RECREATIONAL PROBLEMS IN LOGIC AND ANALYSIS

L.1. For the two-player game of noughts-and-crosses (tictactoe), either one of the players has a winning strategy (*i.e.*, she can ensure that she wins regardless of what her opponent does), or else both players have blocking strategies (*i.e.*, each can prevent her opponent from winning by an intelligent strategy). Note that the X-player plays first and the players move alternately. In the questions below, you will have to decide what "essentially different" means.

(a) What are the essentially different possibilities for the first move?

(b) What are the essentially different configurations of the X and O after each player has made exactly one move?

(c) Tabulate all the essentially different configurations after three moves have occurred, so that the array has X's filled in two positions and an O filled in one position.

(d) For each of the configurations in (c), determine whether the first player can force a win, the second player can force a win, or both players can force a draw.

(e) In the whole game, does either player have a winning strategy, or must both players with good play be able to force a draw?

L.2. On a certain island, knights always tell the truth and knaves always lie. A stranger to the island, knowing these traits but unable to recognize any person as a knight or a knave, encounters three inhabitants: A, B and C. She asks A, "How many knights are among you three?" A mumbles and his reply is not heard by the stranger, although it is heard by the others. So the stranger asks B, "What did A say?" B replies, "A said that there is one knight among us." Then C says, "Do not believe B; he is lying!" What are B and C?

L.3. You are given five billiard balls of identical appearance but all of different weights. Denote them by A, B, C, D and E. Using an equal arms balance a minimum number of times, describe how you would rank them in order of weight.

L.4. Consider the following problem: *You have an equal arms balance, which will permit you to determine which of two weights is heavier or whether both are equal, but is not a scale from which you can read off the actual weight. You also have twelve billiard* *balls which look identical. Eleven of these balls also have exactly the same weight, but the weight of the twelfth differs. Using the balance three times, explain how you can identify the twelfth ball and determine whether it weighs more or less than each of the others.*

(a) Explain why there are 24 different possible situations that might occur for the identity and relative weight of the odd ball. Suppose that you put two balls on each pan of the balance. For each of the three possible outcomes (left pan down, right pan down, balance), determine how many of these possibilities are consistent with it.

(b) Suppose, as in (a), you put a certain number of balls on each pan of the scale, and then you count the number of situations that are consistent with each of the three outcomes. Explain why one of the outcomes must be consistent with at least eight possible situations.

(c) Given that in the first use of the balance, you put the same number of balls in each pan, what is the best number of balls to select for each pan? Explain why.

(d) Determine a solution to the problem.

(e) What do you think is the maximum number of balls for which one can identify and classify an odd ball as in the above problem, if you are allowed two applications of the balance? four applications of the balance?

L.5. Three jars contain 19, 13 and 7 litres respectively. The first is empty and the others full. None of the vessels is graduated. How can one measure out 10 litres, using no other vessels, by pouring fluid from one into another?

L.6. (a) Write down the numbers from 1 to 9 in a 3×3 square array in whatever way you wish. Beside each row, write the largest number in the row; let A be the smallest of these three numbers. Underneath each column, write the smallest number in the column; let B be the largest of these three smallest numbers.

For example, for the magic square

4	3	8
9	5	1
2	7	6

we find that $A = 7$ and $B = 3$. For your array, determine the values of A and B, and, if they are not equal, state which of the two is larger.

(b) If you have not already done so in (a), construct an array for which $A = B$.

(c) Is it possible to find an array for which A is smaller than B?

L.7. Alcuin of York (735-804) was a monk who taught at the Cathedral School in York before being summoned to the Court of Charlemagne in the year 782. In due course, he became the abbot at the abbey of St-Martin of Tours, and, at this time, produced a set of 53 problems entitled *Propositiones ad acuendos juvenes* (Propositions for sharpening youths). Here are two of them:

(a) *Homo quidam debebat ultra fluvium transferre lupum, capram, et fasciculum cauli. Et non potuit aliam navem invenire, nisi quae duos tantum ex ipsis ferre valebat. Praeceptum itaque ei fuerat ut omnia haec ultra illaesa transire potuit?* A certain man needed to take a wolf, a she-goat and a load of cabbage across a river. However, he could only find a boat which would carry two of these at a time. Thus, what rule did he employ so as to get all of them across unharmed? (The condition is that the boat can hold the man and one item.)

(b) *Tres fratres erant qui singulas sorores habebant, et fluvium transire debebant (erat enim unicuique illorum concupiscentia in sorore proximi sui), qui venientes ad fluvium non invenerunt nisi parvum naviculam, in qua non potuerunt, amplius nisi duo ex illis transire. Dicat, qui potest, qualiter fluvium transierunt, ne una quidem earum ex ipsis maculata sit?* There were three men, each having an unmarried sister, who needed to cross a river. Each man was desirous of his friend's sister. Coming to the river, they found only a small boat in which only two persons could cross at a time. Let him say, he who is able, how did they cross the river so that none of the sisters were defiled by the men? (The condition is that no sister is present with other men unless her brother is also present.)

L.8. During the last University College Fireball, it was discovered that, during the course of the evening, no man had danced with every woman, but that each woman had danced with at least one man. Argue that there are two men, M and m, and two women, W and w, for which all of the following hold:

(a) M has danced with W;

(b) m has danced with w;

(c) M has not danced with w;

(d) m has not danced with W.

L.9. Sixteen passengers on a liner discover that they are an exceptionally representative body. Four are Englishmen, four are Scots, four are Irish and four are Welsh. There are also four each of the four different ages: 35, 45, 55, 65, and no two of the same age are of teh same nationality. By profession, also, four are lawyers, four doctors, four soldiers and four clergymen, and no two of the same profession are of the same ago or the same nationality.

It appears, also, that four are bachelors, four married, four widowed and four divorced, and no two of the same marital status are of the same profession, or the same age, or the same nationality. Finally, four are Conservatives, four Liberals, four Socialists and four Fascists, and no two of the same political sympathies are of the same marital status, or the same profession, or the same age, or the same nationality.

Three of the Fascists are known to be an unmarried English lawyer of 65, a married Scots soldier of 55 and a widowed Irish doctor of 45. It is now easy to specify the remaining Fascist.

It is further given that the Irish Socialist is 35, the Conservative of 45 is a Scotsman, and the Englishman of 55 is a Clergyman. What do you know of the Welsh lawyer?

L.10. A number of people meet and certain pairs of them shake hands. of course, no one shakes hands with himself or herself and no pair shakes hands more than once. After this has occurred, each person then states how many people he/she shook hands with.

(a) The numbers that the people give for the number of hand shakes are added together. Explain why the sum has to be even.

(b) Explain why it must be the case that two of the people have shaken hands with the same number of persons.

L.11. Four men are walking late at night. Together they have one flashlight with a weak battery, so that it can only light the immediate vicinity. They come to a bridge, which is so rickety that only two can cross at the same time. Thus, in order to get to the other end of the bridge, two must cross with the flashlight and one must walk back across the bridge to return the flashlight (they cannot risk throwing it), until all are over.

The men can walk at different maximum rates. The slowest needs 10 minutes to cross the bridge, the next 5 minutes, the next 2 minutes and the quickest 1 minute. When two walk together, they must proceed at the pace of the slower. What is the minimum amount of time required to get all four individuals across the bridge?

L.12. (Note that this problem is over 50 years old, so you may have to make some assumptions about the social mores of the time.) In the Stillwater High School, the economics, English, French, history, Latin and mathematics classes are taught, though not necessarily respectively, by Mrs. Arthur, Miss Bascomb, Mrs. Conroy, Mr. Duval, Mr. Eggleston, and Mr. Furness.

The mathematics and Latin teachers were roomates in college.

Eggleston is older than Furness but has not taught as long as the economics teacher.

As students, Mrs. Arthur and Miss Bascomb attended one high school while the others attended a different high school.

Furness is the French teacher's father.

The English teacher is the oldest of the six both in age and in years of service. In fact, he had the mathematics teacher and the history teacher in class when they were students in the Stillwater High School.

Mrs. Arthur is older than the Latin teacher.

What subject does each person teach?

L.13. In an old and much worn book of travels, dating from the times when a man who had journeyed beyond the hills that guarded his own village was an object of awe to his fellows, I once came across a description of a remarkable city of the east. According to the narrator, this city was built upon ten islands connected in the following manner. Five of the islands had a single bridge leading from them to the mainland. Moreover four of the islands had four bridges leading from them, three of the islands had three bridges leading from them, two of the islands had two bridges leading from them, and one island could be reached by only one bridge.

No doubt, in its time this description had provided many simple folk with a vicarious sense of exploration and adventure. In fact, I read it with keen interest until suddenly in a moment of unwonted penetration I perceived that the whole thing was a hoax, and that no such city could exist anywhere in this or in any other world.

What is the fallacy that stamps this description as impossible?

L.14. For two nonzero numbers a and b, we define a $*$-operation as follows:

$a * b = a/b$ (a divided by b)

For example, $35 * 14 = 5/2 = 2.5$, $2 * 5 = 2/5 = 0.4$.

(a) Find $(3 * 7) * (9 * 14)$.

(b) Is the $*$-operation commutative (*i.e.*, $x * y = y * x$ for each x, y)? associative (*i.e.*, $x * (y * z) = (x * y) * z$ for each x, y, z)?

(c) Prove, for any nonzero numbers a, b, that

$$a * (a * b) = 1 * ((a * a) * b) .$$

(d) Prove, for any nonzero numbers a, b, c, that

$$[a * (b * c)] * [(a * b) * c] = c * (1 * c) .$$

L.15. Once upon a time, an aged king, nearing the end of his days and having no heir to succeed him upon the throne, cast about him for a worthy successor. In all corners of his kingdom he caused search to be made for young men of promise. These he gathered together by districts for further judging and selection, and so again and again until the four most gifted men of country were determined and brought before him for a final choice. So near alike in their capabilities were they that no ordinary test could mark one as superior to his companions, and in order to make his final decision the king devised this scheme.

The four were tightly blindfolded and seated around a table. While seated, the king said to them, “In a moment, I shall touch each of you upon the forehead, and mayhap I shall leave upon you a black mark and mayhap a white mark. I shall then cause your bandages to be removed and each of you who, looking upon his companions, sees more black marks than white marks is to stand and remain erect until such time as one of you can state convincingly the colour of the mark he bears. That one I shall name my successor.”

According to his word, the king touched each man upon the forehead. When the blindfolds were removed, each man looked at his fellows and at once rose. For many minutes, each stood silent, pondering. Finally one man spoke, saying, “Sire, I bear a black mark,” and straightway gave convincing argument for his assertion.

How did the king actually mark the men, and how did the successful candidate prove the existence of a black mark upon his forehead?