let (x, y) be a pair of integers for which $1 \leq y < x \leq n$ and $x - y = k$. There are $n - k$ such pairs, $(1, k+1), (2, k+2), \cdots, (n-k, k)$. For each such pair, we consider suitable permutations for which x and y are adjacent in the order (x, y). Then the numbers $1, 2, \cdots, y-1$ must precede and $x+1, \cdots, n$ must follow the pair. The remaining $k-1$ numers from $x+1$ to $x+k-1 = y-1$ can go either before or after the pair; there are 2^{k-1} possibilities. Once it is decided whether each of these goes before or after the pair, there is only one possible arrangement. Hence the number of permutations of the required type is

$$\begin{aligned} \sum_{k=1}^{n-1} (n-k) 2^{k-1} &= \sum_{k=1}^{n-1} [(n-k+1)2^k - (n-k+2)2^{k-1}] \\ &= \sum_{k=1}^{n-1} [(n - \overline{k-1})2^k - (n - \overline{k-2})2^{k-1} \\ &= 2 \cdot 2^{n-1} - (n+1) = 2^n - (n+1) \ . \end{aligned}$$

Solution 4. Let $1 \leq i \leq n-1$ and consider the number of suitable permutations for which $a_i > a_{i+1}$. There are $\binom{n}{i}$ possible choices of $\{a_1, a_2, \cdots, a_i\}$ with $a_1 < a_2 < \cdots < a_i$, and except for the single case of

$\{1, 2, \cdots, i\}$, the maximum element a_i of each of them exceeds the minimum element a_{i+1} of its complement $\{a_{i+1}, \cdots, a_n\}$. Hence the number of permutations is

$$\sum_{i=1}^{n-1} \left[\binom{n}{i} - 1 \right] = \sum_{i=0}^{n} \left[\binom{n}{i} - 1 \right] = 2^n - (n+1) \ .$$

Solution 5. (Variant of Solution 4.) We can form an acceptable permutation in the following way. Let $1 \leq k \leq n$. Select any subset of k numbers in one of $\binom{n}{k}$ ways and place them in ascending order at the beginning of the arrangement and place the other $n - k$ at the end, again in ascending order. This fails to work only when the set chosen is $\{1, 2, \cdots, k\}$. Hence the total number of ways is

$$\sum_{k=1}^{n} \left[\binom{n}{k} - 1 \right] = \left[\sum_{k=1}^{n} \binom{n}{k} \right] - n = (2^n - 1) - n \ .$$

Solution 6. [D. Yu] Let $1 \leq i \leq n - 1$. There are $\binom{n-1}{i}$ ways of selecting a subset U of $\{1, 2, \cdots, n-1\}$ that has i elements. Let $V = \{1, 2, \cdots, n\} \backslash U$. Then we can obtain a suitable permutation by putting the elements of U in ascending order and the elements of V in ascending order, and putting U so ordered before V so ordered, or vice versa. The only time this does not work is when $U = \{1, 2, 3, \cdots, i\}$, when we must put V first. Hence we get $2\binom{n-1}{i} - 1$ suitable permutations. Since every suitable permutation can be obtained in this way for some i, there are

$$\begin{aligned} \sum_{i=1}^{n-1} \left[2\binom{n-1}{i} - 1 \right] &= 2\left[\sum_{i=1}^{n-1} \binom{n-1}{i} \right] - (n-1) \\ &= 2(2^{n-1} - 1) - (n-1) = 2^n - (n+1) \end{aligned}$$

suitable permutations.

201. Let $(a_1, a_2, \cdots, a_n)$ be an arithmetic progression and $(b_1, b_2, \cdots, b_n)$ be a geometric progression, each of n positive real numbers, for which $a_1 = b_1$ and $a_n = b_n$. Prove that

$$a_1 + a_2 + \cdots + a_n \geq b_1 + b_2 + \cdots + b_n \ .$$

Solution 1. The result is obvious if $a_1 = a_n = b_1 = b_n$, as then all of the a_i and b_j are equal. Suppose that the progressions are nontrivial and that the common ratio of the geometric progression is $r \neq 1$. Observe that

$$(r^{n-1} + 1) - (r^{n-k} + r^{k-1}) = (r^{k-1} - 1)(r^{n-k} - 1) > 0 \ .$$

Then

$$\begin{aligned} b_1 + b_2 + \cdots + b_n &= b_1(1 + r + r^2 + r^3 + \cdots + r^{n-1}) \\ &= \frac{b_1}{2} \sum_{k=1}^{n} (r^{n-k} + r^{k-1}) \\ &< \frac{b_1 n}{2}(r^{n-1} + 1) = \frac{n}{2}[b_1 r^{n-1} + b_1] \\ &= \frac{n}{2}[b_n + b_1] = \frac{n}{2}[a_n + a_1] = a_1 + a_2 + \cdots + a_n \ . \end{aligned}$$

Solution 2. For $1 \leq r \leq n$, we have that

$$\begin{aligned} b_r &= b_1^{(n-r)/(n-1)} b_n^{(r-1)/(n-1)} \\ &= a_1^{(n-r)/(n-1)} a_n^{(r-1)/(n-1)} \\ &\leq \frac{n-r}{n-1} a_1 + \frac{r-1}{n-1} a_n = a_r \ , \end{aligned}$$

by the arithmetic-geometric means inequality.

202. For each positive integer k, let $a_k = 1 + (1/2) + (1/3) + \cdots + (1/k)$. Prove that, for each positive integer n,

$$3a_1 + 5a_2 + 7a_3 + \cdots + (2n+1)a_n = (n+1)^2 a_n - \frac{1}{2}n(n+1) \ .$$

Solution 1. Observe that, for $1 \le k \le n$,

$$(2k+1) + (2k+3) + \cdots + (2n+1) = (1 + 3 + \cdots + \overline{2n+1}) - (1 + 3 + \cdots + \overline{2k-1}) = (n+1)^2 - k^2 \ .$$

Then

$$\begin{aligned} 3a_1 + 5a_2 + 7a_3 + &\cdots + (2n+1)a_n \\ &= (3 + 5 + \cdots + \overline{2n+1}) \cdot 1 + (5 + 7 + \cdots + \overline{2n+1}) \cdot \left(\frac{1}{2}\right) + \cdots + (2n+1)\left(\frac{1}{n}\right) \\ &= \sum_{k=1}^{n} [(n+1)^2 - k^2]\left(\frac{1}{k}\right) = (n+1)\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n} k \\ &= (n+1)^2 a_n - \frac{1}{2}n(n+1) \ . \end{aligned}$$

Solution 2. Observe that for each positive integer $k \ge 2$,

$$\begin{aligned} [(k+1)^2 a_k - \frac{1}{2}k(k+1)] - [k^2 a_{k-1} - \frac{1}{2}(k-1)k] \\ &= k^2(a_k - a_{k-1}) + (2k+1)a_k - \frac{1}{2}k(k+1-\overline{k-1}) \\ &= k^2(1/k) + (2k+1)a_k - k = (2k+1)a_k \ . \end{aligned}$$

Hence

$$\begin{aligned} 3a_1 + 5a_2 + &\cdots + (2n+1)a_n \\ &= 3a_1 + \sum_{k=2}^{n}\{[(k+1)^2 a_k - \frac{1}{2}k(k+1)] - [k^2 a_{k-1} - \frac{1}{2}(k-1)k]\} \\ &= 3a_1 + [(n+1)^2 a_n - \frac{1}{2}n(n+1)] - [4a_1 - 1] \\ &= (n+1)^2 a_n - \frac{1}{2}n(n+1) + 1 - a_1 = (n+1)^2 a_n - \frac{1}{2}n(n+1) \ . \end{aligned}$$

Solution 3. We use an induction argument. The result holds for $k = 1$. Suppose it holds for $n = k-1 \ge 1$. Then

$$\begin{aligned} 3a_1 + 5a_2 + \cdots + (2k-1)a_{k-1} &+ (2k+1)a_k \\ &= k^2 a_{k-1} - \frac{1}{2}k(k-1) + (2k+1)a_k \\ &= k^2\left(a_k - \frac{1}{k}\right) - \frac{1}{2}k(k-1) + (2k+1)a_k \\ &= (k+1)^2 a_k - [k + \frac{1}{2}k(k-1)] \\ &= (k+1)^2 a_k - \frac{1}{2}k(k+1) \ . \end{aligned}$$

Solution 4. [R. Furmaniak] Let $a_0 = 0$, Then $a_i = a_{i-1} + (1/i)$ for $1 \le i \le n$, so that

$$\begin{aligned}\sum_{i=1}^{n}(2i+1)a_i &= \sum_{i=1}^{n}[(i+1)^2 - i^2]a_i \\ &= \sum_{i=1}^{n}[(i+1)^2 a_i - i^2 a_{i-1} - i^2(1/i)] \\ &= (n+1)^2 a_n - a_0 - \sum_{i=1}^{n} i = (n+1)^2 a_n - \frac{1}{2}n(n+1) \ .\end{aligned}$$

Solution 5. [A. Verroken] Let $a_0 = 0$. For $n \ge 1$,

$$\begin{aligned}(n+1)^2 a_n &= \sum_{k=0}^{n}(2k+1)a_n \\ &= \sum_{k=0}^{n}(2k+1)\left[a_k + \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{n}\right)\right] \\ &= \sum_{k=1}^{n}(2k+1)a_k + \sum_{k=0}^{n-1}(2k+1)\left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{n}\right) \\ &= \sum_{k=1}^{n}(2k+1)a_k + \sum_{k=0}^{n-1}\left(\frac{1}{k+1}\right)(1 + 3 + \cdots + (2k+1)) \\ &= \sum_{k=1}^{n}(2k+1)a_k + \sum_{k=0}^{n-1}\left(\frac{1}{k+1}\right)(k+1)^2\end{aligned}$$

from which the result follows. (To see the second last equality, write out the sums and instead of summing along the $2k+1$, sum along the $1/(k+1)$.]

Solution 6. [T. Yin] Recall Abel's Partial Summation Formula:

$$\sum_{k=1}^{n} u_k v_k = (u_1 + u_2 + \cdots + u_n)v_n - \sum_{k=1}^{n-1}(u_1 + u_2 + \cdots + u_k)(v_{k+1} - v_k) \ .$$

(Prove this. Compare with integration by parts in calculus.) Applying this to $u_k = 2k+1$ and $v_k = a_k$, we find that $u_1 + \cdots + u_k = (k+1)^2 - 1$ and $v_{k+1} - v_k = 1/(k+1)$, whereupon

$$\begin{aligned}\sum_{k=1}^{n}(2k+1)a_k &= (n+1)^2 a_n - a_n - \sum_{k=1}^{n-1}(k+1) + \sum_{k=1}^{n-1}\frac{1}{k+1} \\ &= (n+1)^2 a_n - a_n - \left[\frac{n(n+1)}{2} - 1\right] + [a_n - 1] \\ &= (n+1)^2 a_n - \frac{n(n+1)}{2} \ .\end{aligned}$$

203. Every midpoint of an edge of a tetrahedron is contained in a plane that is perpendicular to the opposite edge. Prove that these six planes intersect in a point that is symmetric to the centre of the circumsphere of the tetrahedron with respect to its centroid.

Solution 1. Let O be the centre of the circumsphere of the tetrahedron $ABCD$ and G be its centroid. Then

$$\overrightarrow{OG} = \frac{1}{4}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}) \ .$$

Let N be the point determined by

$$\overrightarrow{ON} = 2\overrightarrow{OG} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}) .$$

Let P be the midpoint of the edge AB. Then

$$\overrightarrow{PN} = \overrightarrow{ON} - \overrightarrow{OP} = \overrightarrow{ON} - \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) = \frac{1}{2}(\overrightarrow{OC} + \overrightarrow{OD})$$

and

$$\overrightarrow{PN} \cdot \overrightarrow{CD} = \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OC}) \cdot (\overrightarrow{OD} + \overrightarrow{OC}) = \frac{1}{2}(|\overrightarrow{OD}|^2 - |\overrightarrow{OC}|^2) = 0 .$$

Hence $\overrightarrow{PN} \perp \overrightarrow{CD}$, so that the segment PN is contained in a plane that is orthogonal to CD. A similar result holds for the other five edges. The result follows.

Solution 2. [O. Bormashenko] Let O be the circumcentre and let G be the centroid of the tetrahedron. Let M be the midpoint of the edge AB and N the midpoint of the edge CD. The centroid of the triangle ABC lies at a point E on MC for which $CE = 2EM$, so that $CM = 3EM$. The centroid of the tetrahedron is the position of the centre of gravity when unit masses are placed at its vertices, and so is the position of the centre of gravity of a unit mass placed at D and a triple mass at E. Thus G is on DE and satisfies $DG = 3GE$.

Consider triangle CDE. We have that

$$\frac{CM}{ME} \cdot \frac{EG}{GD} \cdot \frac{DN}{NC} = (-3) \cdot \left(\frac{1}{3}\right) \cdot 1 = -1 ,$$

so that, by the converse to Menelaus' Theorem, G, M and N are collinear. Consider triangle MCN and transversal DGE. By Menelaus' Theorem,

$$-1 = \frac{ME}{EC} \cdot \frac{CD}{DN} \cdot \frac{NG}{GM} = \left(\frac{1}{2}\right) \cdot (-2) \cdot \frac{NG}{GM} ,$$

whence $NG = GM$ and G is the midpoint of MN.

Suppose that K is the point on OG produced so that $OG = GK$. Since OK and MN intersect in G at their respective midpoints, $OMKN$ is a planar parallelogram and $ON \| KM$. Since $OC = OD$, triangle OCD is isosceles, and so $ON \perp CD$. Hence $KM \perp CD$. Therefore, K lies on the plane through the midpoint M of AB and perpendicular to CD. By symmetry, K lies on the other planes through the midpoints of an edge and perpendicular to the opposite edge.

204. Each of $n \geq 2$ people in a certain village has at least one of eight different names. No two people have exactly the same set of names. For an arbitrary set of k names (with $1 \leq k \leq 7$), the number of people containing at least one of the k names among his/her set of names is even. Determine the value of n.

Solution 1. Let P be a person with the least number of names. The remaining $n - 1$ people have at least one of the names not possessed by P, so by the condition of the problem applied to the set of names not possessed by P, $n - 1$ is even and so n is odd. Let x be one of the eight names, and suppose, if possible, that no person has x as his/her sole name. Then all n people have at least one of the remaining names which yields the contradiction that n must be even. Hence, for each name, there is a person with only that name. Suppose there is no person with only a pair $\{x, y\}$ of names. Then there are $n - 2$ people who have a name other than x and y, which yields again a contradiction, since $n - 2$ is odd. Hence, for each pair of names, there is exactly one person possessing those two names.

We can continue the argument. Suppose, if possible, there is no person possessing exactly the three names x, y and z. Then except for the six people with the name sets $\{x\}$, $\{y\}$, $\{z\}$, $\{x, y\}$, $\{y, z\}$, $\{z, x\}$, everyone possesses at least one of the names other than x, y, z, which leads to a contradiction. Eventually, we can argue that, for each nonvoid set of the eight names, there is exactly one person with that set of names. Since there are $255 = 2^8 - 1$ such subsets, there must be 255 people.

Solution 2. [R. Furmaniak] For $1 \leq i \leq 8$, let S_i be the set of people whose names include the ith name. By the condition of the problem for $k = 1$, the cardinality, $\#S_i$, of S_i must be even. Suppose, for $2 \leq k \leq 7$, it has been shown that any intersection of fewer than k of the S_i has even cardinality.

Consider an intersection of k of the S_i, say $S_1 \cap S_2 \cap \cdots \cap S_k$. By the condition of the problem, $\#(S_1 \cup S_2 \cup \cdots \cup S_k)$, the number of people with at least one of the first k names, is even. But, from the Principle of Inclusion-Exclusion, we have that

$$\#(S_1 \cup S_2 \cup \cdots \cup S_k) = \sum_{i=1}^{k} \#S_i - \sum_{i \neq j} \#(S_i \cap S_j) + \sum_{i,j,k} \#(S_i \cap S_j \cap S_k) - \cdots + (-1)^k \#(S_1 \cap S_2 \cap \cdots \cap S_k) \ .$$

By the induction hypothesis, each term in the series on the right but the last is even, and so the last is even as well.

Consider the largest set of names, say $\{i_1, \cdots, i_r\}$ possessed by any one person. This set can appear only once, so that $\cap_{j=1}^{r} S_{i_j}$ is a singleton. By the above paragraph, the intersection must have eight members (no fewer) and so some person possesses all eight names.

If a set of names does not belong to any person, let T be a maximal such set with $k \leq 7$ names, say the first k names. By maximality, each superset of T be be a set of names for someone. The supersets consist of the k names along with all of the $2^{8-k} - 1$ possible subsets of the remaining names. But the superset of names are possessed by all the people in $S_1 \cap S_2 \cap \cdots \cap S_k$, and this set has even cardinality and so cannot have cardinality $2^{8-k} - 1$. This is a contradiction. Thus every possible nonvoid set of names must occur and $n = 2^8 - 1$.

205. Let $f(x)$ be a convex realvalued function defined on the reals, $n \geq 2$ and $x_1 < x_2 < \cdots < x_n$. Prove that

$$x_1 f(x_2) + x_2 f(x_3) + \cdots + x_n f(x_1) \geq x_2 f(x_1) + x_3 f(x_2) + \cdots + x_1 f(x_n) \ .$$

Solution 1. The case $n = 2$ is obvious. For $n = 3$, we have that

$$\begin{aligned} x_1 f(x_2) &+ x_2 f(x_3) + x_3 f(x_1) - x_2 f(x_1) - x_3 f(x_2) - x_1 f(x_3) \\ &= (x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3) - (x_3 - x_1) f(x_2) \\ &= (x_3 - x_1) \left[\frac{(x_3 - x_2)}{(x_3 - x_1)} f(x_1) + \frac{(x_2 - x_1)}{(x_3 - x_1)} f(x_3) - f(x_2) \right] \geq 0 \ . \end{aligned}$$

Suppose, as an induction hypothesis, that the result holds for all values of n up to $k \geq 3$. Then

$$\begin{aligned} x_1 f(x_2) + x_2 f(x_3) + \cdots + x_k f(x_{k+1}) + x_{k+1} f(x_k) \\ &= [x_1 f(x_2) + \cdots + x_k f(x_1)] + [x_k f(x_{k+1}) + x_{k+1} f(x_1) - x_k f(x_1)] \\ &\geq [x_2 f(x_1) + \cdots + x_1 f(x_k)] + [x_{k+1} f(x_k) + x_1 f(x_{k+1}) - x_1 f(x_k)] \\ &= x_2 f(x_1) + \cdots + x_{k+1} f(x_k) + x_1 f(x_{k+1}) \ , \end{aligned}$$

by the result for $n = k$ and $n = 3$.

Solution 2. [J. Kramar] For $1 \leq i \leq n$, let $\lambda_i = (x_i - x_1)/(x_n - x_1)$, so that $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq$

$\lambda_n = 1$ and $x_i = \lambda_i x_n + (1-\lambda_i)x_1$. Then

$$\begin{aligned}
f(x_n)\lambda_{n-1} + (1-\lambda_2)f(x_1) &= (f(x_n) - f(x_1))\lambda_{n-1} + f(x_1)(\lambda_{n-1} + \lambda_n - \lambda_2) \\
&= (f(x_n) - f(x_1))(\lambda_{n-1}\lambda_n - \lambda_1\lambda_2) + f(x_1)(\lambda_n + \lambda_{n-1} - \lambda_2 - \lambda_1) \\
&= (f(x_n) - f(x_1))\sum_{i=2}^{n-1}(\lambda_i\lambda_{i+1} - \lambda_{i-1}\lambda_i) + f(x_1)\sum_{i=2}^{n-1}(\lambda_{i+1} - \lambda_{i-1}) \\
&= \sum_{i=2}^{n-1}[\lambda_{i+1} - \lambda_{i-1}][\lambda_i f(x_n) + (1-\lambda_i)f(x_1)] \\
&\geq \sum_{i=2}^{n-1}[\lambda_{i+1} - \lambda_{i-1}]f(x_i)\ .
\end{aligned}$$

Multiplying by $x_n - x_1$ and rearranging terms yields that

$$x_{n-1}f(x_n) + x_n f(x_1) \geq \left[\sum_{i=2}^{n-1} f(x_i)(x_{i+1} - x_{i-1})\right] + x_1 f(x_n) + x_2 f(x_1)$$

from which the desired result follows.

Solution 3. [D. Yu] Note that the inequality holds for a function $f(x)$ if and only if it holds for $m + f(x)$ for all real constants m. We begin by establishing that a convex function on a closed interval is bounded below.

Proposition. Let $f(x)$ be a convex function defined on the closed interval $[a,b]$. Then there exists a constant M such that $f(x) \geq M$ for $a \leq x \leq b$.

Proof. Let c be the midpoint of $[a,b]$. Then, when $a < c \leq x \leq b$, we have that

$$f(c) \leq \frac{f(a)(x-c) + f(x)(c-a)}{x-a} \leq \frac{|f(a)|(b-c) + f(x)(c-a)}{c-a}$$

whence

$$f(x) \geq f(c) - |f(a)|(b-c)(c-a)^{-1}\ .$$

Similarly, when $a \leq x \leq c < b$, we have that

$$f(c) \leq \frac{f(x)(b-c) + |f(b)|(c-x)}{b-x} \leq \frac{f(x)(b-c) + |f(b)|(c-a)}{b-c}$$

whence

$$f(x) \geq f(c) - |f(b)|(c-a)(b-c)^{-1}\ .$$

We can take M to be the minimum of $f(c) - |f(a)|(b-c)(c-a)^{-1}$ and $f(c) - |f(b)|(c-a)(b-c)^{-1}$. ♠

Return to the problem. Because of the foregoing, it is enough to prove the result when $f(x) \geq 0$ on $[x_1, x_n]$. From the convexity, the graph of f on $[x_1, x_n]$ lies below the line segment joining $(x_1, f(x_1))$ and $(x_n, f(x_n))$. The nonnegative area between this line and the graph is at least as big as the area between the trapezoid with vertices $(x_1, 0)$, $(x_1, f(x_1))$, $(x_n, f(x_n))$, $(x_n, 0)$ and the union of the trapezoids with vertices $(x_i, 0)$, $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, $(x_{i+1}, 0)$ $(1 \leq i \leq n-1)$, and this latter area is equal to

$$\begin{aligned}
\frac{1}{2}\left[(f(x_1) + f(x_n))(x_n - x_1) - \sum_{i=1}^{n-1}(f(x_i) + f(x_{i+1})(x_{i+1} - x_i)\right] \\
= \frac{1}{2}\Bigg[x_n f(x_n) - x_1 f(x_1) + \sum_{i=1}^{n-1}(x_i f(x_i) - x_{i+1} f(x_{i+1})) \\
+ x_n f(x_1) + \sum_{i=1}^{n-1} x_i f(x_{i+1}) - x_1 f(x_n) - \sum_{i=1}^{n-1} x_{i+1} f(x_i)\Bigg]\ .
\end{aligned}$$

The result follows from this.

206. In a group consisting of five people, among any three people, there are two who know each other and two neither of whom knows the other. Prove that it is possible to seat the group around a circular table so that each adjacent pair knows each other.

Solution. Let the five people be A, B, C, D, E. We first show that each person must know exactly two of the others. Suppose, if possible, that A knows B, C, D. Then, by considering all the triples containing A, we see that each pair of B, C, D do not know each other, contrary to hypothesis. Thus, A knows at most two people. On the other hand, if A knows none of B, C and D, then each pair of B, C, D must know each other again yielding a contradiction. Therefore, A knows exactly two people, say B and E. Similarly, each of the others knows exactly two people.

Since A knows B and E, A does not know C and D, so, by considering the triple A, C, D, we see that C and D must know each other, and by considering the triple A, B, E, that B and E do not know each other. Thus, B knows A and one of C and D; suppose, say, that B knows C. Then B knows neither of D and E, so that D must know E. Hence, we can seat the people in the order $A - -B - -C - -D - -E$, and each adjacent pair knows each other.

207. Let n be a positive integer exceeding 1. Suppose that $A = (a_1, a_2, \cdots, a_m)$ is an ordered set of $m = 2^n$ numbers, each of which is equal to either 1 or -1. Let

$$S(A) = (a_1a_2, a_2a_3, \cdots, a_{m-1}a_m, a_ma_1) \ .$$

Define, $S^0(A) = A$, $S^1(A) = S(A)$, and for $k \geq 1$, $S^{k+1} = S(S^k(A))$. Is it always possible to find a positive integer r for which $S^r(A)$ consists entirely of 1s?

Solution 1. For $i > m = 2^n$, define $a_i = a_{i-m}$. Then, by induction, for positive integers r, we can show that the rth iterate of S acting on A is

$$S^r(A) = S(S^{r-1}(A)) = \left(\cdots, \prod_{i=0}^{r} a_{k+i}^{\binom{r}{i}}, \cdots \right) \ .$$

This is clear when $r = 1$. Suppose it holds for the index r. Then the kth term of $S^{r+1}(A)$ is equal to

$$\prod_{i=0}^{r} a_{k+i}^{\binom{r}{i}} \prod_{i=1}^{r+1} a_{k+i}^{\binom{r}{i-1}} = \prod_{i=0}^{r+1} a_{k+i}^{\binom{r+1}{i}} \ .$$

Now let $r = 2^n$. Then, for $1 \leq i \leq 2^{n-1}$,

$$\binom{2^n}{i} = \left(\frac{2^n}{i}\right)\left(\frac{2^n - 1}{1}\right)\left(\frac{2^n - 2}{2}\right)\cdots\left(\frac{2^n - i + 1}{i - 1}\right)$$

is even, since the highest power of 2 that divides $2^n - j$ is that same as the highest power of 2 that divides j for $1 \leq j \leq 2^n - 1$ and 2 divides i to a lower power than it divides 2^n. Hence the kth term of $S^m(A)$ is equal to $a_ka_{k+m} = a_k^2 = 1$, and so $S^m(A)$ has all its entries equal to 1.

Solution 2. [A. Chan] Defining a_i for all positive indices i as in the previous solution, we find that

$$S(A) = (a_1a_2, a_2a_3, a_3a_4, \cdots, a_ma_1)$$

$$S^2(A) = (a_1a_3, a_2a_4, a_3a_5, \cdots, a_ma_2)$$

$$S^4(A) = (a_1a_5, a_2a_6, a_3a_7, \cdots, a_ma_4)$$

$$S^8(A) = (a_1a_9, a_2a_{10}, \cdots, a_ma_8)$$

and so on, until we come to, for $m = 2^n$,

$$S^m(A) = (a_1a_{1+m}, a_2a_{2+m}, \cdots, a_ma_{2m}) = (a_1^2, a_2^2, \cdots, a_m^2) = (1, 1, \cdots, 1).$$

Solution 3. [R. Romanescu] We prove the result by induction on n. The result holds for $n = 1$, since for $A = (a_1, a_2)$, we have that $S(A) = (a_1a_2, a_2a_1)$, and $S^2(A) = (1, 1)$. Suppose, for vectors with 2^n entries, we have shown that $S^{2^n}(A) = (1, 1, \cdots, 1)$ for n-vectors A, for $n \geq 1$. Consider the following vector with 2^{n+1} entries: $A = (a_1, b_1, a_2, b_2, \cdots, a_m, b_m)$ where $m = 2^n$. Then

$$S^2(A) = (a_1a_2, b_1b_2, a_2a_3, b_2b_3, \cdots, a_{m-1}a_m, b_{m-1}b_m)\ ,$$

i.e., applying S twice is equivalent to applying S to the separate vectors consisting of the even entries and of the odd entries. Then, by the induction, applying S^2 2^n times (equivalent to applying S 2^{n+1} times), we get a vector consisting solely of 1s.

208. Determine all positive integers n for which $n = a^2 + b^2 + c^2 + d^2$, where $a < b < c < d$ and a, b, c, d are the four smallest positive divisors of n.

Solution. It is clear that $a = 1$. Suppose, if possible that n is odd; then its divisors a, b, c, d must be odd, and so $a^2 + b^2 + c^2 + d^2$ must be even, leading to a contradiction. Hence n must be even, and so $b = 2$, and exactly one of c and d is odd. Hence

$$n = a^2 + b^2 + c^2 + d^2 \equiv 1 + 0 + 1 + 0 = 2$$

mod 4, and so c must be an odd prime number and d its double. Thus, $n = 5(1 + c^2)$. Since c divides n, c must divide 5, and so $c = 5$. We conclude that $n = 130$.

209. Determine all positive integers n for which $2^n - 1$ is a multiple of 3 and $(2^n - 1)/3$ has a multiple of the form $4m^2 + 1$ for some integer m.

Solution. We first establish the following result: *let p be an odd prime and suppose that $x^2 \equiv -1$ (mod p) for some integer n; then $p \equiv 1$ (mod 4). Proof.* By Fermat's Little Theorem, $x^{p-1} \equiv 1 \pmod p$, since x cannot be a multiple of p. Also $x^4 \equiv 1 \pmod p$. Suppose that $p - 1 = 4q + r$ where $0 \leq r \leq 3$. Since $p - 1$ is even, so is r; thus, $r = 0$ or $r = 2$. Now $x^r \equiv x^rx^{4q} \equiv x^{p-1} \equiv 1 \pmod p$, so $r = 0$. Therefore $p - 1$ is a multiple of 4. ♠

Suppose that 3 divides $2^n - 1$. Since $2^n \equiv (-1)^n \pmod 3$, n must be even. When $n = 2$, $(2^n - 1)/3 = 1$ has a multiple of the form $(2m)^2 + 1$; any value of m will do. Suppose that $n \geq 2$. Let $n = 2^u \cdot v$, with v odd and $u \geq 1$. Then

$$2^n - 1 = (2^v + 1)(2^v - 1)(2^w + 2^{w-2v} + \cdots + 2^{2v} + 1)$$

where $w = n - 2v = 2v(2^{u-1} - 1)$. Suppose that $(2m)^2 \equiv -1 \pmod{(2^n - 1)/3}$. Then, since $2^v + 1$ is divisible by 3, $(2m)^2 \equiv -1 \pmod{2^v - 1}$, If $v \geq 3$, then $2^v - 1$ is divisible by a prime p congruent to 3 (mod 4) and, by the foregoing result, $x^2 \equiv -1 \pmod p$ is not solvable. We are led to a contradiction, and so $v = 1$ and n must be a power of 2.

Now let $n = 2^u$. Then

$$2^n - 1 = (2 - 1)(2 + 1)(2^2 + 1)(2^4 + 1) \cdots (2^{2^{u-1}} + 1)$$

so that

$$\frac{2^n - 1}{3} = \prod_{i=1}^{u-1} (2^{2^i} + 1)\ .$$

We now use the *Chinese Remainder Theorem*: *if* $q_1, q_2, \cdots, q_r$ *are pairwise coprime integers and* $a_i, a_2, \cdots, a_r$ *arbitrary integers, then there exists an integer* x *such that* $x \equiv a_i \pmod{q_1 q_2 \cdots q_r}$ *for* $1 \le i \le r$, *and* x *is unique up to a multiple of* $q_1 q_2 \cdots q_r$. This is applied to $q_i = 2^{2^i} + 1$ $(1 \le i \le u-1)$ and $a_i = 2^{2^{i-1}-1}$. Observe that q_i and q_j are coprime for $i < j$. (For, if $2^{2^i} \equiv -1 \pmod{p}$, then $2^{2^j} \equiv 2^{2^{i+1}} \equiv 1 \pmod{p}$, so that $2^{2^j} + 1 \equiv 2 \pmod{p}$ and $p = 1$.) So there exists an integer m for which

$$m \equiv 2^{2^{i-1}-1} \pmod{2^{2^i} + 1}$$

for $1 \le i \le u - 1$. Therefore

$$4m^2 + 1 \equiv 2^2 \cdot 2^{2^i - 2} + 1 \equiv 2^{2^i} + 1 \equiv 0$$

modulo $\prod_{i=1}^{u-1}(2^{2^i} + 1)$ as desired.

For example, when $u = 3$, we have $m \equiv 1 \pmod 5$ and $m \equiv 2 \pmod{17}$, so we take $m = 36$ and find that $4m^2 + 1 = 61 \times 85 = 61 \times (\frac{1}{3} \times (2^8 - 1))$. When $u = 4$, we need to satisfy $m \equiv 1 \pmod 5$, $m \equiv 2 \pmod{17}$ and $m \equiv 8 \pmod{257}$: when $m = 3606$, $4m^2 + 1 = 52012045 = 2381 \times 5 \times 17 \times 257 = 2381 \times (\frac{1}{3} \times (2^{16} - 1))$.

210. ABC and DAC are two isosceles triangles for which B and D are on opposite sides of AC, $AB = AC$, $DA = DC$, $\angle BAC = 20°$ and $\angle ADC = 100°$. Prove that $AB = BC + CD$.

Solution 1. Produce BC to E so that $CE = CD$. Note that $\angle DCE = 60°$ (why?). Then ΔDCE is isosceles and so $\angle CDE = 60°$. Since $DA = DE$, we have that $\angle DAE = \angle DEA = 10°$. Therefore, $\angle BAE = 60° - 10° = 50°$ and $\angle BEA = 60° = 10° = 50°$, whence $AB = BE$.

Solution 2. Let $a = |AB| = |AC|$, $b = |BC|$, $c = |AD| = |CD|$, and $d = |BD|$. From the Law of Cosines applied to two triangles, we find that $d^2 = b^2 + c^2 + bc = a^2 + c^2 - ac$, whence $0 = b^2 - a^2 + (b + a)c = (b + a)(b - a + c)$. Therefore, $a = b + c$, as desired.

Solution 3. [M. Zaharia] From the Law of Sines, we have that $(\sin 80°)BC = (\sin 20°)AB$ and

$$(\sin 80°)CD = (\sin 100°)CD = (\sin 40°)AC = (\sin 40°)AB \ .$$

Hence

$$(\sin 80°)[BC + CD] = [\sin 20° + \sin 40°]AB = [2 \sin 30° \cos 10°]AB \ .$$

Since $\sin 80° = \cos 10°$ and $\sin 30° = 1/2$, the result follows.

Solution 4. Since, in any triangle, longer sides are opposite larger angles, $AB = AC > AD$. Let E be a point of the side AB for which $AE = AD$. Then ΔAED is isosceles with apex angle $60°$, from which we find that $CD = AD = DE = AE$. Since ΔDEC is isosceles and $\angle EDC = \angle ADC - \angle ADE = 100° - 60° = 40°$, it follows that $\angle DEC = \angle DCE = 70°$, $\angle ACE = 70° - 40° = 30°$ and

$$\angle ECB = 80° - 30° = 50° = 120° - 70° = \angle DEB - \angle DEC = \angle CEB \ .$$

Hence $BE = BC$ and so $AB = AE + EB = CD + BC$.

Solution 5. Since $\angle ABC + \angle ADC = 80° + 100° = 180°$, $ABCD$ is a concyclic quadrilateral. Suppose, wolog, that the circumcircle has unit radius. Since AB, BC and CD subtend respective angles $160°$, $40°$, $80°$ at the centre of the circumcircle, $AB = 2 \sin 80°$, $BC = 2 \sin 20°$ and $CD = 2 \sin 40°$. Since

$$\sin 20° + \sin 40° = 2 \sin 30° \cos 10° = \sin 80° \ ,$$

the result follows.

211. Let ABC be a triangle and let M be an interior point. Prove that

$$\min \{MA, MB, MC\} + MA + MB + MC < AB + BC + CA \ .$$

Solution 1. Let D, E, F be the respective midpoints of BC, AC, AB. Suppose, wolog, M belongs to both of the trapezoids $ABDE$ and $BCEF$. Then

$$MA + MB < BD + DE + EA \quad \text{and} \quad MB + MC < BF + FE + EC$$

whence

$$MA + 2MB + MC < AB + BC + CA \ .$$

To see, for example, that $MA+MB < BD+DE+EA$, construct GH such that G lies on the segment BD, H lies on the segment AE, $GH \| DE$ and M lies on the segment GH. Then

$$\begin{aligned} AM + MB &< AH + HM + MG + GB = AH + HG + GB \\ &< AH + HD + DG + GB = AH + HD + DB \\ &< AH + HE + ED + DB = EA + DE + BD \ . \end{aligned}$$

Solution 2. [R. Romanescu] We first establish that, if W is an interior point of a triangle XYZ, then $XW + WY < XZ + ZY$. To see this, produce YW to meet XZ at V. Then

$$XW + YW < XV + VW + YW = XV + VY < XV + VZ + ZY = XZ + ZY \ .$$

Let AP, BQ, CR be the medians of triangle ABC. These medians meet at the centroid G and partition the triangle into six regions. Wolog, suppose that M is in the triangle AGR. Then $AM + MB < AG + GB$ and $AM + MC < AR + RC$. Hence $2AM + MB + MC < AG + GB + AR + RC$. Since $AP < AR + RP = \frac{1}{2}(AB + AC)$, $AG = \frac{2}{3}AP < \frac{1}{3}(AB + BC)$. Similarly, $BG < \frac{1}{3}(AB + AC)$. Also $CR < \frac{1}{2}(AC + BC)$ and $AR = \frac{1}{2}AB$. Hence

$$\begin{aligned} AG + GB + AR + RC &< \frac{7}{6}AB + \frac{5}{6}AC + \frac{5}{6}BC \\ &< AB + \frac{1}{6}(AC + BC) + \frac{5}{6}AC + \frac{5}{6}BC \\ &= AB + BC + CA \ . \end{aligned}$$

The result now follows.

212. A set S of points in space has at least three elements and satisfies the condition that, for any two distinct points A and B in S, the right bisecting plane of the segment AB is a plane of symmetry for S. Determine all possible finite sets S that satisfy the condition.

Solution. We first show that all points of S lie on the surface of a single sphere. Let U be the smallest sphere containing all the points of S. Then there is a point $A \in S$ on the surface of U. Let B be any other point of S and P be the right bisecting plane of the segment AB. Since this is a plane of symmetry for S, the image V of the sphere U reflected in P must contain all the points of S. Let W be the sphere whose equatorial plane is $P \cap U = P \cap V$. Then $S \subseteq U \cap V \subseteq W \subseteq U \cup V$. Since U is the smallest sphere containing S and W is symmetric about P, $U \subseteq W$, $V \subseteq W$ and $U \cap V = U \cup V$. Hence $U = V$ and P must be an equatorial plane of U. But this means that B must lie on the surface of U.

Consider the case that S is a planar set; then the points of S lie on a circle. Let three of them in order be A, B, C. Since the image of B reflected in the right bisector of AC is a point of S on the arc AC, it can only be B itself. Hence $AB = BC$. Since S is finite, S must consist of the vertices of a regular polygon.

In general, any plane that intersects S must intersect it in the vertices of a regular polygon, so that, in particular, all the faces of the convex hull of S are regular polygons. Let F be one of these faces and G and H be faces adjacent to F sharing the respective edges AB and BC with F. Then G and H are images of each other under the reflection in the right bisector of AC, and so must be congruent. Consider the vertex

B of F; if I is a face adjacent to G and contains the vertex B, then F and I must be congruent. In this way, we can see that around each vertex of the convex hull of S, every second face is congruent. Thus, the polyhedron has all its faces of one or two types of congruent regular polygons. Since every vertex can be carried into every other by a sequence of reflections in right bisectors of edges, each vertex must have the same number of faces that contain it.

Since all the angles of faces meeting at a given vertex must sum to less than 360° and since all the faces are regular polygons, there must be 3, 4 or 5 faces at each vertex. If all the faces are congruent, the convex hull must be a regular polyhedron whenever S has at least four points. If S consists of the vertices of a regular tetrahedron or a regular octahedron, the conditions of the problem are satisfied. Otherside, it is possible to find an edge and a vertex whose plane intersects the polyhedron in a non-equilateral triangle so S cannot be at the vertices of a cube, a regular dodecahedron or a regular icosahedron.

If the polyhedron has two types of faces, then at each vertex, there must be two equilateral triangles and either two squares or two pentagons. Suppose that PQR is one of the triangle faces, and that T is the other end of the edge emanating from R. Then the plane PQT cuts the polyhedron in the non-equilateral triangle PQT (note that all sides have the same length, so there are no other points of S on this plane). Hence, this possibility must be rejected.

213. Suppose that each side and each diagonal of a regular hexagon $A_1A_2A_3A_4A_5A_6$ is coloured either red or blue, and that no triangle $A_iA_jA_k$ has all of its sides coloured blue. For each $k = 1, 2, \cdots, 6$, let r_k be the number of segments A_kA_j $(j \neq k)$ coloured red. Prove that

$$\sum_{k=1}^{6} (2r_k - 7)^2 \leq 54 \ .$$

Solution 1. Suppose, say, $r_1 = 0$. Since every edge emanating from A_1 is blue, every other edge is red, so that $r_2 = r_3 = r_4 = r_5 = r_6 = 4$ and $\sum_{k=1}^{6}(2r_k - 7)^2 = 7^2 + 5 \times 1^2 = 54$.

Suppose, that every vertex is adjacent to at least one red edge, that, say, $r_1 = 1$ and that A_1A_2 is red. Then each of A_3, A_4, A_5, A_6 must be joined to each of the others by a red segment, so that r_3, r_4, r_5 and r_6 are at least equal to 3. Since all of them are joined to A_1 be a blue segment, r_3, r_4, r_5 and r_6 are at most equal to 4. Thus, $(2r_k - 7)^2 = 1$ for $3 \leq k \leq 6$. Since $1 \leq r_k \leq 5$, $\sum_{k=1}^{6}(2r_k - 7)^2 \leq 2 \times 5^2 + 4 \times 1^2 = 54$.

Suppose that $r_k \geq 2$ for each k. Then $2 \leq r_k \leq 5$, so that $(2r_k - 7)^2 \leq 3^2$ for each k and so $\sum_{k=1}^{6}(2r_k - 7)^2 \leq 6 \times 3^2 = 54$.

Solution 2. [A. Feiz Mohammadi] We prove the more general result: *Suppose that each side and each diagonal of a regular $n-$gon $A_1A_2 \cdots A_n$ is coloured either red or blue, and that no triangle $A_iA_jA_k$ has all of its sides coloured blue. For each $k = 1, 2, \cdots, n$, let r_k be the number of segments A_kA_j $(j \neq k)$ coloured red. Then*

$$\sum_{k=1}^{n} \left[2r_k - \left(\frac{3n-4}{2} \right) \right]^2 \leq \frac{n^3}{4} \ .$$

For $1 \leq k \leq n$, let b_k be the number of segments A_kA_j $(j \neq k)$ coloured blue. There are $\binom{b_k}{2}$ pairs of these segments; if A_kA_j and A_kA_i are two of them, then A_iA_j must be coloured red. Hence $\sum_{k=1}^{n}\binom{b_k}{2}$ counts the number of red segments, each as often as there are triangles containing it whose other edges are coloured blue. Suppose that A_uA_v is one of these red segments. There are b_u blue segments emanating from A_u and b_v from A_v, so that the red segments can be counted at most $\min\{b_u, b_v\} \leq \frac{1}{2}(b_u + b_v)$ times.

Hence

$$\sum_{k=1}^{n} \binom{b_k}{2} \leq \sum \left\{ \frac{b_u + b_v}{2} : A_uA_v \ \text{ is coloured red } \right\} .$$

Each b_k will appear in r_k summands, and $r_k = (n-1) - b_k$, so that

$$\begin{aligned}
\frac{1}{2}\left[\sum_{k=1}^n b_k^2 - \sum_{k=1}^n b_k\right] = \sum_{k=1}^n \binom{b_k}{2} &\leq \frac{1}{2}\sum_{k=1}^n r_k b_k \\
&= \frac{1}{2}\sum_{k=1}^n [(n-1-b_k)b_k] = \frac{n-1}{2}\sum_{k=1}^n b_k - \frac{1}{2}\sum_{k=1}^n b_k^2 \\
&\Longrightarrow \sum_{k=1}^n b_k^2 \leq \frac{n}{2}\sum_{k=1}^n b_k \\
&\Longrightarrow \sum_{k=1}^n \left(2b_k - \frac{n}{2}\right)^2 \leq \frac{n^3}{4} \\
&\Longrightarrow \sum_{k=1}^n \left[2r_k - \left(\frac{3n-4}{2}\right)\right]^2 \leq \frac{n^3}{4} \ .
\end{aligned}$$

The upper bound in Feiz Mohammadi's result is actually attained when $r_1 = 0$ and $r_k = n-2$ for $k \geq 2$, and when $r_k = n-1$ for each k.

214. Let S be a circle with centre O and radius 1, and let P_i $(1 \leq i \leq n)$ be points chosen on the (circumference of the) circle for which $\sum_{i=1}^n \overrightarrow{OP_i} = \mathbf{0}$. Prove that, for each point X in the plane, $\sum |XP_i| \geq n$.

Solution 1. Use complex numbers, representing S by the unit circle in the complex plane and the points P_i by complex numbers z_i for which $|z_i| = 1$ and $\sum z_i = 0$. Then

$$\begin{aligned}
\sum |z - z_i| = \sum |z_i||z\overline{z_i} - 1| &= \sum |z\overline{z_i} - 1| \\
&\geq \left|\sum (z\overline{z_i} - 1)\right| = \left|z(\sum \overline{z_i} - 1)\right| \\
&= \left|\overline{\sum z_i} - n\right| = |0 - n| = n \ .
\end{aligned}$$

Solution 2. We have that

$$\begin{aligned}
\sum |XP_i| &= \sum |\overrightarrow{OP_i} - \overrightarrow{OX}||\overrightarrow{OP_i}| \\
&\geq (\overrightarrow{OP_i - OX}) \cdot (\overrightarrow{OP_i}) \\
&= n - \sum \overrightarrow{OX} \cdot \overrightarrow{OP_i} \\
&= n - \overrightarrow{OX} \cdot \sum \overrightarrow{OP_i} = n \ .
\end{aligned}$$

(The inequality is due to the Cauchy-Schwarz Inequality.)

Solution 3. [O. Bormashenko] Let the points $P_i \sim (\cos u_i, \sin u_i)$ be placed on the unit circle of the cartesian plane and let $X \sim (x, y)$. For $1 \leq i \leq n$,

$$\begin{aligned}
(x \sin u_i - y \cos u_i)^2 \geq 0 &\Longleftrightarrow x^2 \sin^2 u_i + y^2 \cos^2 u_i \geq 2xy \cos u_i \sin u_i \\
&\Longleftrightarrow x^2 + y^2 \geq x^2 \cos^2 u_i + 2xy \cos u_i \sin u_i + y^2 \sin^2 u_i \ ,
\end{aligned}$$

so that

$$\begin{aligned}
|XP_i|^2 &= (x - \cos u_i)^2 + (y - \sin u_i)^2 \\
&= x^2 + y^2 + 1 - 2x \cos u_i - 2y \sin u_i \\
&\geq (1 - x \cos u_i - y \sin u_i)^2 \ .
\end{aligned}$$

Thus,

$$\sum_{i=1}^n |XP_i| \geq \sum_{i=1}^n (1 - x\cos u_i - y\sin u_i)$$
$$= n - x\sum_{i=1}^n \cos u_i - y\sum_{i=1}^n \sin u_i = n \ ,$$

because of $\sum_{i=1}^n \overrightarrow{OP_i} = \mathbf{O}$ and the vanishing of the components of this sum in the two coordinate directions.

Solution 4. [A. Mao] Let the equation of the circle S in the cartesian plane be $x^2 + y^2 = 1$. Wolog, we may assume that X lies on the $x-$axis. Let r and s be the lines of equations $x = 1$ and $x = -1$ respectively. If X lies outside the circle, the reflection in the nearer of the lines r and s take X to a point Y for which

$$|OY| = \begin{cases} 2 - |OX|, & \text{for } 1 < |OX| \leq 2; \\ |OX| - 2, & \text{for } |OP| \geq 2. \end{cases}$$

Since Y lies on the same side of the line of reflection as all of the P_i and X lies on the opposite side, $\sum |XP_i| \geq \sum |YP_i|$.

If $1 \leq |OX| < 3$, the first reflection takes X to the interior of the circle. If $|OX| \geq 3$, the first reflection reduces the distance from the origin by 2 and a chain of finitely many reflections will take X into the circle.

Hence, wolog, we may suppose that X lies within or on the circle. Let $X \sim (w, 0)$ with $-1 \leq w \leq 0$ and let $P_i \sim (\cos u_i, \sin u_i)$. Then

$$|XP_i| = \sqrt{(w - \cos u_i)^2 + \sin^2 u_i}$$
$$= \sqrt{w^2 - 2w\cos u_i + 1}$$
$$= \sqrt{(1 - w\cos u_i)^2 + w^2 \sin^2 u_i}$$
$$\geq \sqrt{(1 - w\cos u_i)^2} = 1 - w\cos u_i \ ,$$

since $|w \cos u_i| \leq 1$. Hence

$$\sum |XP_i| \geq n - w \sum \cos u_i = n \ .$$

215. Find all values of the parameter a for which the equation $16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$ has exactly four real solutions which are in geometric progression.

Solution 1. Let $x + (1/x) = t$. Then the equation becomes $f(t) \equiv 16t^2 - at + 2a - 15 = 0$. If the original equation has all real roots, then this quadratic in t must have two real roots t_1 and t_2, both of which have absolute value exceeding 2 (why?). The discriminant of the quadratic is equal to $a^2 - 64(2a - 15) = (a - 8)(a - 120)$, so that its roots are real if and only if $a \leq 8$ or $a \geq 120$. Observe that $f(2) = 49 > 0$, so that 2 does not lie between the roots, t_1 and t_2. Hence the roots are either both less than -2 or both greater than 2.

If both of the roots, t_1 and t_2 are negative, then their sum $a/16$ is less than -4, so that $a < -64$ and $t_1 t_2 = (2a - 15)/64 < 0$. But this yields a contradiction, as the roots have the same sign. Hence, we must have $2 < t_1 < t_2$, say, so that the four roots x_1, x_2, x_3, x_4 of the given equation are positive. Suppose that $x_1 \leq x_2 \leq x_3 \leq x_4$ with x_1 and x_4 the solutions of $x + (1/x) = t_2$ and x_2 and x_3 the solutions of $x + (1/x) = t_1$. (Explain why this alignment of indices is correct.) Note that $x_1 x_4 = x_2 x_3 = 1$. Since the four roots are in geometric progression with common ration $(x_4/x_1)^{1/3} = x_1^{-2/3}$, we find that

$$t_2 = x_1 + \frac{1}{x_1} = \left(x_1^{1/3} + \frac{1}{x_1^{1/3}}\right)\left(\left(x_1^{1/3} + \frac{1}{x_1^{1/3}}\right)^2 - 3\right) = t_1(t_1^2 - 3)$$

so that

$$\frac{a}{16} = t_1 + t_2 = t_1(t_1^2 - 2) \ ,$$

whence

$$a = t_1(16t_1^2 - 32) = t_1(at_1 - 2a + 15 - 32) = at_1^2 - (2a + 17)t_1$$

so that,

$$\begin{aligned} 0 &= -16t_1^2 + a(t_1 - 2) + 15 \\ &= -16t_1^2 + 16t_1(t_1^2 - 2)(t_1 - 2) + 15 \\ &= 16t_1^4 - 32t_1^3 - 48t_1^2 + 64t_1 + 15 \\ &= (2t_1 - 5)(2t_1 + 3)(4t_1^2 - 4t_1 - 1) \ . \end{aligned}$$

Therefore, $t_2 = 5/2$ and so $a = 170$.

Indeed, when $a = 170$, we find that $0 = 16x^4 - 170x^3 + 357x^2 - 170x + 16 = (x-8)(x-2)(2x-1)(8x-1)$.

Solution 2. Let the roots by ur^3, ur, ur^{-1}, ur^{-3}, with $u > 0$. Since the product of the roots is 1, we must have that $u = 1$. From the relationship between the coefficients and the roots, we have that

$$r^3 + r + r^{-1} + r^{-3} = \frac{a}{16}$$

and

$$r^4 + r^2 + 2 + r^{-2} + r^{-4} = \frac{2a + 17}{16} \ .$$

Let $s = r + r^{-1}$ so that $s^3 - 2s = a/16$ and $s^4 - 3s^2 + 2 = (2a + 17)/(16) = 2(s^3 - 2s) + (17/16)$. Hence

$$\begin{aligned} 0 &= s^4 - 2s^3 - 3s^2 + 4s + (15/16) \\ &= (1/16)(4s^2 - 4s - 15)(4s^2 - 4s - 1) = (1/16)(2s + 3)(2s - 5)(4s^2 - 4s - 1) \ . \end{aligned}$$

Since s must be real and its absolute value is not less than 2, $s = 5/2$ and so r is equal to either 2 or 1/2. Therefore

$$a = 16\left(8 + 2 + \frac{1}{2} + \frac{1}{8}\right) = 170 \ .$$

216. Let x be positive and let $0 < a \leq 1$. Prove that

$$(1 - x^a)(1 - x)^{-1} \leq (1 + x)^{a-1} \ .$$

Solution 1. If $x = 1$, the inequality degenerates, but the related inequality $(1 - x^a) \leq (1 + x)^{a-1}(1 - x)$ holds. If $x > 1$, then, with $y = 1/x$, the inequality is equivalent to $(1 - y^a)(1 - y)^{-1} \leq (1 + y)^{a-1}$. (Establish this.) Hence, it suffices to show that the inequality holds when $0 < x < 1$.

By the concavity of the function $(1 + x)^{1-a}$ for $x > -1$, we have that $(1 + x)^{1-a} \leq 1 + (1 - a)x$. (Observe that the tangent to the curve $y = (1 + x)^{1-a}$ at $(0, 1)$ is $y = 1 + (1 - a)x$.) Therefore

$$\begin{aligned} (1 - x) - (1 + x)^{1-a}(1 - x^a) &\geq (1 - x) - [1 + (1 - a)x](1 - x^a) \\ &- x - (1 - a)x + x^a + (1 - a)x^{a+1} = x^a + (1 - a)x^{a+1} - (2 - a)x \ . \end{aligned}$$

By the Arithmetic-Geometric Means Inequality,

$$\begin{aligned} \frac{x^a + (1 - a)x^{a+1}}{2 - a} &\geq x^{a(2-a)^{-1}} x^{(a+1)(1-a)/(2-a)} \\ &= x^{-(1-a)^2/(2-a)} x > x \ , \end{aligned}$$

since $x < 1$. The result now follows.

Solution 2. [A. Feiz Mohammadi] As above, we can restrict to the situation that $0 < x < 1$. Let $f(a) = (1 - x^a)(1 + x)^{1-a}$. Suppose, to begin with, we take $0 < a = m/n < 1$ for some positive integers m and n. Since $m < n$, $m - k < n - k < n$ for $0 < k$. Hence

$$x^n \left(\frac{1 - x^m}{1 - x}\right) = \sum_{k=0}^{m-1} x^n x^k \leq \sum_{k=0}^{m-1} x^{m-k} x^k \leq m x^m < n x^m ,$$

whence (using the binomial expansion),

$$(1 + x^n)^{1/n} \leq 1 + \frac{x^n}{n} \leq 1 + \frac{x^m(1 - x)}{1 - x^m} = \frac{1 - x^{m+1}}{1 - x^m} .$$

This inequality holds if we replace x by $x^{1/n}$. Therefore

$$(1 + x)^{1/n} \leq \frac{1 - x^{(m+1)/n}}{1 - x^{m/n}}$$

$$\Longrightarrow (1 - x^{m/n})(1 + x)^{1-(m/n)} \leq (1 - x^{(m+1)/n})(1 + x)^{1-(m+1)/n} .$$

Thus, $f(m/n) \leq f((m + 1)/n)$.

Let u and v be two rationals with $0 < u < v < 1$, and let n be a common denominator, so that $u = m/n$ and $v = (m + p)/n$ for some positive integers m and p. Then $f(u) \leq f(v)$. Let r_0 be a given rational in $(0, 1)$, and let $\{r_k : k \geq 0\}$ be an increasing sequence for which $\lim_{k\to\infty} r_k = 1$. Since f is an increasing function of rational a,

$$1 - x = f(1) = \lim_{k\to\infty} f(r_k) \geq f(r_0) .$$

Suppose that a is any real with $0 < a < 1$. Suppose, if possible, that $f(a) > 1 - x$ and let $\epsilon = f(a) - (1 - x) > 0$. Since f is continuous at a, there is a positive number δ with $0 < \delta < \min\,(a, 1 - a)$ for which $|f(r) - f(a)| < \epsilon$ whenever $0 < |a - r| < \delta$. Let r be a rational satisfying this condition. Then $0 < r < 1$, $f(r) < f(1)$ and so

$$\epsilon > f(a) - f(r) = (f(a) - f(1)) + (f(1) - f(a)) > f(a) - f(1)$$

yielding a contradiction. The result follows.

Solution 3. [R. Furmaniak] Fix $x > 0$, $x \neq 1$ and let

$$F(a) = (1 - x^a)(1 + x)^{1-a}(1 - x)^{-1}$$

for $a > 0$. Note that $F(a) > 0$. Observe that, by the Arithmetic-Geometric Means Inequality,

$$2x^{(a+b)/2} \leq x^a + x^b$$

so that

$$(1 - x^a)(1 - x^b) \leq (1 - x^{(a+b)/2})^2 .$$

Hence

$$\sqrt{F(a)F(b)} \leq F\left(\frac{a + b}{2}\right)$$

for $a, b > 0$, so that $\log F(a)$ is a concave function on the half-line $(0, \infty)$.

Now $F(1) = 1$ and $F(2) = (1+x)(1+x)^{-1} = 1$, so that $\log F(a)$ vanishes at $a = 1$ and $a = 2$. Hence, by the concavity,

$$F(a) \le 1 \Longleftrightarrow \log F(a) \le 0 \Longleftrightarrow 0 < a \le 1 \quad \text{or} \quad 2 \le a$$

and the result follows.

217. Let the three side lengths of a scalene triangle be given. There are two possible ways of orienting the triangle with these side lengths, one obtainable from the other by turning the triangle over, or by reflecting in a mirror. Prove that it is possible to slice the triangle in one of its orientations into finitely many pieces that can be rearranged using rotations and translations in the plane (but not reflections and rotations out of the plane) to form the other.

Solution 1. There are several ways of doing this problem. Observe that, if a geometric figure has a reflective axis of symmetry, then a rotation of $180°$ about a point on the axis (combined with a translation) will allow it to be superimposed upon its image reflected in an axis perpendicular to the reflective axis. For example, this applies to kites and isosceles triangles. So one strategy is to cut the triangle into finitely many pieces that have such a reflective axis of symmetry.

(a) Cut from the three vertices into the circumcentre of the triangle to obtain three isosceles triangles, which can be rearranged to give the other orientation.

(b) The triangle has at least one internal altitude. Cutting along this altitude yields two right triangles, each of which can be sliced along its median to the hypotenuse to give two isosceles triangles.

(c) Slice along the lines from the incentre of the triangle to the feet of the perpendiculars to the sides from the incentre. This yields three kites that can be moved to give the other orientation.

Solution 2. Superimpose the triangle onto its image obtained by reflecting in a line parallel to its longest side so that the corresponding side of one triangle contains the opposite vertex to this side of the other. Make cuts to produce the quadrilateral common to the triangle and its image. The remaining (isosceles) pieces of the triangle can be rotated to cover the corresponding parts of the image.

218. Let ABC be a triangle. Suppose that D is a point on BA produced and E a point on the side BC, and that DE intersects the side AC at F. Let $BE + EF = BA + AF$. Prove that $BC + CF = BD + DF$.

Solution 1. [O. Bormashenko] Produce CA to W so that $AW = AB$; produce FE to X so that $EX = EB$; produce FC to Y so that $CY = CB$; produce FD to Z so that $DZ = BD$. Then $\angle EXB = \angle EBX = \frac{1}{2}\angle FEB$ (exterior angle), and

$$FW = FA + AW = FA + AB = BE + EF = XE + EF = XF$$

so that $\angle FWX = \angle FXW = \frac{1}{2}\angle CFE$.

$\angle CBY = \angle CYB = \frac{1}{2}\angle BCF \Longrightarrow$

$$\angle XBY = \angle XBE - \angle CBY = \frac{1}{2}(\angle FEB - \angle BCF) = \frac{1}{2}\angle CFE$$

(exterior angle). Hence, $\angle XBY = \angle FWX = \angle YWX$ and $WBXY$ is concyclic.

Also,

$$\begin{aligned}\angle ZBW =& \angle ABW - \angle ABZ = \angle ABW - \angle DBZ = \frac{1}{2}(\angle CAB - \angle XDB) \\ &= \frac{1}{2}\angle DFA = \frac{1}{2}\angle CFE = \angle FXW = \angle ZXW\end{aligned}$$

and so $WBXZ$ is concyclic. Therefore, $WXYZ$ is concyclic and $\angle FZW = \angle XZW = \angle XYW = \angle XYF$.

Consider triangles ZFW and YFX. Since $FW = FX$, $\angle ZFW = \angle YFX$ and $\angle FZW = \angle XYZ$, $\Delta ZFW \equiv \Delta YFX$, and so $FZ = FY$. Therefore,

$$BC + CF = YC + CF = YF = ZF = ZD + DF = BD + DF \ .$$

Solution 2. [A. Feiz Mohammadi] Let $\angle EBF = u_1$, $\angle ABF = u_2$, $\angle BFE = v_1$ and $\angle BFA = v_2$. From the law of sines, we have that

$$EB : EF : BF = \sin v_1 : \sin u_1 : \sin(u_1 + v_1)$$

whence

$$(EB + EF) : BF = (\sin v_1 + \sin u_1) : \sin(u_1 + v_1) \ .$$

Similarly,

$$(AB + FA) : BF = (\sin v_2 + \sin u_2) : \sin(u_2 + v_2) \ .$$

Hence

$$\begin{aligned}
\frac{\sin u_1 + \sin v_1}{\sin(u_1 + v_1)} = \frac{\sin u_2 + \sin v_2}{\sin(u_2 + v_2)} &\Leftrightarrow \frac{\cos \frac{1}{2}(u_1 - v_1)}{\cos \frac{1}{2}(u_1 + v_1)} = \frac{\cos \frac{1}{2}(u_2 - v_2)}{\cos \frac{1}{2}(u_2 + v_2)} \\
&\Leftrightarrow \cos \frac{1}{2}(u_1 - v_1) \cos \frac{1}{2}(u_2 + v_2) = \cos \frac{1}{2}(u_2 - v_2) \cos \frac{1}{2}(u_1 + v_1) \\
&\Leftrightarrow \cos \frac{1}{2}(u_1 + u_2 + v_2 - v_1) + \cos \frac{1}{2}(v_1 + v_2 + u_2 - u_1) \\
&\qquad\qquad = \cos \frac{1}{2}(u_1 + u_2 + v_1 - v_2) + \cos \frac{1}{2}(v_1 + v_2 + u_1 - u_2) \\
&\Leftrightarrow \cos \frac{1}{2}(u_1 + u_2 + v_2 - v_1) - \cos \frac{1}{2}(v_1 + v_2 + u_1 - u_2) \\
&\qquad\qquad = \cos \frac{1}{2}(u_1 + u_2 + v_1 - v_2) - \cos \frac{1}{2}(v_1 + v_2 + u_2 - u_1) \\
&\Leftrightarrow \sin \frac{1}{2}(u_1 + v_2) \sin \frac{1}{2}(v_1 - u_2) = \sin \frac{1}{2}(u_2 + v_1) \sin \frac{1}{2}(v_2 - u_1) \\
&\Leftrightarrow \frac{\sin \frac{1}{2}(v_2 + u_1) \cos \frac{1}{2}(v_2 - u_1)}{\sin \frac{1}{2}(v_2 - u_1) \cos \frac{1}{2}(v_2 - u_1)} = \frac{\sin \frac{1}{2}(v_1 + u_2) \cos \frac{1}{2}(v_1 - u_2)}{\sin \frac{1}{2}(v_1 - u_2) \cos \frac{1}{2}(v_1 - u_2)} \\
&\Leftrightarrow \frac{\sin u_1 + \sin v_2}{\sin(v_2 - u_1)} = \frac{\sin u_2 + \sin v_1}{\sin(v_1 - u_2)} \\
&\Leftrightarrow \frac{\sin \angle FBC + \sin \angle BFC}{\sin \angle FCB} = \frac{\sin \angle FBD + \sin \angle DFB}{\sin \angle FDB} \\
&\Leftrightarrow \frac{FC + BC}{BF} = \frac{DF + DB}{BF} \Leftrightarrow BC + CF = BD + DF \ .
\end{aligned}$$

219. There are two definitions of an ellipse.

(1) An ellipse is the locus of points P such that the sum of its distances from two fixed points F_1 and F_2 (called *foci*) is constant.

(2) An ellipse is the locus of points P such that, for some real number e (called the *eccentricity*) with $0 < e < 1$, the distance from P to a fixed point F (called a *focus*) is equal to e times its perpendicular distance to a fixed straight line (called the *directrix*).

Prove that the two definitions are compatible.

Solution 1. Consider the following set of equivalent equations:

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$$

$$\Leftrightarrow \sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$$

$$\Leftrightarrow x^2+2xc+c^2+y^2 = 4a^2+x^2-2xc+c^2+y^2-4a\sqrt{(x-c)^2+y^2}$$

$$\Leftrightarrow \sqrt{(x-c)^2+y^2} = a - \frac{xc}{a} = e\left(\frac{a}{e}-x\right)$$

where $e = c/a$. In applying the first definition, we may take the foci to be at the points $(c, 0)$ and $(-c, 0)$ and the sum of the focal radii to be $2a$. The final equation in the set describes the locus of a point whose distance from the focus $(c, 0)$ is equal to e times the distance to the line $x = a/e$.

However, in applying the second definition, we can without loss of generality assume that the focus is at $(c, 0)$ and the directrix is given by $x = d$. Where e is the eccentricity, let $a = de$. Then, reading up the equations, note that in going from the third to the second, both sides of the second have the same sign. Then the first equation describes a locus determined by the two foci condition.

Solution 2. In this solution, we start with the standard form of the equation for each definition and show that it describes the other locus.

In applying the first definition, place the foci at the points $(c, 0)$ and $(-c, 0)$, where $c > 0$, and let the ellipse be the locus of points P for which the sum of the distances to the foci is the constant value $2a > 0$. Thus, the equation of the locus is

$$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a$$

$$\Leftrightarrow \sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$$

$$\Rightarrow x^2-2cx+c^2+y^2 = 4a^2-4a\sqrt{(x+c)^2+y^2}+x^2+2cx+c^2+y^2$$

$$\Leftrightarrow a\sqrt{(x+c)^2+y^2} = a^2+cx$$

$$\Rightarrow a^2x^2+2a^2xc+a^2c^2+a^2y^2 = a^4+2a^2cx+c^2x^2$$

$$\Leftrightarrow (a^2-c^2)x^2+a^2y^2 = a^2(a^2-c^2)\ .$$

Let $b^2 = a^2 - c^2$. Then the equation can be written

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1\ .$$

This equation can also be written

$$y^2 = b^2 - \frac{b^2x^2}{a^2}\ .$$

Consider the line $x = d$, where $d > 0$ and let P be a point on the ellipse, F be the focus at $(c, 0)$ and Q be the foot of the perpendicular from P to the line $x = d$. We want to select d so that the ratio $PF^2 : PQ^2$ is independent of $P(x, y)$. Now

$$\begin{aligned}\frac{PF^2}{PQ^2} &= \frac{(x-c)^2+y^2}{(d-x)^2}\\ &= \frac{x^2-2cx+c^2+b^2-(b^2/a^2)x^2}{x^2-2dx+d^2}\\ &= \frac{(a^2-b^2)x^2-2a^2cx+(b^2+c^2)a^2}{a^2x^2-2a^2dx+a^2d^2}\\ &= \frac{c^2}{a^2}\left[\frac{x^2-(2a^2/c)x+(a^4/c^2)}{x^2-2dx+d^2}\right]\ .\end{aligned}$$

The quantity in the square brackets is equal to 1 when $d = a^2/c$. Thus, when $d = a^2/c$, $PF^2 : PQ^2 = c^2 : a^2$, a constant ration. Define $e = c/a$. Note that $e < 1$. Then we find that $PF = ePQ$ and $a = de$.

On the other hand, start with the focus-directrix definition of an ellipse with eccentricity e, focus at $(0,0)$ and directrix $x = d$. Then

$$x^2 + y^2 = e^2(x^2 - 2dx + d^2) \Leftrightarrow (1 - e^2)\left[x + \frac{de^2}{1-e^2}\right]^2 + y^2 = d^2e^2 + \frac{d^2e^4}{1-e^2} = \frac{d^2e^2}{1-e^2}$$

$$\Leftrightarrow \left[x + \frac{de^2}{1-e^2}\right]^2 + \frac{y^2}{1-e^2} = \left(\frac{de}{1-e^2}\right)^2 .$$

Setting $y = 0$, we can check that the curve cuts the $x-$axis at the points $((de)/(1+e), 0)$ and $((-de)/(1-e), 0)$. Define a to be equal to

$$\frac{1}{2}\left(\frac{de}{1+e} + \frac{de}{1-e}\right) = \frac{de}{1-e^2} ,$$

$c = ea$ and $b = \sqrt{a^2 - c^2}$. Then the equation of the focus-directrix locus becomes

$$(x + c)^2 + \frac{y^2}{1 - (c^2/a^2)} = a^2$$

$$\Leftrightarrow \frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1 ,$$

which is a shift of the locus of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

c units to the left.

Since it is not completely clear that the latter form indeed represents the locus according to the two-foci definition, we show that the sum of the distances from any point on the curve to the points $(0,0)$ and $(-2c, 0)$ is constant. Note that $y^2 = (b^2/a^2)[b^2 - x^2 - 2cx]$, from which

$$x^2 + y^2 = \frac{(cx - b^2)^2}{a^2}$$

and

$$\begin{aligned}(x + 2c)^2 + y^2 &= x^2 + 4cx + 4c^2 + (b^2/a^2)[b^2 - x^2 - 2cx] \\ &= (1/a^2)[(a^2 - b^2)x^2 + 2c(2a^2 - b^2)x + 4a^2c^2 + (a^2 - c^2)^2] \\ &= (1/a^2)[c^2x^2 + 2c(a^2 + c^2) + (a^2 + c^2)^2] \\ &= \frac{(cx + a^2 + c^2)^2}{a^2} .\end{aligned}$$

We need to ensure which square root is correct when we calculate the sum of the distances. Note that

$$\frac{b^2}{c} = \frac{a^2}{c} - c = \frac{a}{e} - c = \frac{d}{1-e^2} - \frac{de^2}{1-e^2} = d$$

so that $x < d = b^2/c$. Note also that

$$\frac{a^2 + c^2}{c} - \frac{de}{1-e} = \frac{a^2}{c} + c - \frac{de}{1-e} = \frac{d}{1-e^2}[1 + e^2 - e(1+e)] = \frac{d}{1+e} > 0 .$$

Hence

$$-\frac{a^2 + c^2}{c} < -\frac{de}{1-e} \le x$$

at all points on the curve. Hence

$$\sqrt{x^2+y^2}+\sqrt{(x+2c)^2+y^2}=\frac{1}{a}[(b^2-cx)+(cx+a^2+c^2)]$$
$$=\frac{a^2+b^2+c^2}{a}=\frac{2a^2}{a}=2a\ ,$$

a constant.

220. Prove or disprove: A quadrilateral with one pair of opposite sides and one pair of opposite angles equal is a parallelogram.

Solution 1. The statement is false. To see how to obtain the solution, start with a triangle XYZ with $\angle XYZ < \angle XZY < 90°$. Then it is possible to find a point W on YZ for which $XW = XZ$ (this is the diagram for the *ambiguous case* ASS-congruence situation). There are two ways of gluing a copy of triangle XYW to XYZ (the copy of XW glued along XZ) to give a quadrilateral with an opposite pair of angles equal to $\angle Y$ and an opposite pair of sides equal to $|XY|$. One of these satisfies the condition and is not a parallelogram.

C. Shen followed this strategy with $|XY| = 8$, $\angle XYZ = 60°$, $|YW| = 3$ and $|YZ| = 5$ to obtain a quadrilateral $ABCD$ with $|AB| = 5$, $|BC| = 8$, $|CD| = 3$, $|DA| = 8$, $|BD| = 7$ and $\angle DAB = \angle DCB = 60°$.

Solution 2. The statement is false. Suppose that we have fixed D, A, B and that AB is one of the equal sides and $\angle DAB$ is one of the equal angles. Then C is the intersection of two circles. One of the circles contains the locus of points at which DB subtends an angle equal to $\angle DAB$ and the other circle is that with centre D and radius equal to $|AB|$. The two circles are either tangent or have two points of intersection. One of these points will give the expected parallelogram, so the question arises whether the other point will give a suitable quadrilateral. We show that it can.

Using coordinate geometry, we may take $A \sim (0,0)$, $B \sim (3,0)$, $D \sim (2,2)$ so that $\angle DAB = 45°$. The point E that completes the parallelogram is $(5,2)$, and this will be one of the intersections of the two circles. The circle that subtends an angle of 45° from DB has as its centre the circumcentre of ΔBDE, namely $(7/2, 3/2)$; this circle has equation $x^2 - 7x + y^2 - 3y + 12 = 0$. The circle with centre D and radius $3 = |AB|$ has equation $x^2 - 4x + y^2 - 4y - 1 = 0$. These circles intersect at the points $E \sim (5,2)$ and $C \sim (22/5, 1/5)$. The quadrilateral $ABCD$ satisfies the given conditions but is not a parallelogram.

Comment. Investigate what happens when A, B and D are assigned the coordinates $(0,0)$, $(2,0)$ and (i) $(1,1)$ or (ii) $(2,2)$, respectively.

Comment. Consider the following two "proofs" that the quadrilateral must be a parallelogram.

"Proof" 1. Let $AB = CD$ and $\angle A = \angle C$. Suppose that X and Y, respectively, are the feet of the perpendiculars dropped from B to AD and from D to BC. Then triangles AXB and CYD, having equal acute angles and equal hypotenuses must be congruent. Hence $AX = CY$, and also $BX = DY$, from which it can be deduced that triangles BXD and DYB are congruent. Therefore $XD = YB$ and so $AD = BC$ and the quadrilateral is a parallelogram.

"Proof" 2. Suppose that $AB = CD$ and that $\angle B = \angle D$. Applying the Law of Sines, we find that

$$\frac{DC}{\sin\angle DAC}=\frac{AC}{\sin\angle ADC}=\frac{AC}{\sin\angle ABC}=\frac{AB}{\sin\angle ACB}=\frac{CD}{\sin\angle ACB}\ .$$

Therefore, $\angle DAC = \angle ACB$ so that $\angle DCA = \angle BAC$ and $AB \| DC$.

221. A *cycloid* is the locus of a point P fixed on a circle that rolls without slipping upon a line u. It consists of a sequence of arches, each arch extending from that position on the locus at which the point P rests on the line u, through a curve that rises to a position whose distance from u is equal to the diameter of the generating circle and then falls to a subsequent position at which P rests on the line u. Let v be the straight line parallel to u that is tangent to the cycloid at the point furthest from the line u.

(a) Consider a position of the generating circle, and let P be on this circle and on the cycloid. Let PQ be the chord on this circle that is parallel to u (and to v). Show that the locus of Q is a similar cycloid formed by a circle of the same radius rolling (upside down) along the line v.

(b) The region between the two cycloids consists of a number of "beads". Argue that the area of one of these beads is equal to the area of the generating circle.

(c) Use the considerations of (a) and (b) to find the area between u and one arch of the cycloid using a method that does not make use of calculus.

Solution. (a) Suppose the circle generating the cycloid rotates from left to right. We consider half the arc of the cycloid joining a point T to a point W on v. Let P be an intermediate point on the cycloid and Q be the point on the generating circle as described in the problem. Suppose that the perpendicular dropped from W to u meets u at Y and the perpendicular dropped from T to v meets v at X. Thus $TXWY$ is a rectangle with $|TX| = |WY| = 2r$ and $|TY| = |XW| = \pi r$, where r is the radius of the generating circle.

Let the generating circle touch u and v at U and V, respectively. Then $|\text{arc } (PU)| = |TU|$, so that

$$|\text{arc } VQ| = |\text{arc } VP| = \pi r - |\text{arc } PU| = \pi r - |TU| = |UY| = |VW| \ .$$

This means that Q is on the circle of radius r rolling to the left generating a second cycloid passing through W, Q, T. This second cycloid is the image of the first under a 180° rotation that interchanges the points T and W.

(b, c) Let α be the area of the region within the rectangle $TXWY$ bounded by the two cycloids (one of the "beads"), β be the area above the cycloid TPW and γ the area below the cycloid TQW within the rectangle. Because the region $TXVWP$ is congruent to the region $WYUTQ$, $\beta = \gamma$. Hence

$$\alpha + 2\beta = \alpha + \beta + \gamma = (2r)(\pi r) = 2\pi r^2 \ .$$

At each vertical height between the lines u and v, the length of the chord PQ of the "bead" is equal to the length of the chord at the same height of the generating circle, so that the "bead" can be regarded as being made of infinitesimal slats of the circle that have been translated. Thus, the "bead" has the same area as the generating circle, namely πr^2 (this is due to a principle enunciated by a seventeenth century mathematician, Cavalieri). Thus $\alpha = \pi r^2$ and $2\beta = 2\pi r^2 - \alpha = \pi r^2$. The area under the cycloid and above TY is equal to $\alpha + \beta$ and the area under a complete arch of the cycloid is $2\alpha + 2\beta = 2\pi r^2 + \pi r^2 = 3\pi r^2$, three times the area of the generating circle.

222. Evaluate

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2}{n^2}\right) .$$

Solution 1. Let $a_n = \tan^{-1} n$ for $n \geq 0$. Thus, $0 < a_n < \pi/2$ and $\tan a_n = n$. Then

$$\tan(a_{n+1} - a_{n-1}) = \frac{(n+1) - (n-1)}{1 + (n^2 - 1)} = \frac{2}{n^2}$$

for $n \geq 1$. Then

$$\sum_{n=1}^{m} \tan^{-1} \frac{2}{n^2} = \tan^{-1}(m+1) + \tan^{-1} m - \tan^{-1} 1 - \tan^{-1} 0 \ .$$

Letting $m \to \infty$ yields the answer $\pi/2 + \pi/2 - \pi/4 - 0 = 3\pi/4$.

Solution 2. Let $b_n = \tan^{-1}(1/n)$ for $n \geq 0$. Then

$$\tan(b_{n-1} - b_{n+1}) = \frac{2}{n^2}$$

for $n \geq 2$, whence

$$\sum_{n=1}^{m} \tan^{-1}\frac{2}{n^2} = \tan^{-1} 2 + \sum_{n=2}^{m}(b_{n-1} - b_{n+1}) = \tan^{-1} 2 + \tan^{-1} 1 + \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{m} - \tan^{-1}\frac{1}{m+1}$$

$$= (\tan^{-1} 2 + \cot^{-1} 2) + \tan^{-1} 1 - \tan^{-1}\frac{1}{m} - \tan^{-1}\frac{1}{m+1}$$

$$= \frac{\pi}{2} + \frac{\pi}{4} - \tan^{-1}\frac{1}{m} - \tan^{-1}\frac{1}{m+1}$$

for $m \geq 3$, from which the result follows by letting m tend to infinity.

Solution 3. [S. Huang] Let $s_n = \sum_{k=1}^{n} \tan^{-1}(2/n^2)$ and $t_n = \tan s_n$. Then $\{t_n\} = \{2, \infty, -9/2, -14/5. -20/9, \cdots\}$ where the numerators of the fractions are $\{-2, -5, -9, -14, -20, \cdots\}$ and the denominators are $\{-1, 0, 2, 5, 8, \cdots\}$. We conjecture that

$$t_n = \frac{-n(n+3)}{(n-2)(n+1)}$$

for $n \geq 1$. This is true for $1 \leq n \leq 5$. Suppose that it holds to $n = k-1 \geq 5$, so that $t_{k-1} = -(k-1)(k+2)/(k-3)k$. Then

$$t_k = \frac{t_{k-1} + (2/k^2)}{1 - 2t_{k-1}k^{-2}}$$

$$= \frac{-k^2(k-1)(k+2) + 2(k-3)k}{k^3(k-3) + 2(k-1)(k+2)}$$

$$= \frac{-k(k+3)(k^2 - 2k + 2)}{(k-2)(k+1)(k^2 - 2k + 2)} = \frac{-k(k+3)}{(k-2)(k+1)} \ .$$

The desired expression for t_n holds by induction and so $\lim_{n\to\infty} t_n = -1$. For $n \geq 3$, $t_n < 0$ and $\tan^{-1}(2/n^2) < \pi/2$, so we must have $\pi/2 < s_n < \pi$ and $s_n = \pi - \tan^{-1} t_n$. Therefore

$$\lim_{n\to\infty} s_n = \tan^{-1}(\pi + \lim_{n\to\infty} t_n) = \pi - (\pi/4) = (3\pi)/4 \ .$$

223. Let a, b, c be positive real numbers for which $a + b + c = abc$. Prove that

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2} \ .$$

Solution 1. Let $a = \tan\alpha$, $b = \tan\beta$, $c = \tan\beta$, where $\alpha, \beta, \gamma \in (0, \pi/2)$. Then

$$\tan(\alpha + \beta + \gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha} = \frac{a + b + c - abc}{1 - ab - bc - ca} = 0 \ ,$$

whence $\alpha + \beta + \gamma = \pi$. Then, the left side of the inequality is equal to

$$\cos\alpha + \cos\beta + \cos\gamma = \cos\alpha + \cos\beta - \cos(\alpha + \beta)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - 2\cos^2\left(\frac{\alpha+\beta}{2}\right) + 1$$

$$\leq 2\cos\left(\frac{\alpha+\beta}{2}\right) - 2\cos^2\left(\frac{\alpha+\beta}{2}\right) + 1$$

$$= 2\sin\left(\frac{\gamma}{2}\right) - 2\sin^2\left(\frac{\gamma}{2}\right) + 1$$

$$= \frac{3}{2} - \frac{1}{2}(2\sin(\gamma/2) - 1)^2 \leq \frac{3}{2} \ ,$$

with equality if and only if $\alpha = \beta = \gamma = \pi/3$.

Solution 2. Define α, β and γ and note that $\alpha + \beta + \gamma = \pi$ as in Solution 1. Since $\cos x$ is a concave function on $[0, \pi/2]$, we have that

$$\frac{\cos\alpha + \cos\beta + \cos\gamma}{3} \leq \cos\left(\frac{\alpha+\beta+\gamma}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2} \ ,$$

from which the result follows.

Solution 3. [G. N. Tai] Define α, β, γ as in Solution 1 and let $s = \cos\alpha + \cos\beta + \cos\gamma$. Then

$$s = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 1 - 2\sin^2\frac{\gamma}{2} = 2\sin\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + 1 - 2\sin^2\frac{\gamma}{2} \ .$$

Thus, for each α. β, the quadratic equation

$$2t^2 - 2\cos\frac{\alpha-\beta}{2}\cdot t + (s-1) = 0$$

has at least one real solution, namely $t = \sin(\gamma/2)$. Hence, its discriminant is positive, so that

$$\cos^2\frac{\alpha-\beta}{2} - 2(s-1) \geq 0 \Longrightarrow 2s \leq 2 + \cos^2\frac{\alpha-\beta}{2} \leq 3 \Longrightarrow s \leq 3/2 \ .$$

Equality occurs if and only if $\alpha = \beta = \gamma = \pi/3$.

224. For $x > 0$, $y > 0$, let $g(x,y)$ denote the minimum of the three quantities, x, $y + 1/x$ and $1/y$. Determine the maximum value of $g(x,y)$ and where this maximum is assumed.

Solution 1. When $(x,y) = (\sqrt{2}, 1/\sqrt{2})$, all three functions $x, y + (1/x), 1/y$ assume the value $\sqrt{2}$ and so $g(\sqrt{2}, 1/\sqrt{2}) = \sqrt{2}$.

If $0 < x \leq \sqrt{2}$, then $g(x,y) \leq x \leq \sqrt{2}$. Suppose that $x \geq \sqrt{2}$. If $y \geq 1/\sqrt{2}$, then $g(x,y) \leq 1/y \leq \sqrt{2}$. If $0 < y \leq 1/\sqrt{2}$, then

$$g(x,y) \leq y + (1/x) \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \ .$$

Thus, when $x > 0$, $y > 0$, then $g(x,y) \leq \sqrt{2}$. If either $x \neq \sqrt{2}$ or $y \neq 1/\sqrt{2}$, then the foregoing inequalities lead to $g(x,y) < \sqrt{2}$. Hence $g(x,y)$ assumes its maximum value of $\sqrt{2}$ if and only if $(x,y) = (\sqrt{2}, 1/\sqrt{2})$.

Solution 2. [M. Abdeh-Kolachi] Let u be the minimum of x, $y + (1/x)$ and $1/y$. Then $u \leq x$, $u \leq 1/y$ and $u \leq y + (1/x)$. By the first two inequalities, we also have than $y + (1/x) \leq (1/u) + (1/u) = 2/u$, so that $u \leq 2/u$ and $u \leq \sqrt{2}$. Hence $g(x,y) \leq \sqrt{2}$ for all $x, y > 0$. Since $g(\sqrt{2}, 1/\sqrt{2}) = \sqrt{2}$, g has a maximum value of $\sqrt{2}$ assumed when $(x,y) = (\sqrt{2}, 1/\sqrt{2})$.

We need to verify that this maximum is assumed nowhere else. Suppose that $g(x,y) = \sqrt{2}$. Then $\sqrt{2} \leq x$, $\sqrt{2} \leq 1/y$ and

$$\sqrt{2} \leq y + (1/x) \leq (1/\sqrt{2}) + (1/\sqrt{2}) = \sqrt{2} \ .$$

We must have equality all across the last inequality and this forces both x and $1/\sqrt{y}$ to equal $\sqrt{2}$.

Solution 3. [R. Appel] If $x \leq 1$ and $y \leq 1$, then $g(x,y) \leq x \leq 1$. If $y \geq 1$, then $g(x,y) \leq 1/y \leq 1$. It remains to examine the case $x > 1$ and $y < 1$, so that $y + (1/x) < 2$. Suppose that $\min\,(x, 1/y) = a$ and $\max\,(x, 1/y) = b$. Then $\min\,(1/x, y) = 1/a$ and $\max\,(1/x, y) = 1/b$, so that

$$y + \frac{1}{x} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} \ .$$

Hence $g(x,y) = \min\ (a, (a+b)/(ab))$. Either $a^2 \le 2$ or $a^2 \ge 2$. But in the latter case,

$$\frac{a+b}{ab} \le \frac{2b}{\sqrt{2}b} = \sqrt{2}\ .$$

In either case, $g(x,y) \le \sqrt{2}$. This maximum value is attained when $(x,y) = (\sqrt{2}, 1/\sqrt{2})$.

Solution 4. [D. Varodayan] By the continuity of the functions, each of the regions $\{(x,y) : 0 < x < y + (1/x), xy < 1\}$, $\{(x,y) : 0 < x, y + (1/x) < x, y + (1/x) < (1/y)\}$, and $\{(x,y) : 0 < (1/y) < x, (1/y) < y + (1/x)\}$ is an open subset of the plane; using partial derivatives, we see that none of the three functions being minimized have any critical values there. It follows that any extreme values of $g(x,y)$ must occur on one of the curves defined by the equations

$$x = y + (1/x) \qquad (1)$$

$$x = 1/y \qquad (2)$$

$$y + (1/x) = (1/y) \qquad (3)$$

On the curve (1), $x > 1$ and

$$\begin{aligned} g(x,y) &= \min\left(x, \frac{x}{x^2-1}\right) \\ &= \begin{cases} x, & \text{if } x \le \sqrt{2}; \\ \frac{x}{x^2-1}, & \text{if } x \ge \sqrt{2}. \end{cases} \end{aligned}$$

On the curve (2),

$$\begin{aligned} g(x,y) &= \min\ (x, 2/x) \\ &= \begin{cases} x, & \text{if } x \le \sqrt{2}; \\ 2/x, & \text{if } x \ge \sqrt{2}. \end{cases} \end{aligned}$$

On the curve (3), $0 < y < 1$ and

$$\begin{aligned} g(x,y) &= \min\left(\frac{y}{1-y^2}, \frac{1}{y}\right) \\ &= \begin{cases} \frac{y}{1-y^2}, & \text{if } 0 < y < \frac{1}{\sqrt{2}}; \\ 1/y, & \text{if } \frac{1}{\sqrt{2}} \le y \le 1. \end{cases} \end{aligned}$$

On each of these curves, $g(x,y)$ reaches its maximum value of $\sqrt{2}$ when $(x,y) = (\sqrt{2}.1/\sqrt{2})$.

Solution 5. [J. Sparling] Let $z = 1/y$. For fixed z, let

$$v_z(x) = \min\ \{x, z, (1/x) + (1/z)\}$$

and

$$w(z) = \max\ \{v_z(x) : x > 0\}\ .$$

Suppose that $z \le 1$. Then $(1/x) + (1/z) \ge z$, so $v_z(x) = \min\ \{x, z\}$ and

$$v_z(x) = \begin{cases} x, & \text{for } x \le z; \\ z, & \text{for } x \ge z; \end{cases}$$

so that $w(z) = z$ when $z \le 1$. Suppose that $1 < z \le \sqrt{2}$, so that $z \le z/(z^2 - 1)$. Then

$$v_z(x) = \begin{cases} x, & \text{for } x \le z; \\ z, & \text{for } z \le x < z/(z^2-1); \\ (1/x) + (1/z), & \text{for } z/(z^2-1) \le x; \end{cases}$$

so that $w(z) = z$ when $1 < z \leq \sqrt{2}$. Finally, suppose that $\sqrt{2} > z$. Note that $x \leq (1/x) + (1/z) \Leftrightarrow zx^2 - x - z \leq 0$. Then the minimum of x and $(1/x) + (1/z)$ is x when $zx^2 - x - z \leq 0$, or $x \leq (1 + \sqrt{1 + 4z^2})/2z$. Since

$$\sqrt{2} - \left[\frac{1 + \sqrt{1 + 4z^2}}{2z}\right] = \frac{(2\sqrt{2}z - 1) - \sqrt{1 + 4z^2}}{2z}$$
$$= \frac{4z^2 - 4\sqrt{2}z}{2z[(2\sqrt{2}z - 1) + \sqrt{1 + 4z^2}]}$$
$$= \frac{2(z - \sqrt{2})}{(2\sqrt{2}z - 1) + \sqrt{1 + 4z^2}} \geq 0 \ ,$$

this minimum is always less than z, so that

$$v_z(x) = \begin{cases} x, & \text{for } x \leq \frac{1+\sqrt{1+4z^2}}{2z} \\ \frac{1}{x} + \frac{1}{z}, & \text{for } x \geq \frac{1+\sqrt{1+4z^2}}{2z}, \end{cases}$$

so that $w(z) = (1 + \sqrt{1 + 4z^2})/(2z) \leq \sqrt{2}$ when $\sqrt{2} \leq z$. Hence, the minimum value of $w(z) = \sqrt{2}$ and this is the maximum value of $g(x, y)$, assumed when $(x, y) = (\sqrt{2}, 1/\sqrt{2})$.

Solution 6. For $x > 0$, let

$$h_x(y) = \min\left(x, y + \frac{1}{x}, \frac{1}{y}\right) .$$

Suppose that $x \leq \sqrt{2}$. Then $x - (1/x) \leq (1/x)$ and

$$h_x(y) = \begin{cases} y + \frac{1}{x}, & \text{if } 0 < y \leq x - \frac{1}{x}; \\ x, & \text{if } x - \frac{1}{x} \leq y \leq \frac{1}{x}; \\ \frac{1}{y}, & \text{if } \frac{1}{x} \leq y; \end{cases}$$

so that the minimum value of $h_x(y)$ is x, and this occurs when $x - (1/x) \leq y \leq (1/x)$. Suppose that $x \geq \sqrt{2}$. Then $y + (1/x) \leq (1/y) \Leftrightarrow xy^2 + y - x \leq 0$ and

$$\sqrt{2} - \left[\frac{1 + \sqrt{1 + 4x^2}}{2x}\right] = \frac{(\sqrt{8}x - 1) - \sqrt{1 + 4x^2}}{2x}$$
$$= \frac{4x^2 - 4\sqrt{2}x}{2x[(\sqrt{8}x - 1) + \sqrt{1 + 4x^2})}$$
$$= \frac{2(x - \sqrt{2})}{(\sqrt{8}x - 1) + \sqrt{1 + 4x^2}} \geq 0 \ ,$$

so that

$$\frac{1 + \sqrt{1 + 4x^2}}{2x} \leq \sqrt{2} \leq x \ .$$

$$h_x(y) = \begin{cases} y + \frac{1}{x}, & \text{when } 0 < y \leq \frac{-1+\sqrt{1+4x^2}}{2x}; \\ \frac{1}{y}, & \text{when } \frac{-1+\sqrt{1+4x^2}}{2x} \leq y; \end{cases}$$

so that the minimum value of $h_x(y)$ is $(1 + \sqrt{1 + 4x^2})/(2x)$, and this occurs when $y = (-1 + \sqrt{1 + 4x^2})/(2x)$.

Thus, we have to maximize the function $u(x)$ where

$$u(x) = \begin{cases} x, & \text{if } 0 < x \leq \sqrt{2}; \\ \frac{1+\sqrt{1+4x^2}}{2x}, & \text{if } \sqrt{2} \leq x. \end{cases}$$

By what we have shown, this maximum is $\sqrt{2}$ and is attained when $x = \sqrt{2}$. The result follows.

225. A set of n lighbulbs, each with an *on-off* switch, numbered $1, 2, \cdots, n$ are arranged in a line. All are initially off. Switch 1 can be operated at any time to turn its bulb on of off. Switch 2 can turn bulb 2 on or off if and only if bulb 1 is off; otherwise, it does not function. For $k \geq 3$, switch k can turn bulb k on or off if and only if bulb $k-1$ is off and bulbs $1, 2, \cdots, k-2$ are all on; otherwise it does not function.

(a) Prove that there is an algorithm that will turn all of the bulbs on.

(b) If x_n is the length of the shortest algorithm that will turn on all n bulbs when they are initially off, determine the largest prime divisor of $3x_n + 1$ when n is odd.

Solution. (a) Clearly $x_1 = 1$ and $x_2 = 2$. Let $n \geq 3$. The only way that bulb n can be turned on is for bulb $n-1$ to be off and for bulbs $1, 2, \cdots, n-2$ to be turned on. Once bulb n is turned on, then we need get bulb $n-1$ turned on. The only way to do this is to turn off bulb $n-2$; but for switch $n-2$ to work, we need to have bulb $n-3$ turned off. So before we can think about dealing with bulb $n-1$, we need to get the first $n-2$ bulbs turned off. Then we will be in the same situation as the outset with $n-1$ rather than n bulbs. Thus the process has the following steps: (1) Turn on bulbs $1, \cdots, n-2$; (2) Turn on bulb n; (3) Turn off bulbs $n-2, \cdots, 1$; (3) Turn on bulbs $1, 2, \cdots, n$. So if, for each positive integer k, y_k is the length of the shortest algorithm to turn them off after all are lit, then

$$x_n = x_{n-2} + 1 + y_{n-2} + x_{n-1} \ .$$

We show that $x_n = y_n$ for $n = 1, 2, \cdots$. Suppose that we have an algorithm that turns all the bulbs on. We prove by induction that at each step we can legitimately reverse the whole sequence to get all the bulbs off again. Clearly, the first step is to turn either bulb 1 or bulb 2 on; since the switch is functioning, we can turn the bulb off again. Suppose that we can reverse the first $k-1$ steps and are at the kth step. Then the switch that operates the bulb at that step is functioning and can restore us to the situation at the end of the $(k-1)$th step. By the induction hypothesis, we can go back to having all the bulbs off. Hence, given the bulbs all on, we can reverse the steps of the algorithm to get the bulbs off again. A similar argument allows us to reverse the algorithm that turns the bulbs off. Thus, for each turning-on algorithm there is a turning-off algorithm of equal length, and vice versa. Thus $x_n = y_n$.

We have that $x_n = x_{n-1} + 2x_{n-2} + 1$ for $n \geq 3$. By, induction, we show that, for $m = 1, 2, \cdots$,

$$x_{2m} = 2x_{2m-1} \qquad \text{and} \qquad x_{2m+1} = 2x_{2m} + 1 = 4x_{2m-1} + 1 \ .$$

This is true for $m = 1$. Suppose it is true for $m \geq 1$. Then

$$\begin{aligned} x_{2(m+1)} &= x_{2m+1} + 2x_{2m} + 1 = 2(x_{2m} + 1) + 4x_{2m-1} \\ &= 2(x_{2m} + 2x_{2m-1} + 1) = 2x_{2m+1} \ , \end{aligned}$$

and

$$\begin{aligned} x_{2(m+1)+1} &= x_{2(m+1)} + 2x_{2m+1} + 1 = 2x_{2m+1} + 4x_{2m} + 3 \\ &= 2(x_{2m+1} + 2x_{2m} + 1) + 1 = 2x_{2(m+1)} + 1 \ . \end{aligned}$$

Hence, for $m \geq 1$,

$$3x_{2m+1} + 1 = 4(3x_{2m-1} + 1) = \cdots = 4^m(3x_1 + 1) = 4^{m+1} = 2^{2(m+1)} \ .$$

Thus, the largest prime divisor is 2.

226. Suppose that the polynomial $f(x)$ of degree $n \geq 1$ has all real roots and that $\lambda > 0$. Prove that the set $\{x \in \mathbf{R} : |f(x)| \leq \lambda |f'(x)|\}$ is a finite union of closed intervals whose total length is equal to $2n\lambda$.

Solution. Wolog, we may assume that the leading coefficient is 1. Let $f(x) = \prod_{i=1}^k (x - r_i)^{m_i}$, where $n = \sum_{i=1}^k m_i$. Then

$$\frac{f'(x)}{f(x)} = \sum_{i=1}^k \frac{m_i}{x - r_i} \ .$$

Note that the derivative of this function, $-\sum_{i=1}^{k} m_i(x-r_i)^{-2} < 0$, so that it decreases on each interval upon which it is defined. By considering the graph of $f'(x)/f(x)$, we see that $f'(x)/f(x) \geq 1/\lambda$ on finitely many intervals of the form $(r_i, s_i]$, where $r_i < s_i$ and the r_i and s_j interlace, and $f'(x)/f(x) \leq -1/\lambda$ on finitely many intervals of the form $[t_i, r_i)$, where $t_i < r_i$ and the t_i and r_j interlace. For each i, we have $t_i < r_i < s_i < t_{i+1}$.

The equation $f'(x)/f(x) = 1/\lambda$ can be rewritten as

$$\begin{aligned} 0 =&(x-r_1)(x-r_2)\cdots(x-r_k) - \lambda\sum_{i=1}^{k} m_i(x-r_1)\cdots\widehat{(x-r_i)}\cdots(x-r_k) \\ =& x^k - \left(\sum_{i=1}^{k} r_i - \lambda\sum_{i=1}^{k} m_i\right)x^{k-1} + \cdots \ . \end{aligned}$$

(The "hat" indicates that the term in the product is deleted.) The sum of the roots of this polynomial is

$$s_1 + s_2 + \cdots + s_k = r_1 + \cdots + r_k - \lambda n \ ,$$

so that $\sum_{i=1}^{m}(s_i - r_i) = \lambda n$. This is the sum of the lengths of the intervals $(r_i, s_i]$ on which $f'(x)/f(x) \geq 1/\lambda$. Similarly, we can show that $f'(x)/f(x) \leq -1/\lambda$ on a finite collection of intervals of total length λn. The set on which the inequality of the problem holds is equal to the union of all of these half-open intervals and the set $\{r_1, r_2, \cdots, r_k\}$. The result follows.

227. Let n be an integer exceeding 2 and let $a_0, a_1, a_2, \cdots, a_n, a_{n+1}$ be positive real numbers for which $a_0 = a_n$, $a_1 = a_{n+1}$ and

$$a_{i-1} + a_{i+1} = k_i a_i$$

for some positive integers k_i, where $1 \leq i \leq n$.

Prove that

$$2n \leq k_1 + k_2 + \cdots + k_n \leq 3n \ .$$

Solution. Since $k_i = (a_{i-1}/a_i) + (a_{i+1}/a_i)$ for each i,

$$\sum_{i=1}^{n} k_i = \sum_{i=1}^{n}\left(\frac{a_{i+1}}{a_i} + \frac{a_i}{a_{i+1}}\right) \geq \sum_{i=1}^{n} 2 = 2n \ .$$

As for the other inequality, since the expression has cyclic symmetry, there is no loss in generality in supposing that $a_n \geq a_1$ and $a_n \geq a_{n-1}$ with inequality in at least one case, so that $2a_n > a_{n-1} + a_1$. Therefore, $k_n = 1$ and $a_n = a_{n-1} + a_1$.

We establish the right inequality by induction. For the case $n = 3$, we may suppose that

$$a_2 + a_3 = k_1 a_1 \ ; \quad a_1 + a_3 = k_2 a_2 \ ; \quad a_1 + a_2 = a_3 \ .$$

Substituting for a_3 and rearranging the terms yields the brace of equations

$$2a_2 = (k_1 - 1)a_1 \qquad 2a_1 = (k_2 - 1)a_2$$

whence $4 = (k_1 - 1)(k_2 - 1)$. It follows that $k_1 + k_2 + k_3$ is either $5 + 2 + 1 = 8$ or $3 + 3 + 1 = 7$.

Now suppose the result holds when the index is $n - 1 \geq 3$. Then, supposing that $k_n = 1$ and substituting for a_n, we obtain the $n - 1$ equations

$$a_{n-1} + a_2 = (k_1 - 1)a_1$$

$$a_1 + a_3 = k_2 a_2$$

$$\cdots$$

$$a_{n-3} + a_{n-1} = k_{n-2} a_{n-2}$$

$$a_{n-2} + a_1 = (k_{n-1} - 1) a_{n-1} \ .$$

By the induction hypothesis

$$(k_1 - 1) + k_2 + \cdots + (k_{n-1} - 1) \le 3(n-1) = 3n - 3$$

whence

$$k_1 + k_2 + \cdots + k_n \le (3n - 3) + 2 + 1 = 3n \ .$$

228. Prove that, if $1 < a < b < c$, then

$$\log_a(\log_a b) + \log_b(\log_b c) + \log_c(\log_c a) > 0 \ .$$

Solution. Since $1 < a < b < c$, $\log_a b > 1$, so that

$$\log_a(\log_a b) = \log_a b \cdot \log_b(\log_a b) > \log_b(\log_a b) \ .$$

Also

$$0 < \log_c a = \log_c b \cdot \log_b a < \log_c b < 1 \ ,$$

so that $\log_b(\log_c a) < 0$ and

$$\log_c(\log_c a) = \log_c b \cdot \log_b(\log_c a) > \log_b(\log_c a) \ .$$

Hence,

$$\begin{aligned} &\log_a(\log_a b) + \log_b(\log_b c) + \log_c(\log_c a) \\ &\qquad > \log_b(\log_a b) + \log_b(\log_b c) + \log_b(\log_c a) \\ &\qquad = \log_b(\log_a b \cdot \log_b c \cdot \log_c a) = \log_b 1 = 0 \ . \end{aligned}$$

Comment. As an exercise, you should justify the following fundamental facts about change of basis, beginning with the definition, $\log_p q = r$ iff $p^r = q$ where $0 < p, q$ and $p \neq 1$: (1) $\log_u v \cdot \log_v w = \log_u w$; (2) $\log_u v = 1/(\log_v u)$.

229. Suppose that n is a positive integer and that $0 < i < j < n$. Prove that the greatest common divisor of $\binom{n}{i}$ and $\binom{n}{j}$ exceeds 1.

First solution. Since $\binom{n}{k} = \binom{n}{n-k}$ for $1 \le k \le n-1$, it suffices to prove the result when $0 < i < j \le n/2$, so that $i + j \le n$. Observe that

$$\binom{n}{i} = \left(\frac{n}{n-i}\right)\left(\frac{n-1}{n-1-i}\right)\cdots\left(\frac{n-j+1}{n-j+1-i}\right)\binom{n-j}{i}$$

so that $\binom{n}{i} > \binom{n-j}{i}$, and that

$$\binom{n}{i}\binom{n-i}{j} = \frac{n!}{i!j!(n-i-j)!} = \binom{n}{j}\binom{n-j}{i} \ .$$

Suppose, if possible, that $\binom{n}{i}$ and $\binom{n}{j}$ are coprime. Then, since $\binom{n}{i}$ divides the product of $\binom{n}{j}$ and $\binom{n-j}{i}$, $\binom{n}{i}$ must divide $\binom{n-j}{i}$. But this is impossible, since $\binom{n-j}{i} < \binom{n}{i}$.

Second solution. Observe that, for $1 \le i < j \le n-1$,

$$\binom{n}{i} = \frac{n(n-1)\cdots(j+1)}{(n-i)\cdots(j-i+1)}\binom{j}{i} > \binom{j}{i}$$

and

$$\binom{n}{i}\binom{n-i}{j-i} = \frac{n!}{i!(j-i)!(n-j)!} = \binom{n}{j}\binom{j}{i} \ .$$

If $\binom{n}{i}$ and $\binom{n}{j}$ were coprime, then $\binom{n}{i}$ would divide the smaller $\binom{j}{i}$, an impossibility.

230. Let f be a strictly increasing function on the closed interval $[0,1]$ for which $f(0) = 0$ and $f(1) = 1$. Let g be its inverse. Prove that

$$\sum_{k=1}^{9}\left(f\left(\frac{k}{10}\right) + g\left(\frac{k}{10}\right)\right) \le 9.9 \ .$$

Solution. Observe that $x = g(y)$ and $y = f(x)$ determine the same curve. Sketch a diagram that includes the graph of $y = f(x)$ and the rectangles with vertices $(k/10, f(k/10))$, $(k/10, 0)$, $((k+1)/10, 0)$, $((k+1)/10, f(k/10)$ and areas $(1/10)f(k/10)$, for $1 \le k \le 0$. The area under the graph of $y = f(x)$ and the $x-$axis for $1/10 \le x \le 1$ is at least $(1/10)\sum_{k=1}^{9} f(k/10)$.

Similarly, the area between the graph of $x = g(y)$ and the $y-$axis for $1/10 \le y \le 1$ is at least $(1/10)\sum_{k=1}^{9} g(k/10)$. Since both these regions do not overlap the square with side $1/10$ and opposite vertices at $(0,0)$ and $(1/10, 1/10)$, we must have

$$\frac{1}{100} + \frac{1}{10}\sum_{k=1}^{9}\left(f\left(\frac{k}{10}\right) + g\left(\frac{k}{10}\right)\right) \le 1$$

from which the result follows.

231. For $n \ge 10$, let $g(n)$ be defined as follows: n is mapped by g to the sum of the number formed by taking all but the last three digits of its square and adding it to the number formed by the last three digits of its square. For example, $g(54) = 918$ since $54^2 = 2916$ and $2 + 916 = 918$. Is it possible to start with 527 and, through repeated applications of g, arrive at 605?

Solution. Suppose $n \ge 1000$. Then $g(n) \ge \lfloor n^2/1000 \rfloor \ge 1000$. Since $g(527) = 1006$, the result of each subsequent repeated application of g also exceeds 1000 and so can never be 605.

232. (a) Prove that, for positive integers n and positive values of x,

$$(1 + x^{n+1})^n \le (1 + x^n)^{n+1} \le 2(1 + x^{n+1})^n \ .$$

(b) Let $h(x)$ be the function defined by

$$h(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1; \\ x, & \text{if } x > 1. \end{cases}$$

Determine a value N for which

$$|h(x) - (1 + x^n)^{\frac{1}{n}}| < 10^{-6}$$

whenever $0 \le x \le 10$ and $n \ge N$.

Solution. Recall the power-mean inequality

$$\left(\frac{a^n + b^n}{2}\right)^{1/n} \le \left(\frac{a^{n+1} + b^{n+1}}{2}\right)^{1/(n+1)}$$

for n a positive integer and $a, b > 0$. Applying this to $(a, b) = (1, x)$ yields

$$(1 + x^n)^{n+1} \le 2(1 + x^{n+1})^n \ .$$

If $0 \le x \le 1$, then $x^{n+1} \le x^n$ and

$$(1 + x^{n+1})^n \le (1 + x^n)^n \le (1 + x^n)^{n+1} \ .$$

Let $1 \le x$. Then

$$\left(1 + \frac{1}{x^{n+1}}\right)^n \le \left(1 + \frac{1}{x^n}\right)^n \ .$$

Multiplying by $x^{n(n+1)}$ yields $(x^{n+1} + 1)^n \le (x^n + 1)^{n+1}$, as desired.

(b) Let $0 \le x \le 1$. Then, for each positive integer n, $|1 - (1 + x^n)^{1/n}| = (1 + x^n)^{1/n} - 1 \le 2^{1/n} - 1$. Now let $1 \le x \le 10$. Then, for each positive integer n,

$$|x - (1 + x^n)^{1/n}| = (1 + x^n)^{1/n} - x \le (2x^n)^{1/n} - x = x[2^{1/n} - 1] \le 10(2^{1/n} - 1) \ .$$

It follows that, for $0 \le x \le 10$ and each positive integer n,

$$|h(x) - (1 + x^n)^{1/n}| \le 10(2^{1/n} - 1) \ .$$

Suppose that N is an integer that exceeds $1/\log_2(1 + 10^{-7})$. (N could be 3×10^7 for example.) Then

$$\begin{aligned} n \ge N &\Longrightarrow \frac{1}{n} \le \frac{1}{N} < \log_2(1 + 10^{-7}) \\ &\Longrightarrow 2^{1/n} < (1 + 10^{-7}) \Longrightarrow 10(2^{1/n} - 1) < 10^{-6} \\ &\Longrightarrow |h(x) - (1 + x^n)^{1/n}| < 10^{-6} \end{aligned}$$

for $0 \le x \le 10$.

Comments. The (b) part of this question was badly handled, and solvers did not make the logic of the situation clear. This is a situation, where one works backwards to determine what a suitable value of N might be. Unfortunately, this working backwards involves starting with the desired result, and so the implications are in reverse. For problems of this type, the solution **must** be re-edited to put it into the proper logical form: start with what is given; proceed by justified logical steps to what is desired. The appropriate final form of the solution thus should be: "Let N be equal to $\cdots$. Then (following a sequence of manipulations), $|h(x) - \cdots|$&c." Note that in the above solution, we do a little initial spadework to get an upper bound independent of x for the difference. Having gotten the upper bound, we then define a suitable value of N. The final part of the solution then shows that N does the job, using material that is already known to be true. Those of you who will be studying mathematics at university will undoubtedly in their initial analysis course encounter $\epsilon - \delta$ arguments, which are notoriously difficult for many students to grasp. The present solution is such an argument for a particular value of ϵ, so an attempt to really understand the logical structure at this point will pay dividends for you later on.

There are other ways of establishing (a). For example, when $0 \le x \le 1$,

$$\left(\frac{1 + x^{n+1}}{1 + x^n}\right)^n \le 1 \le 1 + x^n$$

while, if $1 \le x$,

$$\begin{aligned} (x^n + 1)^{n+1} - (x^{n+1} + 1)^n &= \sum_{r=0}^{n} \left[\binom{n+1}{r} x^{n(n+1-r)} - \binom{n}{r} x^{(n+1)(n-r)} \right] + 1 \\ &= \sum_{r=0}^{n} \left[\binom{n+1}{r} x^r - \binom{n}{r} \right] x^{(n+1)(n-r)} + 1 \ge 1 \ . \end{aligned}$$

One could use the Arithmetic-Geometric Means Inequality to obtain

$$(1+x^{n+1})^2 \le (1+x^n)(1+x^{n+2}) \Rightarrow (1+x^{n+1})^{2(n+1)} \le (1+x^n)^{n+1}(1+x^{n+2})^{n+1}$$
$$\Rightarrow \frac{(1+x^{n+1})^{n+2}}{(1+x^{n+2})^{n+1}} \le \frac{(1+x^n)^{n+1}}{(1+x^{n+1})^n}$$

for each positive integer n. When $n = 1$, we have

$$\frac{(1+x)^2}{1+x^2} = 1 + \frac{2x}{1+x^2} \le 2$$

from which one of the inequalities follows.

233. Let $p(x)$ be a polynomial of degree 4 with rational coefficients for which the equation $p(x) = 0$ has *exactly one* real solution. Prove that this solution is rational.

Solution. Suppose that $p(x) = x^4 + tx^3 + ux^2 + vx + w$, where t, u, v, w are all rational. (There is no loss of generality in supposing that the leading coefficient is 1.) Since $p(x) = 0$ has exactly one real solution r and since nonreal solutions come in pairs, there are two possibilities: (a) $p(x) = (x-r)^4$, in which case $r = -t/4$ is rational, or (b) r is a double root and $p(x) = (x-r)^2(x^2+bx+c) = x^4 + (b-2r)x^3 + (c-2br+r^2)x^2 + (br^2-2cr)x + cr^2$. Then $t = b - 2r$, $u = c - 2br + r^2$, $v = br^2 - 2cr$ and $w = cr^2$.

We find that $4r^3 + 3tr^2 + 2ur + v = 0$ (by manipulating the values for t, &c, to eliminate b and c), so that r is a root of the cubic polynomial

$$q(x) = 4x^3 + 3tx^2 + 2ux + v$$

with rational coefficients. Hence, r is a root of the quadratic

$$f(x) \equiv 16p(x) - (4x+t)q(x) = (8u-3t^2)x^2 + (12v-2ut)x + (16w-vt)$$

with rational coefficients. (Use long division to divide $q(x)$ into $p(x)$.) It is not possible for all coefficients of $f(x)$ to vanish, for this would imply that $u = (3/8)t^2$, $v = (3/48)t^3$, $w = (1/4^4)t^4$ and $p(x) = (x + (t/4))^4$, nor can f be a nonzero constant. If $8u - 3t^2 = 0$, then $r = -(16w-vt)/(12v-2ut)$ is rational. If $8u-3t^2 \ne 0$, then we can divide $q(x)$ by $p(x)$ to get a relation $q(x) = f(x)g(x) + h(x)$, where $h(x)$ is a linear polynomial with rational coefficients and the root r. In this case, also, r is rational.

Comment. If you have knowledge of calculus, then you can note that $p(x) = x^4 + tx^3 + ux^2 + vx + w = (x-r)^2(x^2+bx+c)$ implies that

$$\begin{aligned} q(x) = p'(x) &= 4x^3 + 3tx^2 + 2ux + v \\ &= 2(x-r)(x^2+bx+c) + (x-r)^2(2x+b) \\ &= (x-r)[2(x^2+bx+c) + (x-r)(2x+b)] \end{aligned}$$

so that both $p(x)$ and $q(x)$ have root r, We can proceed

234. A square of side length 100 is divided into 10000 smaller unit squares. Two squares sharing a common side are called *neighbours*.

(a) Is it possible to colour an even number of squares so that each coloured square has an even number of coloured neighbours?

(b) Is it possible to colour an odd number of squares so that each coloured square has an odd number of coloured neighbours?

Solution. [Y. Zhao] (a) Yes, it is possible in many ways to perform the task. For example, colour any two nonadjacent squares, and both of them will have zero coloured neighbours. So there are evenly many (2) coloured squares, each with an even number (0) of coloured neighbours.

(b) Suppose, if possible, we could colour an odd number of squares so that each has an odd number of coloured neighbours. Let us count the number of segments or edges that connect two coloured neighbours. Since for each coloured square there is an odd number of coloured neighbours, then the total number of their common sides is the sum of an odd number of odd terms, and so is odd. However, two coloured neighbours share each of these common edges, therefore each coloured neighbour is counted twice in the sum; thus, the sum should be even. This is a contradiction. So, it is impossible to colour an odd number of squares so that each has an odd number of coloured neighbours.

235. Find all positive integers, N, for which:

(i) N has exactly sixteen positive divisors: $1 = d_1 < d_2 < \cdots < d_{16} = N$;

(ii) the divisor with the *index* d_5 (namely, d_{d_5}) is equal to $(d_2 + d_4) \times d_6$ (the product of the two).

Solution. There are some preliminary easy observations:

(1) Since N has exactly sixteen positive divisors and d_5 is an index, $d_5 \leq 16$. On the other hand, d_6 is a proper divisor of d_{d_5}, so $d_6 \leq d_{d_5}$. Thus $6 < d_5 \leq 16$.

(2) If N were odd, all its factors would be odd. But, by (ii), the factor d_{d_5} would be the product of an even and an odd number, and so be even. But this would given N an even divisor and lead to a contradiction.

(3) Recall that, if $N = \prod p_i^{k_i}$ is the prime factor decomposition, then the number of all divisors, including 1 and N is $\prod(1 + k_i)$. [To understand this formula, think how we can form any of the divisors of N; we have to choose its prime factors, each to any of the possible exponents. For an arbitrary prime factor p_i there are $(1 + k_i)$ possibility for the exponent (from 0 to k_i inclusive). In particular, the factor 1 corresponds to taking all exponents 0, and N to taking all exponents to be the maximum k_i.] It can be checked that there are five cases for the prime factorization of N; (i) $N = p^{15}$, $N = p_1^7 p_2$; (iii) $N = p_1^3 p_2 p_3$; (iv) $N = p_1^3 p_2^3$; (v) $N = p_1 p_2 p_3 p_4$.

We now put all of this together, and follow the solution of K.-C. R. Tseng. From (1), $d_2 = 2$.

If d_4 is composite (*i.e.* not a square), then $d_4 = 2d_3$ is even. Since $d_2 + d_4$ divides a factor d_{d_5} of N, it divides N. Since $d_2 + d_4 = 2(1 + d_3)$, $1 + d_3$ divides N. But then $1 + d_3$ would equal $d_4 = 2d_3$, which is impossible. If d_4 were a perfect square, then it must equal either 4 or 9 (since $d_4 < d_5 \leq 16$). In either case, $d_3 = 3$, and 6 must be one of the factors. This excludes the possibility that $d_4 = 9$, since 6 should preceded 9 in the list of divisors. On the other hand, if $d_4 = 4$, then d_5 must be equal to either 5 or 6, which is not possible by (1).

Hence, d_4 must be a prime number, and so one of 3, 5, 7, 11, 13. Since $d_3 \geq 3$, $d_4 \neq 3$.

Suppose that $d_4 = 5$. Then $d_2 + d_4 = 7$ must divide N. Thus d_5 or d_6 must be 7. If $d_5 = 7$, then $d_3 \neq 3$, for otherwise 6 would be a factor between d_4 and d_5. But then $d_3 = 4$, so that $N = 2^2 \cdot 5 \cdot 7 \cdot K$ where K is a natural number. But N must have 16 divisors, and the only way to obtain this is to have 2^3 rather than 2^2 in the factorization. Thus, $d_6 = 8$ and $d_7 = 10$. But then $d_{d_5} = d_7 \neq (d_2 + d_4)d_6$. So $d_5 = 7$ is rejected and we must have $d_6 = 7$. This entails that $d_5 = 6$. But this denies the equality of $d_6 = d_{d_5}$ and $(d_2 + d_4)d_6$. We conclude that $d_4 \neq 5$.

Suppose that $d_4 = 7$. Then $d_2 + d_4 = 9$ is a factor of N, so $d_3 = 3$. Then 6 must be a factor of N; but there is not room for 6, and this case is impossible.

Suppose that $d_4 = 11$. Then $d_2 + d_4 = 13$ divides N, and is either d_5 (when 12 is not a factor) or d_6 (when 12 is a factor). If $d_5 = 13$, then d_3 is either a prime number less than 11 or 4. It cannot be 3, as there is no room to fit the divisor 6. If $d_3 = 4$, then $N = 2^2 \cdot 11 \cdot 13 \cdot K$ and the only way to get 16 divisors is for the exponent of 2 to be 3. Thus, 8 divides N, but there is no room for this divisor. Similarly, if $d_5 = 5$, there is no room for 10.

Finally (with $d_4 = 11, d_5 = 13$), if $d_3 = 7$, we already have four prime divisor of N, and this forces $N = 2 \cdot 7 \cdot 11 \cdot 13 = 2002$. We have that the divisors in increasing order are 1, 2, 7, 11, 13, 14, 22, 26, 77, 91, 143, 154, 182, 286, 1001, 2002, and all the conditions are satified.

When $d_4 = 11$, $d_6 = 13$, then $d_5 = 12$, so that 3, 4, 6 are all factors of N; but there is no room for them between d_2 and d_4.

The remaining case is that $d_4 = 13$, which makes $d_2 + d_4 = 15$ a factor of N; but there is no room for both 3 and 5 between d_2 and d_4. We conclude that $N = 2002$ is the only possibility.

236. For any positive real numbers a, b, c, prove that

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2} \ .$$

Solution. [G.N. Tai] Apply the AM-GM Inequality to get

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq 3\sqrt[3]{\frac{1}{abc(a+b)(b+c)(c+a)}}$$

$$a + b + c \geq 3\sqrt[3]{abc}$$

$$a + b + c = \frac{1}{2}((a+b) + (b+c) + (c+a)) \geq \frac{3}{2}\sqrt[3]{(a+b)(b+c)(c+a)} \ .$$

Multiplying these inequalities together and dividing by $(a+b+c)^2$ yields the result. Equality occurs if and only if $a = b = c$.

237. The sequence $\{a_n : n = 1, 2, \cdots\}$ is defined by the recursion

$$a_1 = 20 \qquad\qquad a_2 = 30$$

$$a_{n+2} = 3a_{n+1} - a_n \qquad \text{for} \ \ n \geq 1 \ .$$

Find all natural numbers n for which $1 + 5a_n a_{n+1}$ is a perfect square.

Solution. [R. Marinov] The first few terms of the sequence are 20, 30, 70, 180, 470, 1230. Observe that

$$0 = (a_{n+1} - a_{n-1})(a_{n+1} + a_{n-1} - 3a_n) \Leftrightarrow a_{n+1}^2 - 3a_{n+1}a_n = a_{n-1}^2 - 3a_n a_{n-1}$$

so that

$$a_{n+1}^2 - 3a_n a_{n+1} + a_n^2 = a_n^2 - 3a_{n-1}a_n + a_{n-1}^2$$

for $n \geq 2$. Hence $a_{n+1}^2 - 3a_{n+1}a_n + a_n^2$ is a constant for $N \geq 2$, and its value is $30^2 - 2 \cdot 30 \cdot 20 + 20^2 = -500$.

Now, $1 + 5a_n a_{n+1} = 501 - 500 + 5a_n a_{n+1} = 501 + (a_{n+1} + a_n)^2$ for each $n \geq 1$. Since $1 + 5a_n a_{n+1} = k^2$ is equivalent to

$$3 \times 167 = 501 = (k - (a_{n+1} + a_n))(k + (a_{n+1} - a_n) \ ,$$

we must have that either (i) $A - (a_{n+1} + a_n) = 1$ and $A + (a_{n+1} + a_n) = 501$ or (ii) $A - (a_{n+1} + a_n) = 3$ and $A + (a_{n+1} + a_n) = 167$. The second possibility leads to $a_{n+1} + a_n = 82$ which is not divisible by 10 and so cannot occur. The first possibility leads to $a_{n+1} + a_n = 250$, which occurs when $n = 3$. Since the sequence is increasing (prove this!), this is the only possibility.

238. Let ABC be an acute-angled triangle, and let M be a point on the side AC and N a point on the side BC. The circumcircles of triangles CAN and BCM intersect at the two points C and D. Prove that the line CD passes through the circumcentre of triangle ABC if and only if the right bisector of AB passes through the midpoint of MN.

Solution. Denote the circumcentres of the triangles ABC, ANC and BMN by O, O_1 and O_2 respectively. Denote also their circumcircles by $\mathfrak{K}$, $\mathfrak{K}_1$ and $\mathfrak{K}_2$ respectively, and the radii of these circles by R, R_1 and R_2 respectively. The common chord CD of $\mathfrak{K}_1$ and $\mathfrak{K}_2$ is perpendicular to O_1O_2. Thus, $O \in CD \Longleftrightarrow CO \perp O_1O_2$.

We prove two lemmata.

`Lemma 1.` Let M_1 be the perpendicular projection of the point M onto AB and N_1 the projection of the point N onto AB. The right bisector of AB, the line S_{AB}, passes through the midpoint of MN if and only if $AN_1 = BM_1$.

`Proof.` Note that MM_1N_1N is a trapezoid with bases parallel to S_{AB}. Recall that the midline of a trapezoid has the following property: *the segment that connects the midpoints of the two nonparallel sides is parallel to the bases and its length is the average of the lengths of the two parallel sides.* As a direct consequence, *a line passing through one of the midpoints of the two nonparallel sides and is parallel to the bases must pass through the midpoint of the other side.* Applying this yields that S_{AB} passes through the midpoint of MN if and only if S_{AB} passes through the midpoint of M_1N_1. Since S_{AB} intersects AB at its midpoint, this is equivalent to S_{AB} passes through the midpoint of $M_1N_1 \Leftrightarrow AB$ and M_1N_1 have the same midpoint, which is equivalent to $AM_1 = BN_1$ or $AN_1 = BM_1$ ♠.

`Lemma 2.` The diagonals d_1 and d_2 of the quadrilateral $PQRS$ are perpendicular if and only if its sides a, b, c, d satisfy the relationship $a^2 + c^2 = b^2 + d^2$. ((a, c) and (b, d) are pairs of opposite sides.)

`Proof.` (To follow the steps of the proof, please draw an arbitrary convex quadrilateral $PQRS$ with the respective lengths of SR, RQ, QP and PS given by a, b, c and d.) Let d_1 and d_2 intersect at I, and let

$$\angle PIQ = \theta \ , \quad |IP| = t \ , \quad |IQ| = z \ , \quad |IR| = y \ , \quad |IS| = x \ .$$

The Law of Cosines applied to triangles PQI, QRI, RSI and SPI yields

$$a^2 = x^2 + y^2 - 2xy\cos\theta$$

$$c^2 = z^2 + t^2 - 2zt\cos\theta$$

$$b^2 = y^2 + z^2 + 2yz\cos\theta$$

$$d^2 = x^2 + t^2 + 2xt\cos\theta \ .$$

As $a^2 + c^2 = b^2 + d^2$ is equivalent to $(xy + zt + yz + xt)\cos\theta = 0$, or $\cos\theta = 0$, the result follows. ♠

Let us return to the problem. Consider (in figure 1) the quadrilateral CO_1OO_2. We already know from the foregoing that

- CD passes through $O \Leftrightarrow CO \perp O_1O_2$;
- $CO \perp O_1O_2 \Leftrightarrow O_1C^2 + OO_2^2 = O_2C^2 + OO_1^2$;
- $AN_1 = BM_1 \Leftrightarrow S_{AB}$ passes through the midpoint of MN.

So to complete the solution, it is necessary to prove that

$$O_1C^2 + OO_2^2 = O_2C^2 + OO_1^2 \Longleftrightarrow AN_1 = BM_1 \ .$$

From the Law of Cosines,

$$OO_1^2 = O_1C^2 + OC^2 - 2O_1C \cdot OC \cdot \cos\angle O_1CO$$

and

$$OO_2^2 = O_2C^2 + OC^2 - 2O_2C \cdot OC \cdot \cos\angle O_2CO$$

from which

$$O_1C^2 + OO_2^2 = O_2C^2 + OO_1^2 - 2OC \cdot (O_2C \cos \angle O_2CO) - O_1C \cos \angle O_1CO) \ .$$

We need to establish that (i) $\angle O_1CO = \angle NAB$ and (ii) $\angle O_2CO = \angle MBA$. (See figure 3.) Ad (i), $\angle AO_1N = 2\angle ACN = 2\alpha$ and $\angle CO_1N = 2\angle CAN = 2\beta$, say, so that $\angle CO_1A = 2(\alpha + \beta)$. The common chord CA of $\mathfrak{K}_1$ and $\mathfrak{K}$ is right bisected by O_1O, so that $\angle CO_1A = 2\angle CO_1O$ and $\angle CO_1O = \alpha + \beta$. On the other hand, $\angle COO_1 = \frac{1}{2}\angle COA = \angle CBA = \gamma$, say. Hence, $\angle O_1CO = 180° - (\alpha + \beta + \gamma)$. Also, $\angle ANB = \alpha + \beta$ and $\angle NAB = 180° - (\alpha + \beta + \gamma) = \angle O_1CO$. Similarly, (ii) can be shown.

From the extended Law of Sines involving the circumradius, we have that $2R_1 = AN/\sin C$ and $2R_2 = MB/\sin C$. It follows that

$$O_2C \cos \angle O_2CO - O_1C \cos \angle O_1CO = 0$$

$$\Leftrightarrow R_2 \cdot \cos \angle MBA - R_1 \cdot \cos \angle NAB = 0$$

$$\Leftrightarrow MB \cos \angle MBA = AN \cos \angle NAB \ .$$

However, $MB \cos \angle MBA = BM_1$ and $AN \cos \angle NAB = AN_1$ (the lengths of the projections on AB). The result now follows, that CD passes through O if and only if S_{AB} passes through the midpoint of MN.

239. Find all natural numbers n for which the diophantine equation

$$(x + y + z)^2 = nxyz$$

has positive integer solutions x, y, z.

Solution. Let $(n; x, y, z) = (n; u, v, w)$ be a solution of the equation. Then the quadratic equation

$$t^2 + (2u + 2v - nuv)t + (u + v)^2 = 0$$

has two solutions, w and a second one w' for which $ww' = (u + v)^2 > 0$ (product of the roots). Since $w + w' = -(2u + 2v - nuv)$, an integer, w' must be a positive integer, and so $(n; x, y, z) = (n; u, v, w')$ is a solution of the equation. If $w > (u+v)$, then $w' < (u+v)$. It follows that, if there is a solution, we can repeat the process long enough using any two of the three variables as fixed to always find solutions $(n; x, y, z)$ of the equation for which $z \le x + y$, $y \le x + z$ and $x \le x + y$. So we impose this additional restriction in our search. Wolog, we can also suppose that $1 \le x \le y \le z$.

Suppose $x = 1$. Since $z \le x+y = 1+y$, $(x, y, z) = (1, r, r)$ or $(1, r, r+1)$. The first leads to $(2r+1)^2 = nr^2$ or $1 = r(nr - 4r - 4)$, whence $(n; r) = (9, 1)$. The second leads to $4(r + 1)^2 = nr(r + 1)$, or $4 = (n - 4)r$; this yields $(n; r) = (8; 1), (6; 2), (5; 4)$. Thus, the four solutions with $x = 1$ are

$$(n; x, y, z) = (5; 1, 4, 5), (6; 1, 2, 3); (8; 1, 1, 2); (9; 1, 1, 1) \ .$$

Suppose $x \ge 2$. Then

$$nxyz = (x + y + z)(x + y + z) \le (z + z + z)(x + y + x + y) = 6z(x + y)$$

so that $nxy \le 6(x + y)$. Rearranging the terms and adding 36 to both sides yields

$$(nx - 6)(ny - 6) \le 36 \ .$$

Since $2 \le x \le y$, we find that $(2n - 6)(2n - 6) \le 36$ so that $0 \le n \le 6$. Checking turns up the additional solutions

$$(n; x, y, z) = (1; 9, 9, 9), (2; 4, 4, 8); (3; 3, 3, 3); (4; 2, 2, 4) \ .$$

Thus, the only natural numbers n for which a solution exists are 1, 2, 3, 4, 5, 6, 8, 9.

240. In a competition, 8 judges rate each contestant "yes" or "no". After the competition, it turned out, that for any two contestants, two judges marked the first one by "yes" and the second one also by "yes"; two judges have marked the first one by "yes" and the second one by "no"; two judges have marked the first one by "no" and the second one by "yes"; and, finally, two judges have marked the first one by "no" and the second one by "no". What is the greatest number of contestants?

Solution. Let n be the number of contestants. Then, the marks of the judges for each of them can be recorded in a column of eight zeros or ones, as follows: there is a 1 on the ith position of the number if the ith judge has marked this contestant by "yes" and there is a 0 in this position if the ith judge has marked this contestant by "no". This way, the information about the marks of the contestants will be recorded in an $n \times 8$ table. Now, the given condition implies that the 2×8 table formed by any two columns of the above table has exactly two rows of each of 00, 01, 10, 11. Denote this property by $(*)$. We will now show that eight columns with any pair having this property do not exist.

Suppose the contrary, and consider a table with eight columns. Interchanging 1 and 0 in any column does not change the property $(*)$, so, wolog, we can assume that the first row consists solely of 0s. Let there be a_i 0s in the ith row. Then $\sum_{i=1}^{8} a_i = 8 \times 4 = 32$ and $\sum_{i=2}^{8} a_i = 32 - 8 = 24$. Next, we will count the number of pairs of two 0s that can appear in the lines of the table in two different ways.

(i) In the ith row, there are a_i 0s. We can choose two of them in $\binom{a_i}{2}$ ways, so the number of possible pairs in all rows is $\sum_{i=1}^{8} \binom{a_i}{2}$.

(ii) There are 8 columns. We can choose two of them in $\binom{8}{2} = 28$ ways. In each selection, there are exactly two rows with 00, so that all the ways to get combinations of two 0s is $2 \times 28 = 56$. Thus,

$$\sum_{i=1}^{8} \binom{a_i}{2} = 56 \ .$$

We have that

$$\begin{aligned}\sum_{i=1}^{8} \binom{a_i}{2} &= \frac{a_1(a_1-1)}{2} + \sum_{i=2}^{8} \frac{a_i(a_i-1)}{2} \\ &= 28 - \frac{1}{2}\sum_{i=2}^{8} a_i + \frac{1}{2}\sum_{i=2}^{8} a_i^2 = 28 - 12 + \frac{1}{2}\sum_{i=2}^{8} a_i^2 \ ,\end{aligned}$$

from which $\sum_{i=2}^{8} a_i^2 = 2(56 - 28 + 12) = 80$. From the ineqaulity of the root mean square and the arithmetic mean, we have that

$$\frac{a_2^2 + \cdots + a_8^2}{7} \geq \left(\frac{a_2 + \cdots + a_8}{7}\right)^2 = \frac{576}{49} \ .$$

whence $80 = \sum_{i=2}^{8} a_i^2 \geq 576/7 > 82$, which is false. Therefore, we must conclude that there cannot be eight columns with condition $(*)$. However, we can realize this condition with a table of seven columns:

0	0	0	0	0	0	0
0	1	1	1	1	0	0
0	1	1	0	0	1	1
0	0	0	1	1	1	1
1	0	1	0	1	0	1
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	0	1	0	0	1

Thanks to Emil Kolev, Sofia, Bulgaria for this problem.

241. Determine $\sec 40° + \sec 80° + \sec 160°$.

Solution 1. The values $40°$, $80°$ and $160°$ all satisfy $\cos 3\theta = -1/2$, or $8\cos^3\theta - 6\cos\theta + 1 = 0$. Thus, $\cos 40°$. $\cos 80°$ and $\cos 160°$ are the roots of the cubic equation $8x^3 - 6x + 1 = 0$, so that their reciprocals $\sec 40°$, $\sec 80°$ and $\sec 160°$ are the roots of the cubic equation $x^3 - 6x^2 + 8 = 0$. The sum of the roots of this cubic is

$$\sec 40° + \sec 80° + \sec 160° = 6 \ .$$

Solution 2. Let $z = \cos 40° + i\sin 40°$. Then $z^9 = 1$. In fact, since $z^9 - 1 = (z-1)(z^2+z+1)(z^6+z^3+1)$ and the first two factors fail to vanish, $z^6+z^3+1 = 0$. Also $1+z+z^2+\cdots+z^8 = (1+z+z^2)(1+z^3+z^6) = 0$. Observe that $\cos 40° = \frac{1}{2}(z+\frac{1}{z})$, $\cos 80° = \frac{1}{2}(z^2+\frac{1}{z^2})$ and $\cos 160° = \frac{1}{2}(z^4+\frac{1}{z^4})$, so that the given sum is equal to

$$\begin{aligned}
2\left[\frac{z}{1+z^2}+\frac{z^2}{1+z^4}+\frac{z^4}{1+z^8}\right] &= 2\left[\frac{z}{1+z^2}+\frac{z^2}{1+z^4}+\frac{z^5}{1+z}\right] \\
&= 2\left[\frac{z(1+z+z^4+z^5)+z^2(1+z+z^2+z^3)+z^5(1+z^2+z^4+z^6)}{(1+z)(1+z^2)(1+z^4)}\right] \\
&= 2\left[\frac{z^7+z^6+3z^5+z^4+z^3+3z^2+z+1}{(1+z)(1+z^2)(1+z^4)}\right] \\
&= 2\left[\frac{(z+1)(z^6+z^3+1)+3z^2(z^3+1)}{(1+z)(1+z^2)(1+z^4)}\right] \\
&= 2\left[\frac{0-3z^8}{1+z+z^2+z^3+z^4+z^5+z^6+z^7}\right] = 2\left[\frac{-3z^8}{-z^8}\right] = 6 \ .
\end{aligned}$$

Solution 3. [T. Liu]

$$\begin{aligned}
\sec 40° + \sec 80° + \sec 160° &= \frac{\cos 40° + \cos 80°}{\cos 40° \cos 80°} + \frac{1}{\cos 160°} \\
&= \frac{2\cos 60° \cos 20°}{\cos 40° \cos 80°} + \frac{1}{\cos 160°} \\
&= \frac{\cos 20° \cos 160° + \cos 40° \cos 80°}{\cos 40° \cos 80° \cos 160°} \\
&= \frac{\cos 180° + \cos 140° + \cos 120° + \cos 40°}{\cos 40°(\cos 240° + \cos 80°)} \\
&= \frac{-1 - 1/2}{(1/2)(-\cos 40° + \cos 120° + \cos 40°)} = \frac{-3/2}{-1/4} = 6 \ .
\end{aligned}$$

Solution 4. Let $x = \cos 40°$, $y = \cos 80°$ and $z = \cos 160°$. Then

$$x + y + z = 2\cos 60° \cos 20° - \cos 20° = 0$$

and

$$\begin{aligned}
xy + yz + zx &= \frac{1}{2}[\cos 120° + \cos 140° + \cos 240° + \cos 80° + \cos 200° + \cos 120°] \\
&= \frac{1}{2}\left[-\frac{3}{2} + x + y + z\right] = -\frac{3}{4} \ .
\end{aligned}$$

Now

$$\begin{aligned}
8\sin 40° \cos 40° \cos 80° \cos 160° &= 4\sin 80° \cos 80° \cos 160° \\
&= 2\sin 160° \cos 160° = \sin 320° = -\sin 40°
\end{aligned}$$

so that $xyz = -1/8$. Then the sum of the problem is equal to $(xy + yz + zx)/(xyz) = 6$.

242. Let ABC be a triangle with sides of length a, b, c oppposite respective angles A, B, C. What is the radius of the circle that passes through the points A, B and the incentre of triangle ABC when angle C is equal to (a) $90°$; (b) $120°$; (c) $60°$. (With thanks to Jean Turgeon, Université de Montréal.)

Solution. $\angle AIB = 180° - \frac{1}{2}(\angle BAC + \angle ABC) = 90° + \frac{1}{2}\angle C$, an obtuse angle. Hence, the side AB of the circle through A, I, B subtends an angle of $180° - \angle C$ at the centre of the circle, so that its radius has length $c/(2\sin(90° - C/2)) = c/(2\cos C/2)$. The radius is equal to $c/\sqrt{2}$, c and $c/\sqrt{3}$ when $\angle C = 90°, 120°, 60°$ respectively.

Comment. a diameter of the circumcircle of ABI is the line joining I to the centre of the escribed circle on side AB.

243. The inscribed circle, with centre I, of the triangle ABC touches the sides BC, CA and AB at the respective points D, E and F. The line through A parallel to BC meets DE and DF produced at the respective points M and N. The midpoints of DM and DN are P and Q respectively. Prove that A, E, F, I, P, Q lie on a common circle.

Solution 1. Since $AF \perp FI$ and $AE \perp EI$, $AEIF$ is concyclic. Since $\Delta ANF \sim \Delta BDF$ and $BD = BF$, then $AF = AN$, Similarly, $AE = AM$, and so A is the midpoint of NM. Thus, $AP \| ND$ and so

$$\angle APE = \angle APM = \angle NDM = \angle FDE = \frac{1}{2}\angle FIE = \angle AIE$$

and $AEPI$ is concyclic. Similarly $AFQI$ is concyclic. Thus P, Q, I all lie on the circle (with diameter AI) through A, E and F.

Solution 2. [T. Yue] Let AQ produced meet CB at R. Then $AQ = QR$ and $NQ = QD$, so that $RD = AN = AE \Longrightarrow CR = CD + DR = CE + AE = CA$. Therefore ΔCAR is isosceles with median CQ. Hence $CQ \perp AR$ and Q lies on the angle bisector of $\angle ACR$. Thus, I, Q, C are collinear with $\angle IQA = \angle IFA = 90°$. Hence $AFQIE$ is concyclic. Also $AFPIE$ is conclyclic and the result follows.

Solution 3. Recall that the *nine-point* circle of a triangle is that circle that contains the midpoints of the sides, the pedal points (feet of altitudes) and the midpoints of the segments joining the orthocentre to the vertices. We show that the six points in question lie on the nine-point circle of triangle MND; indeed, that A, P, Q are the midpoints of the sides, F, E are pedal points and I is the midpoint of the segment joining the orthocentre and D.

$ID \perp AM$, $AF \perp IF$, $AF = AM$, $FI = ID$ and $\angle FAM = 180° - \angle NAF = 180° - \angle FBD = \angle FID$. Hence $\Delta FAM \sim \Delta FID$ and we can transform ΔFAM to ΔFID by a composite of a rotation about F through $90°$ and a dilation with factor $|IF|/|FA|$. Hence $MF \perp ND$ and so F is a pedal point of ΔDMN. Similarly, E is a pedal point. [An alternative argument can be had by noting that A, M, F, E lie on a circle with centre A and diameter NM, so that right angles are subtended at E and F by NM.]

Produce DI to meet the incircle again at H. Since $\angle DFH = 90°$, H lies on FM. Similarly, H lies on EN, so that H is the orthocentre of ΔAMN, and I is the midpoint of DH. The result follows.

244. Let $x_0 = 4$, $x_1 = x_2 = 0$, $x_3 = 3$, and, for $n \geq 4$, $x_{n+4} = x_{n+1} + x_n$. Prove that, for each prime p, x_p is a multiple of p.

Solution. The recursion is satisfied by the sequences whose nth terms are any of a^n, b^n, c^n, d^n, where a, b, c, d are the roots of the quartic equation $t^4 - t - 1 = 0$, and so it is satisfied by $u_n = a^n + b^n + c^n + d^n$. Observe that $u_0 = 4$, $u_1 = a + b + c + d = 0$ (the sum of the roots), $u_2 = a^2 + b^2 + c^2 + d^2 = (a + b + c + d)^2 - 2(ab + ac + ad + bc + bd + cd) = 0 - 0 = 0$ and

$$\begin{aligned} u_3 &= (a^3 + b^3 + c^3 + d^3) \\ &= (a + b + c + d)^3 - 3(a + b + c + d)(ab + ac + ad + bc + bd + cd) + 3(abc + abd + acd + bcd) \\ &= 0 - 0 + 3 = 3\ . \end{aligned}$$

[To check the last, begin with the easier observation that

$$(x^3 + y^3 + z^3) - (x + y + z)^3 + 3(x + y + z)(xy + yz + zx) - 3xyz \equiv 0$$

and note that

$$(a^3 + b^3 + c^3 + d^3) - (a + b + c + d)^3 + 3(a + b + c + d)(ab + ac + ad + bc + bd + cd) - 3(abc + abd + acd + bcd)$$

is a polynomial of degree 3 in four variables that vanishes when any of a, b, c, dequals 0; by the factor theorem, it is divisible by $abcd$. This can happen only if it is identically 0.] Thus, the sequences $\{x_n\}$ and $\{u_n\}$ agree for $n = 0, 1, 2, 3$ and so agree at every index n.

Let p be a prime. Then

$$0 = (a + b + c + d)^p = a^p + b^p + c^p + d^p + pf(a, b, c, d)$$

from the multinomial expansion, where $f(a, b, c, d)$ is a symmetric polynomial that can be written as a polynomial in the symmetric functions $s_1 = a+b+c+d$, $s_2 = ab+ac+ad+bc+bd+cd$, $s_3 = abc+abd+acd+bcd$, $s_4 = abcd$, each of which is an integer. Thus, $a^p + b^p + c^p + d^p = -pf(a, b, c, d)$, where $f(a, b, c, d)$ is an integer and the result follows.

245. Determine all pairs (m, n) of positive integers with $m \leq n$ for which an $m \times n$ rectangle can be tiled with congruent pieces formed by removing a 1×1 square from a 2×2 square.

Solution 1. The tiling can be done for all pairs (m, n) of positive integers for which $m \geq 2$, $n \geq 2$, and either (1) $(m, n) = (2, 3k), (3k, 2), (3, 2k), (2k, 3)$ for some positive integer k, or (2) $m \geq 4$, $n \geq 4$, provided mn is a multiple of 3.

Since each tile is made up of three unit squares, the area of each rectangle must be a multiple of 3, so that $3|mn$. The tiling is impossible if either m or n is equal to 1. If m or n equals 2, then the other variable must be a multiple of 3. Suppose, say, the number of rows m equals 3, and let $n = 2k + 1$. Colour the $k + 1$ odd unit squares (counting from the end) in each of the top and bottom rows. It is impossible for a tile to cover more than one coloured square, so that at least $2(k + 1)$ tiles are necessary. But since the area of the rectangle is $3(2k + 1)$, we do not have room for this many tiles. Thus, if m or n equals 3, the other variable must be even.

We show that the tiling is possible in each of the cases cited. Note that two tiles can be combined to form a 3×2 or 2×3 rectangle, so any rectangle that has one dimension divisible by 3 and the other even can be tiled. In particular, 6×3, 6×2, 2×6, 3×6 rectangles can be tiled, and by combining these, we can tile any rectangle one of whose dimensions is a multiple of 6 and the other dimension exceeds 1.

Suppose that $m = 6k + 3$ where $k \geq 1$. If we can tile a $9 \times n$ rectangle, then by appending tiled $6 \times n$ rectangles, we can tile a $(6k + 3) \times n$ rectangle. A $9 \times n$ rectangle can be tiled when n is even; a 9×3 rectangle cannot be tiled, but a 9×5 rectangle can be tiled (exercise: do it!). It can be deduced that a $9 \times n$ rectangle can be tiled when $n = 2$ or $n \geq 4$. By symmetry, we see that an $m \times (6k + 3)$ rectangle can be tiled whenever $m \geq 4$ and $k \geq 1$.

246. Let $p(n)$ be the number of partitions of the positive integer n, and let $q(n)$ denote the number of finite sets $\{u_1, u_2, u_3, \cdots, u_k\}$ of positive integers that satisfy $u_1 > u_2 > u_3 > \cdots > u_k$ such that $n = u_1 + u_3 + u_5 + \cdots$ (the sum of the ones with odd indices). Prove that $p(n) = q(n)$ for each positive integer n.

For example, $q(6)$ counts the sets $\{6\}$, $\{6, 5\}$, $\{6, 4\}$, $\{6, 3\}$, $\{6, 2\}$, $\{6, 1\}$, $\{5, 4, 1\}$, $\{5, 3, 1\}$, $\{5, 2, 1\}$, $\{4, 3, 2\}$, $\{4, 3, 2, 1\}$.

Solution. A partition of the natural number n can be illustrated by a *Ferrers diagram*, in which there are several rows of symbols, left justified, each row containing no more symbols than the row above it and the numbers of symbols in each row giving a number in the partition, ordered from largest to smallest. For example, if $n = 15$, the partition $15 = 7 + 4 + 3 + 1$ is represented by the diagram

x	x	x	x	x	x	x
x	x	x	x			
x	x	x				
x						

There is a one-one correspondence between partitions of n and diagrams of n symbols in which each row contains no more symbols than its predecessor. We can also get n symbols by counting the symbols in each gnomon (indicated by a, b, c in the diagram below), so that in the present example $15 = 10 + 4 + 1$.

a	a	a	a	a	a	a
a	b	b	b			
a	b	c				
a						

The difficulty is that, if we specify the lengths of the gnomons, there are several possibilities for placing the gnomons to give us a Ferrars diagram. So we need a way of specifying exactly which element of the gnomon is at the turning point. One way to do this is to get a measure of the number of vertical elements in the gnomon, which, we achieve by counting for each gnomon after the first, the elements in the vertical shaft along with the elements above and to the right in the horizontal shaft of the previous gnomon; this is indicated by the symbols y and z in the diagram:

x	y	y	y	y	y	y
x	y	z	z			
x	y	z				
x						

So we insert in the sum $10 + 4 + 1$ the lengths of these hybrid gnomons to get $10 + 8 + 4 + 3 + 1$ where the even terms count the number of y's and z's. On the other hand, given such a sum, we can reconstruct the diagram uniquely.

In the general situation, given a partition of n, construct its Ferrars diagram. To construct a sum counted by $q(n)$, the first term counts the number of symbols in the upper left gnomon, the second the number of symbols in the gnomon formed by the second column and the top row to the right of the first column, the third the number of symbols in the gnomon formed by the second column below the first row and the second row to the right of the first column, and so on. On the other hand, given a sum counted by $q(n)$, we can construct a Ferrars diagram as follows. If the last term is an evenly indexed term, make a horizontal row of that number of symbols; if it is oddly indexed, make a vertical column of that number of symbols to form the lowest rightmost gnomon of the diagram. Now work along the sum from right to left. At each evenly indexed summand, to get the gnomon for the next term to the left, extend the top row by one symbol to the left and make it part of a gnomon with the number of terms of the next summand to the left; at each oddly indexed summand, to get the gnomon for the next term to the left, extend the lect column by one symbol up and make it part of a gnomon with the number of terms of the next summand to the left. In this way, we obtain a one-one correspondence between partitions counted by $p(n)$ and finite sequences counted by $q(n)$.

In the example of the problem, we get the correspondence $[6; \{6, 5\}]$, $[5 + 1; \{6, 4\}]$; $[4 + 2; \{5, 4, 1\}]$, $[4 + 1 + 1; \{6, 3\}]$; $[3 + 3; \{4, 3, 2\}]$, $[3 + 2 + 1; \{5, 3, 1\}]$; $[3 + 1 + 1 + 1; \{6, 2\}]$; $[2 + 2 + 2; \{4, 3, 2\}]$; $[2 + 2 + 1 + 1; \{5, 2, 1\}]$; $[2 + 1 + 1 + 1 + 1; \{6, 1\}]$; $[1 + 1 + 1 + 1 + 1 + 1; \{6\}]$.

247. Let $ABCD$ be a convex quadrilateral with no pairs of parallel sides. Associate to side AB a point T as follows. Draw lines through A and B parallel to the opposite side CD. Let these lines meet CB produced at B' and DA produced at A', and let T be the intersection of AB and $B'A'$. Let U, V, W be points similarly constructed with respect to sides BC, CD, DA, respectively. Prove that $TUVW$ is a parallelogram.

Solution. [T. Yin] Let AB and CD produced intersect at Y. Suppose A' and B' are defined as in the problem. Let the line through C parallel to AD meet AB produced at B'' and the lines through B parallel to AD meet CD produced at C', so that U is the intersection of BC and $B''C'$. Let P be the intersection of AB' and BC' and Q the intersection of $A'B$ and $B''C$. Then $A'B \| AB' \| CD$ and $AD \| BC' \| B''C$, so that $APBA'$ and $CQBC'$ are parallelograms. Hence

$$BT : TA = A'B : AB' = AP : AB' = YC' : YC$$

and

$$BU : UC = BC' : B''C = YB : YB' \ .$$

Since also $YB : YB'' = YC' : YC$, $BT : TA = BU : UC$ and $TU \| AC$. Similarly, $VW \| AC$, $TU \| BD$, $UW \| BD$ and so $TUVW$ is a parallelogram.

248. Find all real solutions to the equation

$$\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1 \ .$$

Solution 1. For the equation to be valid over the reals, we require that $x \geq 1$. Suppose that $y^2 = x - 1$ and $y \geq 0$. Then the equation becomes

$$|y-2| + |y-3| = 1 \ .$$

When $1 \leq x \leq 5$, we have that $0 \leq y \leq 2$ and the equation becomes $(2-y)+(3-y) = 1$ or $y = 2, x = 5$. When $5 \leq x \leq 10$, we have that $2 \leq y \leq 3$ and the equation becomes an identity $(y-2)+(3-y) = 1$. Thus, it holds for all x on the closed interval $[5, 10]$. Finally, when $10 \leq x$, we have that $3 \leq y$ and the equation becomes $(y-2)+(y-3) = 1$ or $y = 3, x = 10$. Thus, the complete set of solutions of the equation is given by $5 \leq x \leq 10$. All these solutions check out.

Solution 2. [Z. Wu] For a solution to exist, we require that $x \geq 1$ and that both $0 \leq x+3-4\sqrt{x-1} \leq 1$ and $0 \leq x+8-6\sqrt{x-1} \leq 1$. These two conditions lead to $(x+2)^2 \leq 16(x-1)$ and $(x+7)^2 \leq 36(x-1)$, which in turn leads to

$$(x-2)(x-10) = (x^2+4x+4) - (16x-16) \leq 0$$

and

$$(x-5)(x-17) = (x^2+14x+49) - (36x-36) \leq 0 \ .$$

These conditions are both satisfied only if $5 \leq x \leq 10$. (Thus, $5 \leq x \leq 10$ is a *necessary* condition for a solution.)

On the other hand, $5 \leq x \leq 10$ implies that $2 \leq \sqrt{x-1} \leq 3$, so that (as in Solution 1) we find that the equation is equivalent to $(\sqrt{x-1}-2)+(3-\sqrt{x-1}) = 1$, which is an identity. Thus, the equation holds exactly when $5 \leq x \leq 10$.

Comment. Your first observation should be that, in order for the equation to make sense, we require that $x \geq 1$. It is important not just to write down a lot of algebraic equations, but to indicate the logical relationships between them; which equations imply which other equations? which pairs of equations are equivalent? This is especially desirable when surd equations are involved, where the operations that lead from one equation to another are not logically reversible and extraneous solutions might be introduced.

249. The non-isosceles right triangle ABC has $\angle CAB = 90°$. Its inscribed circle with centre T touches the sides AB and AC at U and V respectively. The tangent through A of the circumscribed circle of triangle ABC meets UV in S. Prove that:

(a) $ST \| BC$;

(b) $|d_1 - d_2| = r$, where r is the radius of the inscribed circle, and d_1 and d_2 are the respective distances from S to AC and AB.

Solution. Wolog, we may assume that $AB < AC$ so that S and C are on opposite sides of AB. Ad (a), $\angle SVT = \angle SVA = 45°$, $AV = VT$ and SV is common, so that triangles AVS and TVS are congruent. Hence $\angle SAV = \angle STV \Longrightarrow \angle STU = \angle SAU = \angle ACB$ (by the tangent-chord property). Since $TU \| AC$, it follows that $CB \| ST$.

Ad (b), let P and Q be the respective feet of the perpendiculars from S to AB and AC. Note that $SQAP$ is a rectangle so that $\angle PUS = \angle PSU = 45°$ and so $PU = PS$. Then $|QS| - |PS| = |AP| - |PU| = r$.

250. In a convex polygon $\mathfrak{P}$, some diagonals have been drawn so that no two have an intersection in the interior of $\mathfrak{P}$. Show that there exists at least two vertices of $\mathfrak{P}$, neither of which is an enpoint of any of these diagonals.

Solution 1. If no diagonal has been drawn, the result is clear. Suppose that at least one diagonal has been drawn. Let d be a diagonal that has, on one of its sides, the fewest vertices of the polygon. There is at least one such vertex. Then on that side, no further diagonal is drawn, since it cannot cross d and cannot have fewer vertices between its endpoints than d. Hence there is at least one vertex on that side from which no diagonal is drawn.

On the other side of d, select a diagonal g which has the smallest number of vertices between its endpoints on the side opposite to the side of d. By an argument similar to the above, there is at least one vertex on the side of g opposite to d from which no diagonal has been drawn.

Solution 2. [S. King] The result is vacuously true for triangles. Suppose that the polygon has at least four sides. Suppose that a (possibly void) collection of diagonals as specified in the problem is given. We continue adding diagonals one at a time such that each new diagonal does not cross any previous one in the interior of the polygon. At each stage, the polygon $\mathfrak{P}$ is partitioned into polygons with fewer sides all of whose vertices are vertices of the polygon $\mathfrak{P}$. As long as any of the subpolygons has more than three sides, we can add a new diagonal. However, the process will eventually terminate with a triangulation of $\mathfrak{P}$, *i.e.*, a partitioning of $\mathfrak{P}$ into $n-2$ triangles all of whose vertices are vertices of $\mathfrak{P}$. (*Exercise.* Explain why the number of triangles is $n-2$. One way to do this is to note that the sum of all the angles of the triangles is equal to the sum of the angles in the polygon.)

Each triangle must have at most two sides in common with the given polygon. Since there are n sides and $n-2$ triangles, at least two triangles have two sides in common with $\mathfrak{P}$. In each case, the vertex common to the two sides has no diagonal emanating from it (neither an original diagonal nor an added diagonal), and the result follows.

Comment. Many solvers failed to appreciate that the collection of diagonals is given, and that the problem is to establish the desired property no matter what the collection is. A lot of arguments had the students constructing diagonals without indicating how the ones constructed might have anything to do with a given set; in effect, they were giving a particular situation in which the result obtained. Several solvers tried induction, using one diagonal to split $\mathfrak{P}$ into two, but did not handle well the possibility that the loose vertices in the subpolygons might be at the ends of the subdividing diagonal. One way around this is to make the result stronger, and show that one can find two *non-adjacent* vertices that are not the endpoints of diagonals. This is certainly true for quadrilaterals, and using this an induction hupothesis yielded a straightforward argument for polygons of higher order.

251. Prove that there are infinitely many positive integers n for which the numbers $\{1, 2, 3, \cdots, 3n\}$ can be arranged in a rectangular array with three rows and n columns for which (a) each row has the same sum, a multiple of 6, and (b) each column has the same sum, a multiple of 6.

Solution 1. The sum of all the numbers in the array is $3n(3n+1)/2$, so that each column sum must be $3(3n+1)/2$. Since this is divisible by 6, $3(3n+1)$ must be a multiple of 12, and so $3n+1$ is divisible by 4 and $n \equiv 1 \pmod 4$. Since each row sum, $n(3n+1)/2$ is divisible by 6, n must be divisible by 3. Putting this together, we conclude that $n = 12k + 9$ for some value of k.

We now show that, for each n of this form, we can actually construct an array with the desired property. Starting with the magic square, we derive the following array for $n = 9$:

8	1	6	17	10	15	26	19	24
21	23	25	3	5	7	12	14	16
13	18	11	22	27	20	4	9	2

We generalize this for $n = 12k+9$, for k a nonnegative integer. Suppose that an array is possible. Then the sum of all the elements in the array is $(36k+27)(18k+14) = 18(4k+3)(9k+7)$. The sum of the elements in each column is $6(9k+7) = 54k+42$ and the sum in each row is $6(4k+3)(9k+7) = (4k+3)(54k+42)$. If we can achieve this with distinct entries, then we have constructed the array.

We build the array by juxtaposing horizontally $4k+3$ square 3×3 blocks of the form:

$8+9a$	$1+9a$	$6+9a$
$3+9b$	$5+9b$	$7+9b$
$4+9c$	$9+9c$	$2+9c$

where we make $4k+3$ distinct choices of each of a, b, c to ensure that no number is repeated in any row (it is not possible for any repetition to occur down a column). To achieve the column sum, we require that $15 + 9(a+b+c) = 54k+42$, or $a+b+c = 6k+3 = 3(2k+1)$. To achieve the row sum, we require that

$$15(4k+3) + 27\sum a = 15(4k+3) + 27\sum b = 15(4k+3) + 27\sum c$$
$$= (4k+3)(54k+42)$$

so that

$$\sum a = \sum b = \sum c = (4k+3)(2k+1) = 0 + 1 + \cdots + (4k+2) \ ,$$

where each sum is over $4k+3$ distinct elements. It is convenient to let the sets of a's, b's, and c's each consist of the numbers $0, 1, 2, \cdots, 4k+2$ in some order. In the ith 3×3 block, let $0 \le a, b, c, \le 4k+2$ and

$$a = i - 1$$

$$b \equiv (i-1) + 2(k+1) = 2k + i + 1 \pmod{4k+2}$$

$$c = (6k+3) - (a+b) \equiv 4k + 3 - 2i \pmod{4k+3} \ .$$

for $1 \le i \le 4k+3$. It is straightforward to verify that the a's, the b's and the c's each run through a complete set of residues (mod $4k+3$), and we have arranged that $a+b+c = 6k+3$. If $1 \le i \le 2k+1$, then $2k+2 \le b = 2k+i+1 \le 4k+2$ and $2k+2 \le a+b = 2(k+i) \le 6k+2$, so that $1 \le c \le 4k+1$. If $2k+2 \le i \le 4k+3$, then $0 \le b = (2k+i+1) - (4k+3) = i - 2k - 2 \le 2k+1$ and $2k+1 \le a+b = 2i - 2k - 3 \le 6k+3$, so that $0 \le c \le 4k+2$. With this choice of the variables a, b, c we can construct the array as desired.

For example, when $n = 45, k = 3$, there are 15 blocks and the choice of a, b, c for these blocks can be read along the rows of

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
8	9	10	11	12	13	14	0	1	2	3	4	5	6	7
13	11	9	7	5	3	1	14	12	10	8	6	4	2	0

It is left as an exercise for the reader to construct the 3×45 array.

Solution 2. [Y. Zhao] We can form the 3×9 array:

4	9	2	13	18	11	22	27	20
12	14	16	21	23	25	3	5	7
26	19	24	8	1	6	17	10	15

Suppose, as an induction hypothesis, we can build a $3 \times n$ array for some positive integer n. Duplicate this array five times and put them side by side in a row. Partition the $3 \times 5n$ array into fifteen $1 \times n$ subarrays, and to the elements of each of the fifteen subarrays add a constant number as indicated by the positions in the following 3×5 table:

+0	+3b	+6n	+9n	+12n
+6n	+9n	+12n	+0	+3
+12n	+6n	+0	+9n	+3n

The row sum of the numbers added is $30n$ and the column sum is $18n$, so the $3 \times 5n$ array preserves the divisibility by 6 properties of the $3 \times n$ array. Therefore, we can see by induction that an array is constructibel whenever $n = 9 \times 5^k$ for $0 \leq k$.

Solution 3. [J. Zhao] For the time being, neglect the conditions involving divisibility by 6, and focus only on the condition that the numbers $1, 2, \cdots, 3n$ be used and that the row sums and the column sums be each the same. Then, when $n = 3$, a magic square will serve.

Suppose that, for some $k \geq 1$, we have found a suitable 3×3^k matrix M. Let A be the $3 \times 3^{k+1}$ matrix obtained by placing three copies of M side by side and B the $3 \times 3^{k+1}$ matrix determined by placing side by side the 3×3^k matrices B_1, B_2, B_3 where each column of B_1 is (the transpose of) (0, 1, 2), of B_2 is (1, 2, 0), and of B_3 is (2, 0, 1). Each of A and B has constant row sums and constant column sums.

Let $N = A + 3^{k+1}B$. Then N not only has constant row and column sums, but consists of the numbers $1, 2, \cdots, 3^{k+2}$ (why?). The row sums of M are each $(1/6)(3^{k+1})(3^{k+1} + 1)$, so that the row sums of N are each

$$\begin{aligned} 3 \times (1/6)(3^{k+1})(3^{k+1} + 1) + 3^{k+1}(3^k) + 3^{k+1}(2 \times 3^k) &= (1/6)[3^{k+2}(3^{k+1} + 1)] + (3^{2k+1} \times 3) \\ &= (1/6)(3^{k+2})(3^{k+1} + 1 + 6 \times 3^k) = (1/6)(3^{k+2})(3^{k+2} + 1) \ . \end{aligned}$$

The column sums of M are each $(3/2)(3^{k+1} + 1)$ and so the column sums of N are each

$$(3/2)(3^{k+1} + 1) + 3^{k+1} + 2 \times 3^{k+1} = (1/2)(3^{k+2} + 3 + 2 \times 3^{k+2}) = (3/2)(3^{k+2} + 1) \ .$$

We now require that each of $(1/6)(3^{k+1})(3^{k+1}+1)$ and $(3/2)(3^{k+1}+1)$ be divisible by 6. This will occur exactly when $3^{k+1} + 1 \equiv 0 \pmod 4$, so that k must be even. Thus, we can obtain an array as desired when $n = 9^m$ for some positive integer m. (Note that $9^m \equiv 9 \pmod{12}$.)

252. Suppose that a and b are the roots of the quadratic $x^2 + px + 1$ and that c and d are the roots of the quadratic $x^2 + qx + 1$. Determine $(a - c)(b - c)(a + d)(b + d)$ as a function of p and q.

Solution 1. From the theory of the quadratic, we have that $a + b = -p$, $c + d = -q$ and $ab = cd = 1$. Then

$$\begin{aligned} (a - c)(b - c)(a + d)(b + d) &= (a - c)(b + d)(b - c)(a + d) \\ &= (ab - cd + ad - bc)(ba - cd + bd - ca) \\ &= (ad - bc)(bd - ca) = abd^2 - a^2cd - b^2cd + abc^2 \\ &= d^2 - a^2 - b^2 + c^2 = [(c + d)^2 - 2cd] - [(a + b)^2 - 2ab] \\ &= (q^2 - 2) - (p^2 - 2) = q^2 - p^2 \ . \end{aligned}$$

Solution 2. Using $a + b = -p$, $c + d = -q$ and $ab = cd = 1$, we obtain that

$$\begin{aligned} (a - c)(b - c)(a + d)(b + d) &= [ab - (a + b)c + c^2][ab + (a + b)d + d^2] \\ &= (1 + pc + c^2)(1 - pd + d^2) = (2 + c^2 + d^2) - p^2 \\ &= (c + d)^2 - p^2 = q^2 - p^2 \ . \end{aligned}$$

253. Let n be a positive integer and let $\theta = \pi/(2n+1)$. Prove that $\cot^2\theta$, $\cot^2 2\theta$, $\cdots$, $\cot^2 n\theta$ are the solutions of the equation

$$\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \binom{2n+1}{5}x^{n-2} - \cdots = 0 \ .$$

Solution 1. From de Moivre's Theorem that

$$\cos m\theta + i\sin m\theta = (\cos\theta + i\sin\theta)^m \ ,$$

we obtain from a comparison of imaginary parts that

$$\sin m\theta = \binom{m}{1}\cos^{m-1}\theta\sin\theta - \binom{m}{3}\cos^{m-3}\sin^3\theta + \cdots \ ,$$

for each positive integer m. Hence

$$\sin(2n+1)\theta = \sin^{2n+1}\theta\left[\binom{2n+1}{1}\cot^{2n}\theta - \binom{2n+1}{3}\cot^{2n-2}\theta + \cdots\right] \ .$$

When $\theta = (k\pi)/(2n+1)$ for $1 \le k \le n$, $\sin(2n+1)\theta = 0$ while $\sin\theta \ne 0$. The desired result follows.

Solution 2. [Y. Zhao] Observe that, for each complex a,

$$\frac{1}{2}[(a+1)^{2n+1} - (a-1)^{2n+1}] = \binom{2n+1}{1}a^{2n} + \binom{2n+1}{3}a^{2n-2} + \binom{2n+1}{5}a^{2n-4} + \cdots \ .$$

Suppose that $a = i\cot k\theta$ with $\theta = \pi/(2n+1)$ and $1 \le k \le n$. Note that $\sin k\theta \ne 0$. Then

$$\begin{aligned}\binom{2n+1}{1}(-\cot^2 k\theta)^n + \binom{2n+1}{3}(-\cot^2 k\theta)^{n-1} + \cdots &= \frac{1}{2}[(i\cot k\theta + 1)^{2n+1} - (i\cot\theta - 1)^{2n+1}] \\ &= \frac{1}{2}\left(\frac{i}{\sin k\theta}\right)^{2n+1}[(\cos k\theta - i\sin k\theta)^{2n+1} - (\cos k\theta + i\sin k\theta)^{2n+1}] \\ &= \left(\frac{i}{\sin k\theta}\right)^{2n+1}[-\sin(2n+1)k\theta] = \left(\frac{i}{\sin k\theta}\right)^{2n+1}[-\sin k\pi] = 0 \ ,\end{aligned}$$

and the result follows.

254. Determine the set of all triples (x,y,z) of integers with $1 \le x,y,z \le 1000$ for which $x^2+y^2+z^2$ is a multiple of xyz.

Solution. Suppose that $x^2+y^2+z^2 = kxyz$, for a positive integer k. It can be checked that if the equation is satisfied by (x,y,z) then it is also satisfied by $(x,y,kxy-z)$. Since $z^2 < x^2+y^2+z^2 = kxyz$, it follows that $z < kxy$, If $z > \frac{1}{2}kxy$, then $kxy - z < \frac{1}{2}kxy$. Suppose that we start with a solution. If we have, say z exceeding $\frac{1}{2}xy$, then we can replace z by a new value less than $\frac{1}{2}xy$. We can do the analogous thing with x and y. Every such operation reduces the sum $x+y+z$, so it can be performed at most finitely often, and we reach a situation where it cannot be done any more. Thus, we arrive at a solution where, say, $1 \le x \le y \le z \le kxy/2$, so that, in particular $kx \ge 2$. We can also start with such a solution and go backwards to achieve any given solution.

Since

$$x^2 + y^2 + \left(\frac{kxy}{2} - z\right)^2 = \left(\frac{kxy}{2}\right)^2 \ ,$$

it follows that

$$x^2 + y^2 + \left(\frac{kxy}{2} - y\right)^2 \geq \left(\frac{kxy}{2}\right)^2 ,$$

so that

$$3y^2 \geq x^2 + 2y^2 \geq kxy^2$$

and $kx \leq 3$. Thus $kx = 2$ or $kx = 3$.

The case $kx = 2$ leads to $x^2 + (y - z)^2 = 0$ which has no solutions as specified. Hence $kx = 3$ and $k = 1$ or $k = 3$. For these two cases, we find that the base solutions are respectively $(x, y, z) = (3, 3, 3)$ and $(x, y, z) = (1, 1, 1)$.

Suppose that $k = 1$. Modulo 3, any square is congruent to 0 or 1. If, say, $x \equiv 0 \pmod 3$, then $y^2 + z^2 \equiv 0 \pmod 3$; this can occur only if y and z are multiples of 3. Hence $(x, y, z) = (3u, 3v, 3v)$ for some integers u, v, w. But then $9u^2 + 9v^2 + 9w^2 = 27uvw$, or $u^2 + v^2 + w^2 = 3uvw$. Contrarily, any solution (u, v, w) of this equation gives rise to a solution (x, y, z) of $x^2 + y^2 + z^2 = xyz$. Therefore there is a one-one correspondence between solutions of $x^2 + y^2 + z^2 = xyz$ where all numbers are multiples of 3 and solutions of $x^2 + y^2 + z^2 = 3xyz$. We will obtain these solutions below.

The only other possibility is that none of x, y, z is divisible by 3. But then, $x^2 + y^2 + z^2$ would be a multiple of 3 and xyz not a multiple of 3; thus there are no solutions of this type.

Suppose that $k = 3$. From the above, we know that every solution arises from a solution for which $1 \leq x \leq y \leq z \leq 3xy/2$ and for such a solution $x = 1$. Let $(x, y, z) = (1, y, ty)$ where $1 \leq y \leq 3/2$. Then $1 + (1 + t^2)y^2 = 3ty^2$, so that

$$y^2 = \frac{1}{3t - 1 - t^2} = \frac{1}{\frac{5}{4} - (t - \frac{3}{2})^2} .$$

The denominator is not less than 1, so that $y^2 \leq 1$. Hence the only solution that can generate the rest is $(x, y, z) = (1, 1, 1)$.

To get a handle on the situation, fix $x = u$ and consider a sequence of solutions $(x, y, z) = (u, v_{n-1}, v_n)$. The solution $w = v_{n-1}$ satisfies the quadratic equation

$$u^2 + w^2 + v_n^2 = 3uwv_n$$

and so also does a second value $w = v_{n+1}$. By the theory of the quadratic, we have that

$$v_{n+1} + v_{n-1} = 3uv_n \tag{1}$$

and

$$v_{n+1}v_{n-1} = u^2 + v_n^2 . \tag{2}$$

If we start off with a solution $(x, y, z) = (u, v_0, v_1)$, we can use either (1) or (2) to determine the sequence $\{v_n\}$. Note that, since

$$v_{n+1} - v_n = v_n - v_{n-1} + (3u - 2)v_n > v_n - v_{n-1} ,$$

if $v_1 \geq v_0$, then the sequence $\{v_n\}$ is increasing. Note also, that the equations (1) and (2) are symmetric in v_{n-1} and v_{n+1}, so we can extend the sequence backwards as well as forwards.

Using the recursion $v_{n+1} = 3uv_n - v_{n-1}$, we get the following sequences for various values of u:

$$u = 1 \ : \quad \{v_n\} = \{1, 1, 2, 5, 13, 34, 89, 233, 610\}$$

$$u = 2 \ ; \quad \{v_n\} = \{1, 1, 5, 29, 169, 985\}$$

$$u = 5 \ ; \quad \{v_n\} = \{194, 13, 1, 2, 29, 433\}$$

$$u = 13 \ ; \quad \{v_n\} = \{34, 1, 5, 194\}$$

$$u = 29\ ;\quad \{v_n\} = \{433, 5, 2, 169\}$$

$$u = 34\ ;\quad \{v_n\} = \{13, 189\}$$

and so on. This yields the following solutions with $1 \leq x \leq y \leq z \leq 1000$: $(x, y, z) = (1, 1, 1)$, $(1, 1, 2)$, $(1, 5, 13)$, $(1, 13, 34)$, $(1, 34, 89)$, $(1, 89, 233)$, $(1, 233, 610)$, $(2, 5, 29)$, $(2, 29, 169)$, $(2, 169, 985)$, $(5, 13, 194)$, $(5, 29, 433)$.

255. Prove that there is no positive integer that, when written to base 10, is equal to its kth multiple when its initial digit (on the left) is transferred to the right (units end), where $2 \leq k \leq 9$ and $k \neq 3$.

Solution 1. Note that the number of digits remains the same after multiplication. Thus, if $k \geq 5$, the left digit of the number must be 1 and so the multiple must end in 1. This is impossible for $k = 5, 6, 8$. If $k = 7$ or 9, then the number must have the form $10^m + x$ where $x \leq 10^n - 1$. Then $k(10^m + x) = 10x + 1$, so that

$$x = \frac{k \cdot 10^m - 1}{10 - k} \geq \frac{7 \cdot 10^m - 1}{3} > 2 \times 10^m\ ,$$

an impossibility.

If $k = 4$, the first digit of the number cannot exceed 2, and so must be even to achieve an even product. Thus, for some positive integers m and $x \leq 10^m - 1$, we must have $4(2 \times 10^m + x) = 10x + 2$, whence

$$x = \frac{4 \times 10^m - 1}{3} > 10^m\ ,$$

again an impossibility. Finally, if $k = 2$, then $d \leq 4$ and $2(d \cdot 10^m + x) = 10x + d$, whence $d(2 \cdot 10^m - 1) = 8x$. Since $2 \cdot 10^m - 1$ is odd, 8 must divide d, which is impossible. The desired result follows.

Solution 2. [A. Critch] Suppose that multiplication is positive for some $k \neq 3$. Let the number be $d \cdot 10^m + u$ for a positive digit d, a positive integer m and a nonnegative integer $u < 10^m - 1$. Then $k(d \cdot 10^m + u) = 10u + d$, whence

$$(10^m - 1)k < k \cdot 10^m - 1 \leq d(k \times 10^m - 1) = (10 - k)u \leq (10 - k)(10^m - 1)\ ,$$

so that $k < 10 - k$ and k is equal to 2 or 4. Since k is even, d must be even. Since

$$10 - k = d\left(\frac{k \times 10^m - 1}{u}\right) > d\frac{k \times 10^m - k}{10^m - 1} = dk\ ,$$

$d < (10/k) - 1$. When $k = 2$, d must be 2, and we get $2(2 \times 10^m - 1) = 8u$, or $2 \times 10^m - 1 = 4u$, an impossibility. When $k = 4$, we get $d < 1.5$, which is also impossible. Hence the multiplication is not possible.

Comment. When $k = 3$, the first digit must be 1, 2 or 3. It can be shown that 2 and 3 do not work, so that we must have $3(10^m + x) = 10x + 1$ for $x = (3 \times 10^m - 1)/7$. This actually gives a result when $m \equiv 5 \pmod 6$. Indeed, when $m = 5$, we obtain the example 142857.

256. Find the condition that must be satisfied by y_1, y_2, y_3, y_4 in order that the following set of six simultaneous equations in x_1, x_2, x_3, x_4 is solvable. Where possible, find the solution.

$$x_1 + x_2 = y_1y_2 \qquad x_1 + x_3 = y_1y_3 \qquad x_1 + x_4 = y_1y_4$$

$$x_2 + x_3 = y_2y_3 \qquad x_2 + x_4 = y_2y_4 \qquad x_3 + x_4 = y_3y_4\ .$$

Solution. We have than $y_1(y_2 - y_3) = x_2 - x_3 = y_4(y_2 - y_3)$, whence $(y_1 - y_4)(y_2 - y_3) = 0$. Similarly, $(y_1 - y_2)(y_3 - y_4) = 0 = (y_1 - y_3)(y_2 - y_4)$. From this, we deduce that three of the four y_i must be equal. Suppose, wolog, that $y_1 = y_2 = y_3 = u$ and $y_4 = v$. Then the system can be solved to obtain $x_1 = x_2 = x_3 = u^2/2$ and $x_4 = uv - (u^2/2) = \frac{1}{2}u(2v - u)$. (This includes the case $u = v$.)

257. Let n be a positive integer exceeding 1. Discuss the solution of the system of equations:

$$ax_1 + x_2 + \cdots + x_n = 1$$
$$x_1 + ax_2 + \cdots + x_n = a$$
$$\cdots$$
$$x_1 + x_2 + \cdots + ax_i + \cdots + x_n = a^{i-1}$$
$$\cdots$$
$$x_1 + x_2 + \cdots + x_i + \cdots + ax_n = a^{n-1} \, .$$

Solution 1. First, suppose that $a = 1$. Then all of the equations in the system become $x_1+x_2+\cdots+x_n = 1$, which has infinitely many solutions; any $n-1$ of the x_i's can be chosen arbitrarily and the remaining one solved for.

Henceforth, assume that $a \neq 1$. Adding all of the equations leads to

$$(n - 1 + a)(x_1 + x_2 + \cdots + x_n) = 1 + a + a^2 + \cdots + a_{n-1} = \frac{1 - a^n}{1 - a} \, .$$

If $a = 1 - n$, then the system is viable only if $a^n = 1$. This occurs, only if $a = -1$ and n is a positive integer *i.e.*, when $(n, a) = (2, -1)$. In this case, both equations in the system reduce to $x_2 - x_1 = 1$, and we have infinitely many solution. Otherwise, when $a = 1 - n$, there is no solution to the system.

When $a \neq 1 - n$, then

$$x_1 + x_2 + \cdots + x_n = \frac{1 - a^n}{(1 - a)(n - 1 + a)} \, .$$

Taking the difference between this and the ith equation in the system leads to

$$(a - 1)x_i = a^{i-1} - \left(\frac{1 - a^n}{(1 - a)(n - 1 + a)} \right)$$

for each i and the system is solved.

Solution 2. As above, we dispose first of the case $a = 1$. Suppose that $a \neq 1$.Taking the difference of adjacent equations leads to $(a - 1)(x_{i+1} - x_i) = a^i - a^{i-1}$, so that $x_{i+1} = x_i + a^{i-1}$ for $1 \leq i \leq n - 1$. Hence $x_i = x_1 + (1 + a + \cdots + a^{i-2})$ for $2 \leq i \leq n$. From the first equation, we find that

$$(n - 1 + a)x_1 + 1 + (1 + a) + (1 + a + a^2) + \cdots + \cdots (1 + a + \cdots + a^{n-2}) = 1$$
$$\Longrightarrow (n - 1 + a)x_1 + \frac{(1 - a^2) + \cdots + (1 - a^{n-1})}{1 - a} = 0$$
$$\Longrightarrow (n - 1 + a)x_1 + \frac{n - 2 - a^2(1 + a + \cdots + a^{n-3})}{1 - a} = 0$$
$$\Longrightarrow (n - 1 + a)x_1 + \frac{(n - 2)(1 - a) - a^2(1 - a^{n-2})}{(1 - a)^2} = 0 \, .$$

Suppose that $n = 1 - a$. Then

$$0 = (n - 2)(1 - a) - a^2(1 - a^{n-2}) = -(1 + a)(1 - a) - a^2(1 - a^{n-2}) = a^{n-2} - 1 \, ,$$

so that a must be -1 and $n = 2$, The system reduces to a single equation with an infinitude of solutions. If $n \neq 1 - a$, then we can solve for x_1 and then obtain the remaining values of the x_i.

Comment. Beware of the "easy" questions. Many solvers had only a superficial analysis which did not consider the possibility that a denominator might vanish, and almost nobody picked up the $(n,a) = (2,-1)$ case. When you write up your solution, it is good to dispose of the singular cases first before you get into the general situation.

258. The infinite sequence $\{a_n; n = 0, 1, 2, \cdots\}$ satisfies the recursion

$$a_{n+1} = a_n^2 + (a_n - 1)^2$$

for $n \geq 0$. Find all rational numbers a_0 such that there are four distinct indices p, q, r, s for which $a_p - a_q = a_r - a_s$.

Solution. The recursion can be rewritten as

$$a_{n+1} = 2a_n^2 - 2a_n + 1 \Leftrightarrow 2a_{n+1} - 1 = (2a_n - 1)^2 \ .$$

Let $b_n = 2a_n - 1$, so that $a_n = \frac{1}{2}(b_n + 1)$. Then $a_p - a_q = a_r - a_s$ is equivalent to $b_p - b_q = b_r - b_s$. Since $b_{n+1} = b_n^2$ for each nonnegative integer n, we have that $b^n = b_0^{2^n}$. If $b_p - b_q = b_r - b_s$, then b_0 must be the rational solution of a polynomial equation of the form,

$$x^{2^p} - x^{2^q} - x^{2^r} + x^{2^s} = 0$$

where the left side consists of four distinct monomials. One possibility is $b_0 = 0$. Suppose now that $b_0 \neq 0$. Dividing by the monomial with the smallest exponent, we obtain a polynomial equation for b_0 whose leading coefficient and constant coefficients are each 1. So the numerator of b_0 written in lowest terms, dividing the constant term, must be ± 1 and the denominator, dividing the leading coefficient, must also be ± 1. Hence, the only possibilities for b_0 are -1, 0 and 1. These correspond to the possibilities $0, \frac{1}{2}, 1$ for a_0, and each of these choices leads to a sequence for which $a_n = a_1$ for $n \geq 1$ and for which there are two pairs of terms with the same difference (0).

259. Let ABC be a given triangle and let $A'BC$, $AB'C$, ABC' be equilateral triangles erected outwards on the sides of triangle ABC. Let Ω be the circumcircle of $A'B'C'$ and let A'', B'', C'' be the respective intersections of Ω with the lines AA', BB', CC'.

Prove that AA', BB', CC' are concurrent and that

$$AA'' + BB'' + CC'' = AA' = BB' = CC' \ .$$

Solution. A rotation of 60° about the vertex A takes triangle ACC' to the triangle $AB'B$, and so $BB' = CC'$. Similarly, it can be shown that each of these is equal to AA'. Suppose that BB' and CC' intersect in F. From the rotation, $\angle BFC' = 60^\circ = \angle BAC'$, so that $AFBC'$ is concyclic.

hence $\angle C'FB = \angle C'AB = 60^\circ$. Also $\angle AFC' = \angle ABC' = 60^\circ$, $\angle AFB' = 60^\circ$ and so $\angle BFC = \angle C'FB' = 120^\circ$. Since $\angle BFC + \angle BA'C = 180^\circ$, the quadrilateral $BFCA'$ is concyclic and $\angle BFA' = \angle BCA' = 60^\circ$. Hence $\angle AFA' = \angle AFC' + \angle C'FB + \angle BFA' = 180^\circ$, so that A, A' and F are collinear, and AA', BB' and CC' intersect at F.

From Ptolemy's Theorem, $AB \cdot C'F = AF \cdot BC' + FB \cdot AC'$, whence $C'F = AF + BF$. Similarly, $A'F = BF + CF$ and $C'F = AF + BF$. Indeed, $AA' = BB' = CC' = AF + A'F = AF + BF + CF$.

[J. Zhao] Let O be the circumcentre of triangle $A'B'C'$ and let the respective midpoints of $A'A''$, $B'B''$, $C'C''$ be X, Y, Z. Since $OX \perp A'A''$, $OX \perp FX$. Similarly, $OY \perp FY$ and $OZ \perp FZ$, so that X, Y, Z lie on the circle with diameter OF. Suppose, wolog, that F lies on the arc ZX. Then $\angle XZY = \angle XFY = \angle A'FB'' = 60^\circ$ and $\angle ZXY = \angle ZFY = 60^\circ$, so that XYZ is an equilateral triangle and Ptolemy's theorem yields that $FY = FX + FZ$.

Hence

$$\begin{aligned} AA'' + BB'' + CC'' &= (A'A'' + B'B'' + C'C'') - (AA' + BB' + CC') \\ &= 2(A'X + B'Y + C'Z) - (AA' + BB' + CC') \\ &= 2(A'X \pm FX + B'Y \mp FY + C'Z \pm FZ) - (AA' + BB' + CC') \\ &= 2(A'F + B'F + C'F) - (AA' + BB' + CC') \\ &= 4(AF + BF + CF) - 3(AF + BF + CF) \\ &= AF + BF + CF = AA' = BB' = CC' \ . \end{aligned}$$

260. $TABC$ is a tetrahedron with volume 1, G is the centroid of triangle ABC and O is the midpoint of TG. Reflect $TABC$ in O to get $T'A'B'C'$. Find the volume of the intersection of $TABC$ and $T'A'B'C'$.

Solution. Denote by X' the reflection of a point X in O. In particular, $T' = G$. Let D be the midpoint of BC. Since $TT' = TG$ and AA' intersect at O, the points A, G, D, T, A' are collinear. Let A_1 be the intersection of DT and GA'. Since the reflection in O takes any line to a parallel line, $A'G \| AT$, so that (from triangle DTA), $DA_1 : DT = DG : DA = 1 : 3$ and A_1 is the centroid of triangle TBC. Also

$$GA_1 : GA' = GA_1 : AT = DA_1 : DT = 1 : 3$$

so that $GA_1 = (1/3)GA'$.

Applying the same reasoning all around, we see that each side of one tetrahedron intersects a face of the other in its centroid one third of the way along its length. Thus GA' intersects TBC in A_1, GB' intersects TAC in B_1, GC' intersects TAB in C_1, TA intersects $GB'C'$ in A_2, TB intersects $GA'C'$ in B_2 and TC intersects $GA'B'$ in C_2. Note that the $A'_i = A_j$, $B'_i = B_j$, $C'_i = C_j$ for $i \neq j$.

The intersection of the two tetrahedra is a parallelepiped with vertices $T, A_2, B_2, C_2, A_1, B_1, C_1, G$ and faces $TA_2C_1B_2$, $TB_2A_1C_2$, $TC_2B_1A_2$, $GA_1C_2B_1$, $GB_1A_2C_1$, $GC_1B_2A_1$ (to see that, say, $TB_2A_1C_2$ is a parallelogram, note that a dilation with centre T and factor 3/2 takes it to a parallelogram with diagonal TD). The volume of this parallelpiped is three times that of the skew pyramid $TB_2A_1C_2A_2$ with base $TB_2A_1C_2$ and altitude dropped from A_2, which in turn is twice that of tetrahedron $TA_2B_2C_2$. But the volume of tetrahedron $TA_2B_2C_2$ is $1/27 = (1/3)^3$ that of $TABC$ since it can be obtained from $TABC$ by a dilation with centre T and factor 1/3. Hence the volume of the parallelpiped common to both tetraheda $TABC$ and $GA'B'C'$ is $6 \times (1/27) = 2/9$ is the volume of either of these tetrahedra.

261. Let $x, y, z > 0$. Prove that

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(x+y)(y+z)}} + \frac{z}{z + \sqrt{(x+z)(y+z)}} \leq 1 \ .$$

Solution. Observe that

$$(x+y)(x+z) - (\sqrt{xy} + \sqrt{xz})^2 = x^2 + yz - 2x\sqrt{yz} = (x - \sqrt{yz})^2 \geq 0$$

(with equality iff $x^2 = yz$). Hence

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} \leq \frac{x}{x + \sqrt{xy} + \sqrt{xz}} = \frac{\sqrt{x}}{\sqrt{x} + \sqrt{y} + \sqrt{z}} \ ,$$

with a similar inequality for the other two terms on the left side. Adding these inequalities together leads to the desired result.

262. Let ABC be an acute triangle. Suppose that P and U are points on the side BC so that P lies between B and U, that Q and V are points on the side CA so that Q lies between C and V, and that R and W are points on the side AB so that R lies between A and W. Suppose also that

$$\angle APU = \angle AUP = \angle BQV = \angle BVQ = \angle CRW = \angle CWR \; .$$

The lines AP, BQ and CR bound a triangle T_1 and the lines AU, BV and CW bound a triangle T_2. Prove that all six vertices of the triangles T_1 and T_2 lie on a common circle.

Solution 1. Note that the configuration requires the feet of the altitudes to be on the interior of the sides of the triangle and the orthocentre to be within the triangle. Let θ be the common angle referred to in the problem. Let XYZ be that triangle with sides parallel to the sides of triangle ABC and A on YZ, B on ZX and C on XY. Then A, B, C are the respective midpoints of YZ, ZX, XY and the orthocentre H or triangle ABC is the circumcentre of triangle XYZ. [Why?] Let ρ be the common length of HX, HY, HZ.

Let K be the intersection of AP and BQ (a vertex of T_1). Since $\angle KPC + \angle KQC = 180°$, $CQKP$ is concyclic. Hence $\angle AKB + \angle AZB = \angle PKQ + \angle PCQ = 180°$ and $AKBZ$ is concyclic. Since $AH \perp BC$ and $BH \perp AC$, the angle between AH and BH is equal to $\angle ACB = \angle XZY$, so that $AHBZ$ is also concyclic. Thus, A, H, K, B, Z lie on a common circle, so that $\angle HKZ = \angle HBC = 90°$.

Now $\theta = \angle APC = \angle ZAK = \angle ZHK$, so that, in the right triangle HKZ, $|HK| = \rho \cos\theta$. Similarly, it can be shown that the distance from each vertex of triangle T_1 and T_2 from H is $\rho\cos\theta$ and the result follows.

Solution 2. [R. Dan] Let H be the orthocentre of triangle ABC, let AP and BQ intersect at D, and let AU and BQ intersect at E. Triangle APU is isosceles with $AP = AU$, and AH a bisector of $\angle PAU$ and a right bisector of PU. Suppose $X = AH \cap BC$, $Y = BH \cap AC$ and $Z = CH \cap AB$.

Since triangles APU and BQV are similar, $\angle PAH = \angle QBH$, so that $ABDH$ is concyclic and $\angle ADH = \angle ABH$. Similarly, $ACEH$ is concyclic and $\angle AEH = \angle ACH$. Since the quadrilateral $BZYC$ has right angles at Z and Y, $BZYC$ is concyclic and

$$\angle ABH = \angle ABY = \angle ZBY = \angle ZCY = \angle ZCA = \angle ACH \; .$$

Therefore, $\angle ADH = \angle ABH = \angle ACH = \angle AEH$.

Since AH is common, $\angle ADH = \angle AEH$ and $\angle DAH = \angle EAH$, triangles ADH and AEH are congruent (ASA) and $HD = HE$. Thus, H is equidistant from the intersections of BQ with both AP and AU. Similarly, H is equidistant from the intersection of AP and both BQ and BV. Following around, we can show that H is equidistant from all the vertices of triangle T_1 and T_2, and the result follows.

263. The ten digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are each used exactly once altogether to form three positive integers for which the largest is the sum of the other two. What are the largest and the smallest possible values of the sum?

Solution 1. Since the sum has at least as many digits as either of the summands, the sum must have at least four digits. However, the number of digits of the sum cannot exceed one more than the number of digits of the larger summand. Hence, the sum can have at most five digits. However, a five-digit sum must arise from the sum of a four-digit number which is at most 9876 and a single-digit number which is at most 9. Since this means that the sum cannot exceed 9885, we see that a five-digit sum is impossible.

A four-digit sum can arise either as the sum of two three-digit numbers or as the sum of a four-digit and a two-digit number. In the former case, the sum must exceed 1000 and be less than 2000 and, in the latter case, it must be at least 2000.

Thus, the smallest possible sum must be obtained by adding two three-digit numbers to get a four-digit sum. Since the digits of the sum are all distinct, the smallest possible sum is at least 1023. Since

$589 + 437 = 1026$, the smallest sum is at most 1026. We may assume that each digit in the first summand exceeds the corresponding digit in the second summand. The only possibilities for a lower sum are

$$5pq + 4rs = 1023 \ , \quad 6pq + 3rs = 1024 \ , \quad 6pq + 3rs = 1025 \ ,$$

for digits p, q, r, s. One can check that none of these works.

For the largest sum, let the first summand have four digits and the second two. The hundreds digit of the first summand is 9 and the thousands digit of the sum exceeds the thousands digit of the first summand by 1. Since $5987 + 34 = 6021$, the largest sum is at least 6021. The only possibilities to consider for a larger sum are

$$79ab + cd = 80ef \ , \quad 69ab + cd = 70ef \ , \quad 59ab + cd = 60ef \ ,$$

for digits a, b, c, d, e, f. It can be checked that none of these works.

Thus, the smallest sum is 1026 and the largest is 6021.

Solution 2. [C. Shen] As in Solution 1, we eliminate the possibility of a five-digit sum. Suppose that we have

$$a9bc + de = f0gh$$

with digits a, b, c, d, e, f, g, h and $f = a + 1$. There must be a carry from adding the tens digits and we have two possibilities:

$$c + e = h \ , \quad b + d = 10 + g \ ; \qquad (1)$$

$$c + e = 10 + h \ , \quad b + d = 9 + g \ . \qquad (2)$$

In case (1), we have that

$$\begin{aligned} 36 &= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = a + b + c + d + e + f + g + h \\ &= a + 10 + 2g + 2h + f = 2(5 + g + h + a) + 1 \ , \end{aligned}$$

which is impossible, as the two sides have different parities. In case (2), we have that

$$36 = a + 9 + 2g + 10 + 2h + f = 2(10 + a + g + h) \ ,$$

so that $a + g + h = 8$. Since a, g, h are all positive integers, $a \leq 5$ and we have the case $59bc + de = 60gh$ with $c + e \geq 11$. The only possibilities for (c, e) are $(8, 3)$, $(8, 4)$, $(7, 4)$, and these lead to

$$5978 + 43 = 6021 \ , \quad 5978 + 34 = 6012 \ , \quad 5987 + 34 = 6021 \ .$$

The largest sum is 6021.

The smallest sum is at least 1023 and at most $1026 = 589 + 437$. Suppose that

$$pqr + uvw = 102x$$

with $3 \leq x \leq 6$. Since $r + w \geq 3 + 4 = 7$ and $q + v \geq 7$, we have

$$r + w = 10 + x \ , \quad q + v = 11 \ , \quad p + u = 9 \ .$$

Hence

$$\begin{aligned} 42 &= 3 + 4 + 5 + 6 + 7 + 8 + 9 = p + q + r + u + v + w + x \\ &= 9 + 11 + 2x + 10 = 30 + 2x \ , \end{aligned}$$

so that $x = 6$ and 1026 is the smallest sum.

264. For the real parameter a, solve for real x the equation

$$x = \sqrt{a + \sqrt{a + x}} \ .$$

A complete answer will discuss the circumstances under which a solution is feasible.

Solution 1. Suppose that $y = \sqrt{a+x}$. Note that x and y are both nonnegative. Then $x^2 - a = y$ and $y^2 - a = x$, whence

$$0 = (x^2 - y^2) + (x - y) = (x - y)(x + y + 1) \ .$$

Since $x + y + 1 \geq 1$, it follows that $y = x$ and so

$$0 = x^2 - x - a = (x - (1/2))^2 - ((1/4) + a) \ .$$

For a real solution, we require that $a \geq -1/4$. For $-1/4 \leq a \leq 0$, both the sum and the product of the solutions are nonnegative and we get the candidates

$$x = \frac{1 \pm \sqrt{1+4a}}{2} \ .$$

When $a > 0$, the equation has a positive and a negative solution, and only the positive solution

$$x = \frac{1 + \sqrt{1+4a}}{2}$$

is up for consideration.

We check that these solutions work. When $a \geq -1/4$, $x = \frac{1}{2}(1 + \sqrt{1+4a})$ and

$$a + x = \frac{2a + 1 + \sqrt{1+4a}}{2} = \frac{4a + 2 + 2\sqrt{4a+1}}{4} = \left(\frac{1 + \sqrt{4a+1}}{2}\right)^2 ,$$

so that

$$a + \sqrt{a+x} = \frac{2a + 1 + \sqrt{1+4a}}{2} = \left(\frac{1 + \sqrt{4a+1}}{2}\right)^2 = x^2 \ .$$

When $-1/4 \leq a \leq 0$, $x = \frac{1}{2}(1 - \sqrt{1+4a})$,

$$a + x = \frac{2a + 1 - \sqrt{1+4a}}{2} = \left(\frac{1 - \sqrt{4a+1}}{2}\right)^2 .$$

so that

$$a + \sqrt{a+x} = a + \left(\frac{1 - \sqrt{4a+1}}{2}\right) = \left(\frac{1 - \sqrt{4a+1}}{2}\right)^2 = x^2 \ .$$

(Note that, when $a > 0$,

$$\sqrt{a+x} = \frac{\sqrt{4a+1} - 1}{2}$$

and we get an extraneous solution.)

Solution 2. $x = \sqrt{a + \sqrt{a+x}} \Longrightarrow x^2 - a = \sqrt{a+x}$

$$\Longrightarrow 0 = x^4 - 2ax^2 - x + a^2 - a = (x^2 - x - a)(x^2 + x - a + 1) \ .$$

We analyze the possibilities from $x^2 - x - a = 0$ as in Solution 1. If, on the other hand, $x^2 + x - (a-1) = 0$, then $x = \frac{1}{2}(-1 \pm \sqrt{4a-3})$, which is real when $a \geq 3/4$. The possibility $x = \frac{1}{2}(\sqrt{4a-3} - 1)$ leads to

$$x + a = \left(\frac{\sqrt{4a-3} + 1}{2}\right)^2$$

and

$$a + \sqrt{a+x} = \frac{2a + 1 + \sqrt{4a-3}}{2} \neq \left(\frac{\sqrt{4a-3} - 1}{2} \right)^2 .$$

Thus, $x = \frac{1}{2}(\sqrt{4a-3} - 1)$ is extraneous. Since $\frac{1}{2}(-1 - \sqrt{4a-3}) < 0$, $x = \frac{1}{2}(-1 - \sqrt{4a-3}) < 0$ is also extraneous,

Solution 3. For a solution, we require that $x \geq 0$. By squaring twice, we are led to the equation

$$0 = x^4 - 2ax^2 - x + a^2 - a = a^2 - (2x^2 + 1)a + (x^4 - x) .$$

Solving for a yields

$$\begin{aligned} a &= \frac{(2x^2+1) + \sqrt{(2x^2+1)^2 - 4(x^4 - x)}}{2} = \frac{(2x^2+1) + \sqrt{4x^2 + 4x + 1}}{2} \\ &= \frac{(2x^2+1) + (2x+1)}{2} = x^2 + x + 1 , \end{aligned}$$

or

$$a = \frac{(2x^2+1) - (2x+1)}{2} = x^2 - x .$$

(Note that the proper square root has been extracted since $x \geq -1/2$.) In the first case

$$\sqrt{a + \sqrt{a+x}} = \sqrt{x^2 + x + 1 + \sqrt{x^2 + 2x + 1}} = \sqrt{x^2 + 2x + 2} > x .$$

In the second case,

$$\sqrt{a + \sqrt{a+x}} = \sqrt{x^2 - x + \sqrt{x^2}} = \sqrt{x^2} = x .$$

Thus, only the case, $a = x^2 - x$ leads to a valid solution. Note that $a = x^2 - x = (x - \frac{1}{2})^2 - \frac{1}{4}$, so that $a \geq -\frac{1}{4}$ for a solution to work. Since we require $x \geq 0$ and $a = x(x-1)$, we see from the graph of this equation that there are two valid values of x when $-\frac{1}{4} \leq a \leq 0$ and one valid value of x when $0 < a$.

Solution 4. [J. Zhao] For any real x, one of the following must hold:

$$x > \sqrt{a+x} \; ; \qquad x < \sqrt{a+x} \; ; \qquad x = \sqrt{a+x} \; .$$

In case of the first,

$$x > \sqrt{a+x} > \sqrt{a + \sqrt{a+x}} \neq x ,$$

so that such x does not satisfy the equation. Similarly, we can reject any x satisfying the second condition as a solution of the equation. Hence for every solution of the given equation, we must have that $x = \sqrt{a+x}$ or $x^2 - x - a = 0$. We now finish off as in the previous solutions.

Comment. Many solvers did not pay attention to the feasibility of the solutions. Solution 3 was particularly insidious, because it was easy to skip the analysis that only one of the values of x gave a solution when $a > 0$. Surd equations is a dandy topic for students to lose points they should gain because of carelessness or a superficial treatment.

265. Note that $959^2 = 919681$, $919 + 681 = 40^2$; $960^2 = 921600$, $921 + 600 = 39^2$; and $961^2 = 923521$, $923 + 521 = 38^2$. Establish a general result of which these are special instances.

Solution. Let $b \geq 2$ be a base of enumeration. Then we wish to investigate solutions of the system

$$(b^k - u)^2 = b^k v + w \tag{1}$$

$$(u-1)^2 = v + w \tag{2}$$

where k, u, v are positive integers and the integer w satisfies $0 \leq w \leq b^k - 1$. The numerical examples given correspond to $(b, k, u) = (10, 3, 41), (10, 3, 40)$ and $(10, 3, 39)$. Subtracting (2) from (1) yields

$$(b^{2k} - 1) - 2(b^k - 1)u = (b^k - 1)v$$

whence $v = (b^k + 1) - 2u$ and $w = u^2 - b^k$. We require that $b^k \leq u^2 \leq 2b^k - 1$ in order to get a generalization. So, to generate examples of the phenomenon, first select a base b and a parameter k for the number of digits; then select u to satisfy the foregoing inequality. Then one can check, with v and w determined, the desired system of equations holds.

Consider first the situation $b = 10$. When $k = 1$, we have that $u = 4$ and we get the case $6^2 = 36, 3+6 = 3^2$. When $k = 2$, we have that $10 \leq u \leq 14$ and find that $86^2 = 7396, 73+96 = 13^2$ and so on up to $90^2 = 8100, 81+0 = 9^2$. When $k = 3$, we have that $32 \leq u \leq 44$, and find that $956^2 = 913936, 913+936 = 43^2$ and so on up to $968^2 = 937024, 937 + 24 = 31^2$.

Examples from base 3 are $5^2 = (221)_3$, $(2 + 21)_3 = 3^2$; $6^2 = (1100)_3$, $(11 + 0)_3 = 2^2$; $20^2 = (112211)_3$, $(112 + 211)_3 = 6^2$; $21^2 = (121100)_3$, $(121 + 100)_3 = 5^2$.

Comment. In the above system, we could replace $u - 1$ by $u - d$ and get other instances. For example, with $(b, k) = (10, 2)$, we can get the instances $(27^2 = 729, 7 + 29 = 6^2)$, $(29^2 = 841, 8 + 41 = 7^2)$, $(30^2, 3^2)$, $(39^2, 6^2)$, $(40^2, 4^2)$, $(50^2, 5^2)$, $(57^2, 9^2)$. $(60^2, 6^2)$, $(70^2, 7^2)$, $(75^2, 9^2)$, $(78^2, 12^2)$, $(80^2, 8^2)$ and $(98^2, 10^2)$.

Another formulation is to note that the numerical equations are special instances of the system $n^2 = b^k x + y$; $(n - x)^2 = x + y$ with $0 \leq y \leq b^k - 1$ and $0 \leq n < b^k$, where $(n, b, x, y) = (959, 10, 919, 681)$, $(960, 10, 921, 600)$, $(961, 10, 923, 521)$. These equations imply that $x(2n - x) = (b^k - 1)x$, whence $x = 2n - (b^k - 1)$. Thus,

$$n^2 = 2b^k n - b^{2k} + b^k + y \Longrightarrow y = (b^k - n)^2 - b^k \ ,$$

and we require that $b^k \leq (b^k - n)^2 < 2b^k - 1$. The analysis can be continued from here.

This question was, on the whole, badly done. In describing the generalization, one needs to provide a road map whereby one can make the appropriate subsitutions to obtain further examples. Many solvers were content to write down some equations of which the numerical examples were an instance without analyzing the conditions under which the equations could be used to obtain further examples. In effect, no further information was provided to show where other examples might be found.

266. Prove that, for any positive integer n, $\binom{2n}{n}$ divides the least common multiple of the numbers $1, 2, 3, \cdots, 2n - 1, 2n$.

Solution. We first establish that

$$0 \leq \lfloor 2x \rfloor - 2\lfloor x \rfloor \leq 1$$

for each positive real x. ($\lfloor \cdot \rfloor$ refers to "the greatest integer not exceeding".) If, for some integer s, $2s \leq 2x \leq 2s + 1$, then $s \leq x < s + \frac{1}{2}$ and $\lfloor 2x \rfloor = 2\lfloor x \rfloor$; if $2s + 1 \leq 2x < 2s + 2$, then $s + \frac{1}{2} \leq x < s + 1$ and $\lfloor 2x \rfloor = 2s + 1 = 2\lfloor x \rfloor + 1$.

Let p be a prime divisor of $\binom{2n}{n}$, so that $p \leq 2n$. Suppose that p^k is the highest power of p that divides an integer not exceeding $2n$. Then $p^k \leq 2n < p^{k+1}$. The exponent of p in the prime factorization of $\binom{2n}{n}$ is equal to

$$\left(\left\lfloor \frac{2n}{p} \right\rfloor + \left\lfloor \frac{2n}{p^2} \right\rfloor + \left\lfloor \frac{2n}{p^3} \right\rfloor + \cdots + \left\lfloor \frac{2n}{p^k} \right\rfloor \right) - 2\left(\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{n}{p^k} \right\rfloor \right)$$

$$= \sum_{i=1}^{k} \left(\left\lfloor \frac{2n}{p^i} \right\rfloor - 2\left\lfloor \frac{n}{p^i} \right\rfloor \right) \leq k \ .$$

Hence the exponent of p in the prime factorization of $\binom{2n}{n}$ does not exceed the exponent of p in the prime factorization of the least common multiple of the first $2n$ positive integers, for each prime divisor of $\binom{2n}{n}$. The result follows.

267. A non-orthogonal reflection in an axis a takes each point on a to itself, and each point P not on a to a point P' on the other side of a in such a way that a intersects PP' at its midpoint and PP' always makes a fixed angle θ with a. Does this transformation preserves lines? preserve angles? Discuss the image of a circle under such a transformation.

Solution. We suppose that $\theta \neq 90°$. The transformation preserves lines. This is clear for any line parallel to a. Let AB be a line through A that meets a at P, and let A' and B' be the reflective images of A and B. Since $AA' \| BB'$ and a is a median from P of triangles PAA' and PBB', it follows that A', B' and P are colllinear. Thus, any point on the line AP gets carried to a point on the line $A'P$. However, angles are not preserved. A line perpendicular to a is carried to a line making an angle not equal to a right angle with a (while a is kept fixed). [What is the angle of intersection?]

Suppose that the axis of reflection is the y axis. Let A and B be mutual images with A to the left and B to the right of the axis, AB meeting the axis at P and the upper right (and lower left) angles of intersection being θ. If $A \sim (x, y)$ (with $x \leq 0$), then $P \sim (0, y - x\cot\theta)$ and $B \sim (-x, y - 2x\cot\theta)$. If $B \sim (u, v)$ (with $u \geq 0$), then $P \sim (0, v - u\cot\theta)$ and $A \sim (-u, v - 2u\cot\theta)$. Thus the transformation is given by

$$(x, y) \longrightarrow (X, Y) \equiv (-x, y - 2x\cot\theta) \ .$$

Consider the particular case of the circle with equation $x^2 + y^2 = 1$. The image curve has equation

$$1 = (-X)^2 + (Y - 2X\cot\theta)^2 = (1 + 4\cot^2\theta)X^2 - 4XY\cot\theta + Y^2$$

and this does not represent a circle. Thus, circles are not preserved under the transformation. In fact, a general circle with equation of the form $(x - a)^2 + (y - b)^2 = r^2$ gets carried to an second degree curve in the plane which turns out to be an ellipse.

Comment. A synthetic way of analyzing the image of a circle is to note that two chords of a circle that bisect each other must be diameters, and so have the same length. Using this, one can argue that a circle with one diameter along the axis and other perpendicular to the axis does not go to a circle.

268. Determine all continuous real functions f of a real variable for which

$$f(x + 2f(y)) = f(x) + y + f(y)$$

for all real x and y.

Solution 1. First, we show that $f(u) = 0$ if and only if $u = 0$. Suppose that $f(u) = 0$. Then, for all x,

$$f(x) = f(x + 2f(u)) = f(x) + u + f(u) = f(x) + u$$

so that $u = 0$. On the other hand, let $v = f(0)$. Taking $(x, y) = (-2v, 0)$ in the condition yields that

$$v = f(0) = f(-2v + 2v) = f(-2v) + 0 + v \ ,$$

whence

$$0 = f(-2v) = f(-2v + 2f(-2v)) = 0 - 2v + 0 = -2v$$

and $v = 0$.

Setting $y = x$ yields

$$f(x + 2f(x)) = x + 2f(x)$$

for all x. Let $g(x) \equiv x + 2f(x)$. Then $f(g(x)) = g(x)$ so that

$$\frac{1}{2}[g(g(x)) - g(x)] = g(x)$$

whence

$$g(g(x)) = 3g(x) \ .$$

Note also that $g(0) = 0 + 2f(0) = 0$. If $g(x) \equiv 0$, then $f(x) = -x/2$, and this is a valid solution. Suppose that $g(z) = a \neq 0$ for some z and a. Then, as $g(0) = 0$ and g is continuous, g assumes all values between 0 and a (by the intermediate value theorem). But $g(a) = g(g(z)) = 3g(z) = 3a$, so by the same argument, g assumes all values between 0 and $3a$. We can continue on to argue that g assumes all values between 0 and $3^k a$ for each positive integer k. Thus g assumes all positive values if $a > 0$ and assumes all negative values if $a < 0$.

Suppose that the former holds. Then, for all $x \geq 0$, we have that $g(x) = 3x$ and so $x + 2f(x) = 3x$, whence $f(x) = x$. Therefore, when x is arbitrary and $y \geq 0$, $f(x + 2y) = f(x) + 2y$. In particular, $0 = f(-2y + 2y) = f(-2y) + 2y$ so that $f(-2y) = -2y$. Hence, for all x, we must have that $f(x) = x$. A similar argument can be followed to show that $f(x) \equiv x$ when $a < 0$. Therefore, the only two solutions are $f(x) = x$ and $f(x) = -x/2$.

Solution 2. [S. Eastwood] Setting $x = y$, we find that $f(x + 2f(x)) = x + 2f(x)$ for all real x. Let

$$A = \{x + f(x) : x \in \mathbf{R}\} \ .$$

Then A is a nonvoid set. Suppose that $a \in A$. Then $f(a) = a$, so that $3a = a + 2f(a) \in A$. Hence $a \in A \Rightarrow 3a \in A$. Since $x \to x + 2f(x)$ is continuous, it satisfies the intermediate value theorem and so A must be of one of the following types: $A = \{0\}$; $A = [b, \infty)$, $A = [-b, \infty)$, $A = \mathbf{R}$ for some nonnegative value of b.

Suppose that $A = \{0\}$. Then, for all real x, $x + 2f(x) = 0$ and so $f(x) = -x/2$. This is a valid solution.

Suppose that A has a nonzero element a. Then

$$a = f(a) = f(-a + 2a) = f(-a + 2f(a)) = f(-a) + a + f(a) = f(-a) + 2a \ ,$$

whence $f(-a) = -a$ and $-3a = -a + 2f(-a) \in A$. Hence A must contain numbers that are both positive and negative, and so must consist of the whole set of reals. Hence, for all real x, $f(x) = x$, and this also is valid.

Solution 3. [J. Zhao] Suppose that $f(u) = f(v)$. Then, for each x,

$$f(x) + u + f(u) = f(x + f(u)) = f(x + f(v)) = f(x) + v + f(v)$$

so that $u = v$. Hence, f is one-one. Since f is continuous, f is always strictly increasing or always strictly decreasing on $\mathbf{R}$.

Suppose that f is increasing. Then

$$\lim_{x \to +\infty} x + 2f(x) = +\infty$$

and

$$\lim_{x \to -\infty} x + 2f(x) = -\infty \ ,$$

so that $x + f(x)$ assumes every real value (by the intermediate value theorem). Suppose that z is any real number. Select y such that $y + 2f(y) = z$. Then

$$f(z) = f(y + 2f(y)) = f(y) + y + f(y) = z \ ,$$

and so $f(x) \equiv x$. This works.

Suppose that f is decreasing. Suppose, if possible, that p and q are such that $p + 2f(p) < q + 2f(q)$. Then $f(p + 2f(p)) > f(q + 2f(q))$. But, we get a contradiction since $f(p + 2f(p)) = p + 2f(p)$ and

$f(q+2f(q)) = q+2f(q)$. Hence, there is a constant c such that, for all real x, $x+2f(x) = c$. Hence, $f(x) = -\frac{1}{2}(-x+c)$. Plugging this into the functional equation, we find that $c = 0$, and so we obtain the solution $f(x) = -x/2$.

Comments. If we assume that $f(x)$ is a polynomial, then it can be shown that its degree must be 1. Assuming a solution $f(x) = cx+d$ for constants c and d leads to the equations $d = 0 = 2c^2 - c - 1 = (2c+1)(c-1)$. Thus, it is not hard to get a partial solution.

There were a number of approaches to ascertaining that $f(0) = 0$. A. Critch began with the observation that $f(2f(0)) = f(0+2f(0)) = f(0)+0+f(0) = 2f(0)$. Let $a = 2f(0)$, so that $f(a) = a$. Furthermore,

$$f(2a) = f(a+a) = f(a+2f(0)) = a+f(0) = (3a/2)$$

and

$$f(2a) = f(0+2a) = f(0+2f(a)) = f(0)+a+f(a) = (5a/2) .$$

This leads to $a = 0$.

R. Dan noted that $f(2f(y)) = y+f(y)$, and then went on to derive

$$f(-2f(y)+2f(y)) = f(-2f(y))+y+f(y) = f(-2f(y))+f(2f(y)) .$$

Along with the property that $f(0) = 0$, one can then show that f assumes both positive and negative values.

269. Prove that the number

$$N = 2\times 4\times 6\times \cdots \times 2000\times 2002 + 1\times 3\times 5\times \cdots \times 1999\times 2001$$

is divisible by 2003.

Solution 1. We will start with more general observations. Let k be a natural number, $A = 2\times 4\times 6\times \cdots \times (2k)$, $B = 1\times 3\times 5\times \cdots \times (2k-1)$, $C = 2k+1$ and $M = A+B$. Since $1 = C-2k$, $3 = C-(2k-2)$ and so on, $B = (C-2k)(C-(2k-2))\cdots(C-2)$. Upon expansion, we find that the only term in the right side that does not contain C is $(-1)^k \times 2\times 4\times \cdots \times (2k)$. Thus

$$M = C\times \text{ natural number} + (1+(-1)^k)\times A ,$$

so that, when k is odd (for example, when $k = 1001$), M is divisible by C. The result follows.

Solution 2. [T. Yue] Modulo 2003,

$$\begin{aligned}
&2\times 4\times 6\times \cdots \times 2000\times 2002 \\
&\qquad \equiv (-2001)\times(-1999)\times(-1997)\times\cdots\times 3\times 1 \\
&\qquad = -(2001\times 1999\times 1997\times\cdots\times 3\times 1) .
\end{aligned}$$

Therefore, $N \equiv 0 \pmod{2003}$, *i.e.*, N is divisible by 2003.

270. A straight line cuts an acute triangle into two parts (not necessarily triangles). In the same way, two other lines cut each of these two parts into two parts. These steps repeat until all the parts are triangles. Is it possible for all the resulting triangle to be obtuse? (Provide reasoning to support your answer.)

Solution 1. It is clear that if in the final step there are k cuts, made as required, they form $k+1$ triangles. Assume, if possible, that all of these triangles be obtuse. Note the total number of acute or right angles in the configuration after each cut. When the cutting line intersects an existing side of a triangle, it forms two new angles with a sum of $180°$, so that at least one of them is acute or right. When the cutting line passes through a vertex of a triangle, it forms two new angles, dividing the existing angle (smaller than $180°$) into smaller angles, so that there is one more acute or right angle than before. Hence at each step, the total number of acute and right angles in the configuration increases at least by 2. Starting from a configuration with three such angles, after k steps, we get at least $2k+3$ acute or right angles. On the other hand, in $k+1$ obtuse triangles, there must be exactly $2(k+1)$ non-obtuse angles. This contradicts our assumption, so that the answer to the question of the problem is "no".

Solution 2. Suppose that there were a way to cut the given triangle into t obtuse triangles. According to the required procedure of cutting, if two triangles with a common vertex appear after one cut, then they will lie on the same side of the plane with respect to another line segment (say, a side of the triangle or a previous cut). Denote by n the number of points that are vertices of the obtuse triangles but not vertices of the given triangle. On the one hand, the sum of the interior angles in all the triangles is $180t^\circ$. On the other hand, for each of the n points, the sum of all triangular angles at a vertex there is 180°. So the sum of all the interior angles of the triangles will be $(180n+180)^\circ$ (we must add the sum of the angles of the original triangle). Hence $t = n+1$. However, only the n interior vertices can be vertices of an obtuse angle, and each of them can be the vertex of at most one obtuse angle. Hence $t \leq n$, yielding a contradiction. Thus, it is impossible to cut the original triangle into obtuse triangles only.

271. Let x, y, z be natural numbers, such that the number

$$\frac{x - y\sqrt{2003}}{y - z\sqrt{2003}}$$

is rational. Prove that

(a) $xz = y^2$;

(b) when $y \neq 1$, the numbers $x^2 + y^2 + z^2$ and $x^2 + 4z^2$ are composite.

Solution. (a) Since the given number is rational, it can be represented as a reduced fraction p/q, where p and $q \neq 0$ are two coprime integers. This yields

$$xq - yp = (yq - zp)\sqrt{2003} \ .$$

Since the left side is rational, the right must be as well. Since $\sqrt{2003}$ is irrational, both sides must vanish. Thus $xq - yp = yq - zp = 0$, whence $x/y = y/z = p/q$, so that $xz = y^2$.

(b) Let $M = x^2 + y^2 + z^2$ and $N = x^2 + 4z^2$. We will prove that M and N are both composite, provided that $y \neq 1$. Since $xz - y^2$,

$$M = x^2 + y^2 + z^2 = x^2 + 2xz + z^2 - y^2 = (x+z)^2 - y^2 = (x+z-y)(x+z+y) \ .$$

For M to be composite, the smaller factor, $x+z-y$ must differ from 1. (It cannot equal -1. Why?) Since y is a natural number distinct from 1, $y > 1$. As $xz = y^2$, at least one of x and z is not less than y. Say that $x \geq y$. If $x = y$, then $z = y$ and $x + z - y = y > 1$; if $x > y$, then $x + z - y \geq z > 1$. Thus in all possible cases, $x + y - z > 1$ and M is the product of two natural numbers exceeding 1.

Similarly,

$$N = x^2 + 4z^2 = x^2 + 4xz + 4z^2 - 4y^2 = (x+2z)^2 - (2y)^2 = (x+2z-2y)(x+2z+2y) \ .$$

To prove that N is composite, it suffices to show that the smaller factor $x + 2z - 2y$ exceeds 1. (Why cannot this factor equal -1?) We prove this by contradiction. Suppose, if possible, that $x + 2z - 2y = 1$. Then $x + 2z = 2y + 1$, whence

$$x^2 + 4xz + 4z^2 = 4y^2 + 4y + 1 \Leftrightarrow x^2 + 4z^2 = 4y + 1 \ .$$

However, it is clear that $x^2 + 4z^2 \geq 4xz = 4y^2$, from which it follows that $4y + 1 \geq 4y^2$. But this inequality is impossible when $y > 1$. Thus, we conclude that $x + 2z - 2y \neq 1$ and so N is composite.

272. Let $ABCD$ be a parallelogram whose area is 2003 sq. cm. Several points are chosen on the sides of the parallelogram.

(a) If there are 1000 points in addition to A, B, C, D, prove that there always exist three points among these 1004 points that are vertices of a triangle whose area is less that 2 sq. cm.

(b) If there are 2000 points in addition to A, B, C, D, is it true that there always exist three points among these 2004 points that are vertices of a triangle whose area is less than 1 sq. cm?

Solution. (a) Since there are 1000 points on the sides of a parallelogram, there must be at least 500 points on one pair of adjacent sides, regardless of the choice of points. Wolog, let these points be on the sides AB and BC of the parallelogram. and let m of the points $P_1, P_2, \cdots, P_m$ be on AB and k of the points $Q_1, Q_2, \cdots, Q_k$ be on BC. Let P_1 and Q_1 be the points closest to B. Connect the vertex C to $P_1, P_2, \cdots, P_m$ and the point P_1 to $Q_1, Q_2, \cdots, Q_k$ to get $m + k + 1$ triangles the sum of whose areas equals the area of ABC. Thus $[ABC] = \frac{1}{2}[ABCD] = 1001.5$ sq cm. Let us assume that each of these $m + k + 1$ triangles has an area that exceeds 2 sq cm. Then $[ABC] \geq 501 \times 2 = 1002 > 1001.5$, a contradiction. Therefore, at least one of these triangles must have an area of less than 2 sq cm.

(b) No, this is not always true. We will construct a counterexample to justify this answer. Let us choose 2000 points on the sides of $ABCD$ so that 1000 of them are on AB and 1000 of them are on CD. We will consider the first set of 1000 points, and then do symmetrical constructions and considerations for the second set. Using the notation from (a), let $m = 1000, k = 0$ and select the points so that $BP_1 = P_1P_2 = P_2P_3 = \cdots = P_{1000}A$. Then the triangle CBP_1, CP_1P_2, $\cdots$, $CP_{1000}A$ have the same area, say s sq cm. However, $s = [ABC]/1001 = (1001.5)/(1000) > 1$; thus, this choice of the first 1000 points allows a construction of triangles such that the area of each of them exceeds 1 sq cm. Similarly, all triangles formed symmetrically with vertices among the other set of 1000 points have an area which exceeds 1 sq cm. So it is not true that there always exists three points among the chosen 2000 points and the points A, B, C, D that are vertices of a triangle whose area is less than 1 sq cm.

Comments. (1) It was not specified in the text of the question that the three points chosen to be the vertices of a triangle have to be non-collinear. Otherwise, we get the trivial case of a "triangle" with an area of 0, which is not interesting, because $0 < 1$, $0 < 2$ and such a triangle will be an example of existence in both cases. However, it is expected that candidates will make a reasonable interpretation of the problem that renders it nontrivial.

(2) Looking into possible interpretations of this problem, Michael Lipnowski came up with a different, but very similar, and interesting problem. *Let $ABCD$ be a parallelogram whose area is 2003 sq cm. Several points are chosen inside the parallelogram. (a) If there are 1000 points in addition to A, B, C, D, prove that there are always three points among these 1004 points that are vertices of a triangle whose area is less than 2 sq cm. (b) If there are 2000 points in addition to A, B, C, D, is it true that there are always three points among the 2004 points that are vertices of a triangle whose area is less than 1 sq cm.* We provide a solution to this problem. Please note that the answer to (b) differs from the answer of the corresponding part of the original question.

(a) Let $|AB| = x$, $|AD| = y$. Let P and Q lie on AB and CD respectively, so that $PQ \| AD$ and $|AP| = |DQ| = (4/2003)x$. This way, we have a parallelogram "cut" from $ABCD$. Construct analogous parallelograms with respect to the sides AB, BC and CD, drawing lines parallel to these sides, so that each of them has a width of $(4x)/2003$ or $(4y)/2003$ respectively. (1) If at least one of the points lies within the parallelograms "cut", say, R, is within $APQD$, then $[ARD] < (1/2)(4/2003)[ABCD] = (1/2)(4/2003)(2003) = 2$, so this proves what is required. (2) Let us assume that all 1000 points (without the vertices of course) lie within the interior parallelogram $KLMN$ whose vertices are the intersection points of the four lines drawn before. Clearly, it is similar to $ABCD$, and the coefficient of proportionality is 1995/2003, so its area is $(1995/2003)^2 \cdot (2003) = (1995^2)/(2003)$. Divide $KLMN$ into 499 congruent parallelograms (for example, by drawing 498 equally spaced lines parallel to KL). Then, since $1000 = 2 \times 499 + 2$ points lie inside $KLMN$, at least one of the 499 parallelograms contains at least three of them, according to the extended pigeonhole principle. Consider the triangle formed by them. Since each of these parallelograms has an area equal to $(1/499)[KLMN] = (1/499)(1995^2/2003) < (1995 \cdot 1996)/(499 \cdot 2003) = 4 \cdot (1995/2003) < 4$, then the area of the triangle will not exceed half of 4, namely 2. So there must be at least one triangle inside $ABCD$ of area less than 2.

(b) Yes, it is always true that there exists three among the 2004 points that are vertices of a triangle with area less than 1. Proceed as in (a) except for the following differences: (1) Construct the parallel lines so that the width of the "cut" parallelograms is $(2x)/2003$ or $(2y)/2003$, respectively. Now, the parallelogram $KLMN$ is similar to $ABCD$, with a coefficient of proportionality $1999/2003$ and an area of $(1999^2)/2003$. (2) Divide $KLMN$ into 999 congruent parallelograms. Since $2000 = 2\times 999+2$ points lie within 999 regions, at least one region contains at least three of the points. Similar calculations show that in this case, the area of the triangle formed by these three points has area less than 1. The result holds.

273. Solve the logarithmic inequality

$$\log_4(9^x - 3^x - 1) \geq \log_2 \sqrt{5} \ .$$

Solution. Let $3^x = y$. Then $y > 0$ and the given inequality is equivalent to $\log_4(y^2 - y - 1) \geq \log_2 \sqrt{5}$. Since the logarithmic function is defined only for positive numbers, we must have $y^2 - y - 1 > 0$. In this domain, the inequality is equivalent to $y^2 - y - 1 \geq 5$ or $(y+2)(y-3) \geq 0$. The solution of the last inequality consists of all numbers not less than 3 (since $y > 0$). Hence $3^x \geq 3$ or $x \geq 1$. Thus, the inequality is satisfied if and only if $x \geq 1$.

Comment. It is very important before starting to solve the inequality to determine the domains so that all functions are well-defined. It is mandatory to take these restrictions into consideration for the final answer as well as along the way in making transformations.

274. The inscribed circle of an isosceles triangle ABC is tangent to the side AB at the point T and bisects the segment CT. If $CT = 6\sqrt{2}$, find the sides of the triangle.

Solution. Denote the midpoint of CT by K, and the tangent point of the inscribed circle and BC by L. Then, from the given information,

$$CK = \frac{1}{2} CT \ . \tag{1}$$

We will use the standard notation a, b, c for the lengths of BC, CA and AB, respectively. It is not specified which two sides of the isosceles triangle are equal, so there are two possible cases.

Case 1. $AC = BC$ or $a = b$. Then T is also the midpoint of AB. By the tangent-secant theorem, $CL^2 = CK \cdot CT$, which together with (1) implies that $(a - (c/2))^2 = CL^2 = (1/2)CT^2 = 36$. Hence $a = 6 + (c/2)$ (2).

On the other hand, from the Pythagorean theorem applied to triangle BCT, we get that $a^2 = (c^2/4)+72$. Using (2), we obtain that

$$\left(6 + \frac{c}{2}\right)^2 = \frac{c^2}{4} + 72 \Leftrightarrow 36 + 6c = 72 \Leftrightarrow c = 6 \ ,$$

whence $a = b = 9$. So the lengths of the triangle are $(a, b, c) = (9, 9, 6)$.

Case 2. $AB = AC$ or $c = b$. Now L is the midpoint of the side BC so that

$$CL^2 = CK \cdot CT = (1/2)CT^2 \Leftrightarrow (a^2/4) = (1/2)(6\sqrt{2})^2 = 36 \Leftrightarrow a = 12 \ . \tag{3}$$

Next we have to calculate the lengths of AB and AC. From the cosine law, applied to triangle BCT with $\beta = \angle ABC$,

$$\begin{aligned} CT^2 &= BT^2 + BC^2 - 2BT \cdot BC \cos\beta \\ &\Leftrightarrow (6\sqrt{2})^2 = (a^2/4) + a^2 - a^2 \cos\beta \\ &\Leftrightarrow 72 = 36 + 144 - 144 \cos\beta \Leftrightarrow \cos\beta = 3/4 \ . \end{aligned}$$

On the other hand, the cosine law for triangle ABC leads to

$$\begin{aligned} b^2 &= c^2 + a^2 - 2ca\cos\beta = b^2 + a^2 - 2ba\cos\beta \\ &\Leftrightarrow \cos\beta = a/2b \ . \end{aligned}$$

This, with (3), implies that $c = b = 8$. Therefore, $(a, b, c) = (8, 8, 12)$.

275. Find all solutions of the trigonometric equation

$$\sin x - \sin 3x + \sin 5x = \cos x - \cos 3x + \cos 5x \ .$$

Solution 1. [M. Lipnowski] Note that, if $x = \theta$ satisfies the equation, then so does $x = \theta + \pi$. Thus, it suffices to consider $0 \leq x \leq \pi$. A simple computation shows that $x = \pi/2$ is not a solution, so that we may assume that $\cos x \neq 0$. Multiplying both sides of the equation by $2\cos x \neq 0$ yields that

$$\begin{aligned} &\sin x - \sin 3x + \sin 5x = \cos x - \cos 3x + \cos 5x \\ &\quad \Leftrightarrow 2\sin x\cos x - 2\sin 3x\cos x + 2\sin 5x\cos x = 2\cos^2 x - 2\cos 3x\cos x + 2\cos 5x\cos x \\ &\quad \Leftrightarrow \sin 2x - (\sin 4x + \sin 2x) + \sin 6x + \sin 4x) = 1 + \cos 2x - (\cos 4x + \cos 2x) + (\cos 6x + \cos 4x) \\ &\quad \Leftrightarrow \sin 6x - \cos 6x = 1 \ . \end{aligned}$$

Squaring both sides of the last equation, we get

$$\sin^2 6x - 2\sin 6x\cos 6x + \cos^2 6x = 1 \Leftrightarrow \sin 12x = 0 \ .$$

This equation has as a solution $x = k\pi/12$ for k an integer. Checking each of these for validity, we find that the solutions are $x = \pi/12$, $2\pi/12$, $5\pi/12$, $9\pi/12$, $10\pi/12$, and the general solution is obtained by adding a multiple of π to each of these.

Solution 2. The given equation is equivalent to

$$2\sin 3x\cos 2x - \sin 3x = 2\cos 3x\cos 2x - \cos 3x$$

$$\Leftrightarrow (2\cos 2x - 1)(\sin 3x - \cos 3x) = 0 \ .$$

Thus, either $\cos 2x = \frac{1}{2}$ in which case $x = \pm(\pi/6) + k\pi$ for some integer k, or $\cos 2x \neq \frac{1}{2}$. In the latter case, we must have $\cos 3x \neq 0$ (why?), so that $\tan 3x = 1$ and $x = (\pi/12) + (k\pi/3)$. Thus, all solutions of the equation are $x = (\pi/12) + k\pi$, $(\pi/6) + k\pi$, $(5\pi/12) + k\pi$, $(3\pi/4) + k\pi$ and $(5\pi/6) + k\pi$ where k is an integer.

276. Let a, b, c be the lengths of the sides of a triangle and let $s = \frac{1}{2}(a + b + c)$ be its semi-perimeter and r be the radius of the inscribed circle. Prove that

$$(s - a)^{-2} + (s - b)^{-2} + (s - c)^{-2} \geq r^{-2}$$

and indicate when equality holds.

Solution 1. Let the angles of the triangle at A, B, C be 2α, 2β, 2γ, respectively, Then $(s - a)^{-1} = (\tan\alpha)/r$, *etc.*, and the ineqaulity is equivalent to

$$\tan^2\alpha + \tan^2\beta + \tan^2\gamma \geq 1 \ .$$

Since $\alpha + \beta + \gamma = 90°$,

$$\begin{aligned} 1 &= \tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma + \tan\alpha \\ &\leq \sqrt{\tan^2\alpha + \tan^2\beta + \tan^2\gamma}\sqrt{\tan^2\beta + \tan^2\gamma + \tan^2\alpha} \ , \end{aligned}$$

by the Cauchy-Schwarz Inequality. Equality occurs if and only if $\tan\alpha = \tan\beta = \tan\gamma = 1/\sqrt{3}$, *i.e.*, when the triangle is equilateral.

Solution 2. Let $u = (s-a)^{-1}$, $v = (s-b)^{-1}$ and $w = (s-c)^{-1}$. Then $(u-v)^2 + (v-w)^2 + (w-u)^2 \geq 0$ implies that $u^2 + v^2 + w^2 \geq uv + vw + wu$. Hence

$$\begin{aligned}
&\frac{1}{(s-a)^2} + \frac{1}{(s-b)^2} + \frac{1}{(s-c)^2} \\
&\qquad \geq \frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)} \\
&\qquad = \frac{(s-a)+(s-b)+(s-c)}{(s-a)(s-b)(s-c)} = \frac{s}{(s-a)(s-b)(s-c)} \ .
\end{aligned}$$

Since the area of the triangle is $rs = \sqrt{s(s-a)(s-b)(s-c)}$, we have that $(s-a)(s-b)(s-c) = r^2 s$, and the desired result follows. Equality occurs if and only if $u = v = w \Leftrightarrow a = b = c$.

Solution 3. [S. Seraj] Let $a = v+w$, $b = w+u$, $c = u+v$; the condition that a, b, c are sides of a triangle is equivalent to u, v, w being all positive. By the Arithmetic-Geometric Means Inequality, we have that

$$x^2y^2 + z^2x^2 \geq 2x^2yz$$

for any positive reals x, y, with equality if and only if $y = z$. Applying this to the three numbers u, v, w in cyclic order and adding, we find that

$$u^2v^2 + v^2w^2 + w^2u^2 \geq (u+v+w)uvw$$

from which we find that

$$(s-a)^2(s-b)^2 + (s-b)^2(s-c)^2 + (s-c)^2(s-a)^2 \geq s(s-a)(s-b)(s-c) = r^2s^2 \ .$$

Dividing by $(s-a)^2(s-b)^2(s-c)^2 = r^4s^2$ yields the desired result. Equality occurs if and only if $u = v = w$, *i.e.* when the triangle is equilateral.

Comment. We can also use the Arithmetic-Geometric Means Inequality to obtain that

$$\frac{1}{(s-a)^2} + \frac{1}{(s-b)^2} \geq \frac{2}{(s-a)(s-c)} = \frac{2(s-c)}{(s-a)(s-b)(s-c)} \ ,$$

etc., and add the inequalities to get the result. Many solvers neglected to mention when equality occurred.

277. Let m and n be positive integers for which $m < n$. Suppose that an arbitrary set of n integers is given and the following operation is performed: select any m of them and add 1 to each. For which pairs (m, n) is it always possible to modify the given set by performing the operation finitely often to obtain a set for which all the integers are equal?

Solution. If the task can be completed, then it can be completed in particular when the sum of the integers in the set is equal to 1 (for example, if there is one 1 and the rest all 0). Suppose that the operation is performed x times, so that the sum is increased by m each time, until all the numbers are equal to y. Then we must have $1 + mx = ny$, from which it follows that the greatest common divisor of m and n is equal to 1.

Conversely, suppose that the greatest common divisor of m and n is equal to 1. Then it is possible to find positive integers u and v for which $mu = nv + 1$. For convenience, let us suppose that the numbers in the set are arranged in a ring. We show that it is possible to increase any of these numbers by one more than we increase the rest of the numbers if the operation is repeated suficiently often. Suppose the numbers are $a_1, a_2, \cdots, a_n$ in order around the ring, and we wish to increase a_1 by one more than the rest. Begin by adding 1 to each of $a_1, a_2, \cdots, a_m$; then add 1 to each of a_{m+1}, a_{m+2} and so on for m numbers. Each time, increase a run of m numbers by 1, starting off immediately after the last number increased on the previous round. Doing this u times, we find that each number is increased by v except for a_1 which is increased by $v + 1$.

To achieve our task, begin with a sequence of operations, each of which increases the minimum number of the set by one more than each other number. After a finite number of times of doing this, the difference between the maximum and minimum numbers of the set will be reduced by 1. Eventually, this difference will be reduced to zero and the job will be done.

Comment. One approach is to increase the smallest m numbers of the set by 1 each time around. This looks as though it should succeed, but it seems difficult to establish that this is so.

278. (a) Show that $4mn - m - n$ can be an integer square for infinitely many pairs (m, n) of integers. Is it possible for either m or n to be positive?

(b) Show that there are infinitely many pairs (m, n) of positive integers for which $4mn - m - n$ is one less than a perfect square.

Solution 1. (a) Two possible solutions are $(m, n) = (-(5k^2 \pm 2k), -1)$ and $(m, n) = (-a^2, 0)$. Suppose, if possible, that $4mn - m - n = x^2$ with at least m and n positive. Then $(4m-1)(4n-1) = 4x^2 + 1$. There must be at least one prime q congruent to -1 modulo 4 which divides $4m - 1$ and so $4x^2 + 1$. Therefore, $(2x)^2 \equiv -1 \pmod{q}$, whence $(2x)^4 \equiv 1 \pmod{q}$. By Fermat's Little Theorem, $(2x)^{q-1} \equiv 1 \pmod{q}$. Observe that neither $2x$ nor $(2x)^3$ is congruent to $\pm 1 \pmod{q}$, so that 4 is the minimum positive value of r for which $(2x)^r \equiv 1 \pmod{q}$. Let $q = 4s + 3$. Then

$$(2x)^{q-1} = (2x)^{4s} \cdot (2x)^2 \equiv (2x)^2 \not\equiv 1$$

$\pmod{q}$, which contradicts the Fermat result. Hence, it is not possible for either m or n to be positive.

(b) One set of solutions is given by $(m, n) = (3k^2, 1)$.

Solution 2. Examples of solutions can be given as in the foregoing solution. Suppose if possible, there exist m, n, k with m a positive integer, for which $(4m - 1)(4n - 1) = 4k^2 + 1$. Then this means that $4k^2 + 1$ has a positive factor congruent to $-1 \pmod 4$. Let r be the smallest positive value of k for which such a factor of $4k^2 + 1$ exists. Then

$$4r^2 + 1 = ab$$

where a, b are positive integers exceeding 1 and congruent to -1 modulo 4 and $a \leq b$. Then $4r^2 + 1 \geq a^2 \Rightarrow 2r > a \Rightarrow r > a - r$. Clearly, $r - a < r$, whence $|r - a| < r$. Then

$$4(r - a)^2 + 1 = (4r^2 + 1) + 4a(a - 2r) = a(4a + b - 8r)$$

so that $|r - a|$ is a smaller value of k than r for which $4k^2 + 1$ has a factor congruent to -1 modulo 4. (Why is $r - a$ not 0?) This contradicts the definition of r, and so no solution with m, n positive is possible.

279. (a) For which values of n is it possible to construct a sequence of abutting segments in the plane to form a polygon whose side lengths are $1, 2, \cdots, n$ exactly in this order, where two neighbouring segments are perpendicular?

(b) For which values of n is it possible to construct a sequence of abutting segments in space to form a polygon whose side lengths are $1, 2, \cdots, n$ exactly in this order, where any two of three sucessive segments are perpendicular?

Solution. (a) Since the direction of the sides alternates around the polygon, n must be even. Let $n = 2k$. Suppose that the odd sides of the polygon are parallel to the $x-$axis and the even sides to the $y-$axis. Then along each odd side, the abscissa of the vertices increases or decreases by an odd integer, and for all the sides, the net increase of the abscissae is zero. In other words, modulo 2,

$$0 = \pm 1 \pm 3 \pm 5 \pm \cdots \pm (2k-1) \equiv 1 + 3 + 5 + \cdots + (2k-1) = k^2 ,$$

whence k is even, and n is a multiple of 4. Similarly, consideration of the ven sides leads to, modulo 4,

$$0 = 2(\pm 1 \pm 2 \pm 3 \pm \cdots \pm k) \equiv 2(1 + 2 + \cdots + k) = k(k+1)$$

from which we infer that k must be a multiple of 4. Hence, it is necessary that n is a multiple of 8.

Now, suppose that n is a multiple of 8. For $n = 8$, we can construct the octagon with vertices $(0,0)$, $(1,0)$, $(1,2)$, $(-2,2)$, $(-2,-2)$, $(-7,-2)$, $(-7,-8)$, $(0,-8)$, corresponding to the sums

$$1 - 3 - 5 + 7 = 2 - 4 - 6 + 8 = 0 .$$

In general, for $n = 8r$ $(r \geq 1)$, we can take the polygon corresponding to the sums

$$1 + 3 + \cdots + (2r-1) - (2r+1) - \cdots - (6r-1) + (6r+1) + \cdots + (8r-1) = 0$$

and

$$2 + 4 + \cdots + 2r - (2r+2) - \cdots - (6r) + (6r+2) + \cdots + 8r = 0 .$$

(b) By the condition of the problem, the sides of the parallelogram must cycle through the three coordinate directions in turn, so n must be a multiple of 3, say $n = 3k$. As in (a), we can argue that, for some choice of signs, with the congruence taken modulo 2,

$$0 = \pm 1 \pm 4 \pm \cdots \pm (3k-2) \equiv 1 + 4 + \cdots + (3k-2) = \frac{k(3k-1)}{2} ,$$

$$0 = \pm 2 \pm 5 \pm \cdots \pm (3k-1) \equiv 2 + 5 + \cdots + (3k-1) = \frac{k(3k+1)}{2} ,$$

$$0 = \pm 3 \pm 6 \pm \cdots \pm 3k \equiv 3(1 + 2 + \cdots + k) = \frac{3k(k+1)}{2} .$$

Hence, $k(3k-1)$, $k(3k+1)$ and $3k(k+1)$ are all divisible by 4. This is possible if and only if k is a multiple of 4, say $4r$, so that $n = 12r$ is a multiple of 12.

On the other hand, suppose that $n = 12r$, for $r \geq 1$. We can construct polyons corresponding to the sums

$$1 + 4 + \cdots + (3r-2) - (3r+1) - \cdots - (9r-2) + (9r+1) + \cdots + (12r-2) = 0 ,$$

$$2 + 5 + \cdots + (3r-1) - (3r+2) - \cdots - (9r-1) + (9r+2) + \cdots + (12r-1) = 0 ,$$

$$3 + 6 = \cdots + 3r - (3r+3) - \cdots - 9r + (9r+3) + \cdots + 12r = 0 ,$$

for the lengths of the sides in the three respective coordinate directions.

280. Consider all finite sequences of positive integers whose sum is n. Determine $T(n,k)$, the number of times that the positive integer k occurs in all of these sequences taken together.

Solution. Each ordered partition of n corresponds to a placement of vertical lines between certain adjacent pairs of dots in a line of n dots. For example, the sequence $\{4,2,3,1\}$ partitioning 10 corresponds to

$$\cdot\cdot\cdot\cdot|\cdot\cdot|\cdot\cdot\cdot|\cdot \qquad .$$

Suppose, first that $k = n$. Then there is one possible sequence, and so $T(n,n) = 1$. If $k = n-1$, then there are two sequences ($\{1, n-1\}$ and $\{n-1, 1\}$), so $T(n, n-1) = 2$. Henceforth, let $2 \leq k \leq n-2$.

If k is the initial term of the sequence, the first vertical line will occur after the kth dot, and there are $n-k-1$ position between adjacent remaining dots in which lines might be placed to signify the sequences beginning with k. There are 2^{n-k-1} such possibilities. Similarly, there are 2^{n-k-1} possibilities where k is the final term of the sequence. Thus, k occurs 2^{n-k} times as the first or the last term of a sequence.

Suppose that that k occurs in an intermediate position, and that the sum of the terms preceding k is equal to s and the sum of the terms succeeding k is equal to $n-k-s>0$. By an argument similar to that in the previous paragraph, there are $2^{s-1} \cdot 2^{n-k-s-1} = 2^{n-k-2}$ sequences where the terms before k have a sum s. Since we have that $1 \leq s \leq n-k-1$, there are $(n-k-1)2^{n-k-2}$ occurrences of k in an intermediate position in the sequences.

Therefore, the total number of occurrences of k in all sequences is

$$2^{n-k} + (n-k-1)2^{n-k-2} = (n-k+3)2^{n-k} ,$$

for $1 \leq k \leq n-1$, and $T(n,n) = 1$.

281. Let a be the result of tossing a black die (a number cube whose sides are numbers from 1 to 6 inclusive), and b the result of tossing a white die. What is the probability that there exist real numbers x, y, z for which $x+y+z=a$ and $xy+yz+zx=b$?

Solution. Eliminating z from the system, we obtain the equation

$$x^2 + (y-a)x + (y^2 - ay + b) = 0 .$$

This is solvable for real values of x if and only if

$$(y-a)^2 - 4(y^2 - ay + b) \geq 0 \Longleftrightarrow -3y^2 + 2ay + (a^2 - 4b) \geq 0 .$$

Since $-3y^2+2ay+(a^2-4b)$ is negative for large values of y, the inequality $-3y^2+2ay+(a^2-4b) \geq 0$ is solvable for real values of y if and only there are real solutions to the quadratic equation $3y^2 - 2ay - (a^2-4b) = 0$, and the condition for this is $a^2 \geq 3b$.

So, if $a^2 \geq 3b$, we can find a solution for the inequality in y, and then solve the equation for x, and then set $z = a - x - y$. So the system is solvable if and only if $a^2 \geq 3b$, and this occurs when $a \geq 5$ or when $(a,b) = (2,1), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), (4,5)$. (In particular, when $(a,b) = (3,3)$, $(x,y,z) = (a/3, a/3, a/3)$ necessarily.) The required probability is $21/36 = 7/12$.

Solution 2. The system of equations is equivalent to the system: $xy + (x+y)[a-(x+y)] = b$; $z = a-(x+y)$. The first of these equations can be written as

$$3(2x+y-a)^2 + (3y-a)^2 = 4(a^2-3b) .$$

Hence, we require that $a^2 \geq 3b$. On the other hand, if $a^2 \geq 3b$, then we can determine a real pair (u,v) for which $3u^2 + v^2 = 4(a^2-3b)$. Then

$$(x,y,z) = \left(\frac{3u - v + 2a}{6}, \frac{a+v}{3}, \frac{2a - 3u - v}{6} \right)$$

satisfies the system. There are 21 of the possible 36 outcomes of casting the dice for which $a^2 \geq 3b$, so the desired probability is 7/12.

Comment. The necessity of the condition $a^2 \geq 3b$ also follows from

$$a^2 - 3b = (x^2+y^2+z^2) - (xy+yz+zx) = \frac{1}{2}[(x-y)^2 + (y-z)^2 + (z-x)^2] \geq 0 .$$

282. Suppose that at the vertices of a pentagon five integers are specified in such a way that the sum of the integers is positive. If not all the integers are non-negative, we can perform the following operation: suppose that x, y, z are three consecutive integers for which $y < 0$; we replace them respectively by the integers $x + y, -y, z + y$. In the event that there is more than one negative integer, there is a choice of how this operation may be performed. Given any choice of integers, and any sequence of operations, must we arrive at a set of nonnegative integers after a finite number of steps?

For example, if we start with the numbers $(2, -3, 3, -6, 7)$ around the pentagon, we can produce $(1, 3, 0, -6, 7)$ or $(2, -3, -3, 6, 1)$.

Solution. Let x_1, x_2, x_3, x_4, x_5 be the five numbers in order around the pentagon at some particular point, and suppose that $x_3 < 0$ and we change the numbers to $x_1, x_2 + x_3, -x_3, x_4 + x_3, x_5$. Observe that under the operation, the sum of the numbers remains unchanged, and so always positive, Let S be the sum of the squares of the differences

$$S = (x_1 - x_3)^2 + (x_2 - x_4)^2 + (x_3 - x_5)^2 + (x_4 - x_1)^2 + (x_5 - x_2)^2 \ ,$$

and T be the corresponding sum of squares after the operation has been performed:

$$T = (x_1 + x_3)^2 + (x_2 - x_4)^2 + (x_3 + x_5)^2 + (x_4 + x_3 - x_1)^2 + (x_5 - x_2 - x_3)^2 \ .$$

Then $S - T = -2x_3(x_1 + x_2 + x_3 + x_4 + x_5) > 0$ since $x_3 < 0$ and $x_1 + x_2 + x_3 + x_4 + x_5 > 0$.

Each time we perform the operation, the sum S of the squares decreases. Since this sum is a positive integer, this can happen only finitely often, so we must come to a stage at which there is no negative number available to operate on. The result follows.

Comment. It appears to be the case that the number of operations required and the final configuration when all the numbers are nonnegative is independent of the choice of the negative number on which to operate at each stage.