BOOK REVIEW

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Liping Ma, *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States.* Lawrence Erlbaum Associates, Inc., 10 Industrial Avenue, Mahwah, NJ 07430-2262 (`orders@erlbaum.com`), 1999 (ISBN 0-8058-2909-1) xxv+166 pages

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Before its appearance as a book, Liping Ma's thesis was an underground publication enthusiastically read and commented upon by several prominent American mathematicians actively interested in school education. Dismayed at the apparent de-emphasis of technical proficiency in educational reform, they found in Liping Ma's work affirmation of their sense that conceptual development and competence can both be accommodated in the curriculum, provided that teachers are up to the job.

Thirty years ago, the author, a middle-school student in Shanghai, was shipped off to the countryside to "learn from the peasants" as part of the notorious Cultural Revolution in China. Within months of her arrival, the village leader asked her to teach at the local elementary school. This set her on a career that saw her become principal of the school, then superintend the elementary schools of the county and eventually earn a master's degree at East China Normal University. In 1989, she began studying at Michigan State University, where she participated in a survey of the mathematical understanding of elementary teachers.

She was puzzled at the apparent systemic lack of thorough knowledge of mathematics among American teachers compared to their Chinese counterparts, despite having had a much higher level of formal education. The exploration of this phenomenon constituted the research for her doctoral dissertation under Lee S. Shulman at Stanford University. Her study involved 23 "better than average" teachers participating in professional development or graduate intern programs in the USA and a sample of 72 teachers from five elementary schools of varying reputation in China. The American teachers reported themselves to be comfortable with mathematics, while the Chinese teachers were more broadly representative.

The teachers were given four scenarios to discuss mathematically and pedagogically:

(1) teaching subtraction of two-digit numbers with regrouping;

(2) dealing with the erroneous multiplication:

$$\begin{array}{cr} & 123 \\ \times & 645 \\ & --- \\ & 615 \\ & 492 \\ & 738 \\ & --- \\ & 1845 \end{array}$$

(3) determining $1\frac{3}{4} \div \frac{1}{2}$ and creating a story or model for which this division is pertinent;

(4) responding to the student who asserts that, because a 4×4 rectangle has perimeter 16 and area 16 while a 4×8 rectangle has perimeter 24 and area 32, the area of a closed figure increases as the perimeter increases.

The first four chapters analyze the responses of the teachers to these situations, and the last two deal with *Profound Understanding of Elementary Mathematics* or *PUFM*: what it is, how one determines whether teachers have it, and how it can be attained. A short conclusion and a substantial bibliography wrap up the monograph.

What were the findings that so excited our mathematical colleagues? On the whole, the Chinese teachers saw the items as part of *knowledge packages* that had to be strategically presented to pupils in order for them to grasp the fundamental principles. While American teachers tended to appeal to a rote borrowing rule in substraction, Chinese teachers were more likely to refer to the decomposition of higher place values and to ground the algorithm in basic subtraction within 10 and within 20. Many American teachers treated the erroneous multiplication as a procedural mishap, while most of the Chinese teachers sought to diagnose the mistake and give an explanation that focussed on the properties of the place value system. Here, for example, is the comment of Teacher Wong:

> We need to deepen students' understanding of place value. Their concept of place value used to be pretty straightforward. The basic unit of a number is always the one at the ones place. When they saw a number 492, it always meant 492 ones. When they saw a number 738, it always meant 738 ones. But now the place value of the basic unit is no longer a unique one. It changes according to the context. For example, the place value of the 4 in the problem is ten. When we multiply 123 by the 4, we regard it as 4 tens. The tens becomes the place value of the product 492. It is not 492 ones, like it is in the students' work, but 492 tens. That is why we put the 2 at the tens place. The same happens when we multiply 123 by the 6, which we regard as 6 hundreds. [$\cdots$] To correct the students' mistake we should expand their understanding of place value, to help them to think of the concept in a flexible way. (p. 43)

The differences between the national groups were more striking in the fraction example. All 72 Chinese teachers gave the correct answer; of the US teachers, eleven had the correct algorithm but two did not give a complete answer, and the reaction of the remainder varied between uncertainty and incomprehension. The Chinese teachers elaborated on the validity of the invert-and-multiply rule. Only one of the 23 American teachers generated a correct representation for the meaning of the equation, while 65 of the Chinese teachers created among them more than 80 problems, picking up different themes, such as the measurement and the partitive models of division, and the determination of a factor that when multiplied by $\frac{1}{2}$ will give $1\frac{3}{4}$. Ms. D., for example, said,

> The equation of $1\frac{3}{4} \div \frac{1}{2} =$ can be represented from different perspectives. For instance, we can say, here is $1\frac{3}{4}$ kg sugar and we want to wrap it into packs of $\frac{1}{2}$ kg each. How many packs can we wrap? Also, we can say that here we have two packs of sugar, one of white sugar and the other of brown sugar. The white sugar is $1\frac{3}{4}$ kg and the brown sugar is $\frac{1}{2}$ kg. How many times is the weight of white sugar of that of brown sugar? Still, we can say that here is some sugar on the table that weighs $1\frac{3}{4}$ kg; it is $\frac{1}{2}$ of the sugar we now have at home, so how much sugar do we have at home? All three stories are about sugar, and all of them represent $1\frac{3}{4} \div \frac{1}{2}$. But the numerical models they illustrate are not the same. I would put the three stories on the board and invite my students to compare the different meanings they represent. After discussion I would ask them to try to make up their own story problems to represent the different models of division by fractions. (p. 80)

Ma notes that American and Chinese teachers seem to operate from different concepts of fractions. While the US teachers generally deal with "real" and "concrete" wholes and their fractions, Chinese teachers go beyond this. Teaching operations with fractions, they tend to use "abstract" and "invisible" wholes, such as the length of a particular stretch of road or the length of time taken to complete a task. Similar differences were manifest in the perimeter-area scenario. While the American teachers tried to engage the student in exploring the conjecture, their own lack of appreciation of the issues did not allow this to happen in more than a superficial way; two accepted the student's statement and only one investigated the matter thoroughly enough to reach a sound conclusion. The Chinese teachers started out similarly uncertain, but then approached the situation more systematically. Their responses ranged from simply providing a counterexample to analyzing under what circumstances the conjecture might hold and indicating where a counter-example might be found. About one fifth of the Chinese teachers failed to produce a fine enough analysis to get a correct solution.

For each of the four scenarios, the author provides a conceptual map involving related topics and techniques that provides a solid critical foundation for discussing the teachers' approaches. As for the teachers temselves, she comes to this conclusion:

> Considered as a whole, the knowledge of the Chinese teachers seemed clearly coherent while that of the US teachers was clearly fragmented. Although the four topics in this study are located at various levels and subareas of elementary mathematics, while interviewing the Chinese teachers I could perceive interconnections among their discussions of each topic. From the US teachers' responses, however, one can hardly see any connection among the four topics. Intriguingly, the fragmentation of the US teachers' mathematical knowledge coincides with the fragmentation of mathematics curriculum and teaching in the US found by other researchers as major explanations for unsatisfactory mathematics learning in the United States. [$\cdots$] From my perspective, however, this fragmentation and coherence are effects, not causes. Curricula, teaching, and teachers' knowledge reflect the terrains of elementary mathematics in the United States and in China. What caused the coherence of the Chinese' knowledge, in fact, is the mathematical substance of their knowledge. (pp. 107-108)

Even though the American teachers, unlike the Chinese ones, had a college education that included some advanced mathematics, it is not to such courses and exposure to abstract structures that we look to provide this substance. The cause is much more subtle, and a major strength of Ma's book is her ability to put a finger on it. The better Chinese teachers exhibited *Profound Knowledge of Fundamental Mathematics*; "profound" here carries the triple connotation of *deep*, *vast* and *thorough*. This has four characteristics. The first is a sense of the connectedness of mathematics; the teachers have knowledge packages with a central core linked to ancillary topics. Those with *PUFM* entertain multiple perspectives, can analyze their advantages and disadvantages, and lead students to a flexible understanding. Thirdly, there is an awareness of the importance of simple but powerful ideas, which are especially stressed and developed. Finally, their teaching exhibits longitudinal coherence, with a sense of the elementary curriculum as a whole, so they can exploit what students have studied already and lay the foundations for what is to follow.

When and how is *PUFM* achieved? To answer this question, Ma interviewed two additional groups in China, 26 preservice teachers and 20 ninth-grade students "from a mediocre school in Shanghai". While both groups were equally competent algorithmically, the students were more divergent but less sound, and the prospective teachers seemed to grasp the concepts more firmly. Both groups performed better than the American teachers on the third and fourth scenarios. However, they lacked the maturity of the Chinese teachers, and Ma interviewed three of the latter to find out why there was a difference. They pointed to the value of teaching a spectrum of grades over their careers, studying intently teaching materials, particularly the textbook and the government curriculum, interacting with colleagues, solving problems on their own, and being willing to learn from their students. The better Chinese teachers are products of a virtuous circle that begins with a solid school education and continues with their preparation for teaching and the substantial mathematics they pass on to *their* students. In the United States, elementary mathematics seems to be undervalued and teachers have less to build on; the positive feedback does not occur.

To remedy the North American situation, Ma would recognize the interdependence of improving the school curriculum and teacher knowledge, working on both *simultaneously*. She would enhance the interaction between teachers' study of school mathematics and how to teach it; she would like to have college programs more pertinent to the mastery of the elementary curriculum. She concludes the book with some pithy observations on the reform movement, pointing out that while Chinese teachers may not look as they were being very modern, their students may still often be actively engaged in enquiry, problem solving and making critical judgments.

As this is a qualitative investigation that involves only a limited number of teachers, it would be wrong to interpret it as a general investigation of the knowledge of teachers in either China or the United States. Also, this is certainly not the first time on this continent that researchers have concerned themselves with what teachers know and feel about mathematics; Ma's work fits into a much broader pattern of American investigation, particularly that carried out by Deborah Ball. (See **(1)** for more discussion of this point.) But Ma's purpose was not to make national comparisons or just record different perceptions of elementary teachers, but to understand more fully where they were and what was possible; for this, it was useful to go outside the American cultural setting for a referent. As a result, we have a work that can serve as a powerful guide to some of the factors that we should be examining in addressing what should be done in North American mathematical education. The socalled "math wars" in the United States have left people in opposing hostile camps despite the attempts of many prominent educators and mathematicians to find a common ground. Ma's book helps to define that ground. Those who are strong proponents of solid mastery will be reassured that this need not be neglected in successful teaching, while those who see salvation in the active engagement of students in formulating concepts, investigations and problem solving will see that these too are key. Improvement of mathematical education will not come from hopeful purple prose in curriculum documents and threats of hellfire on teachers and schools whose students do not measure up in some way, but only through a corps of knowledgable, reflective and meticulous teachers with the time and environment that supports continuous professional growth.

Our best teachers often toil in isolation, not well supported by their principals or superintendents; teachers-in-training might have taken innovative and strong courses either as undergraduates or as teachers-in-training, but are not given the mentoring in the field to incorporate these into their own coherent vision. The temptation is to respond to a crisis in education by spending money on new texts, testing and technology. Ma's book reminds us how good teachers must be at the core of a solid mathematics education.

Every prospective elementary teacher, university library, board and ministry of education in the country should have a copy of this fine clearly argued monograph.

Reference

1. Roger Howe, Book review: Knowing and teaching elementary mathematics *Notices of the American Mathematical Society* 46:8 (1999) 881-887