TALENT SEARCH PROBLEMS FOR ELEMENTARY CHILDREN

Ed Barbeau
Department of Mathematics
University of Toronto
Toronto, ON M5S 3G3
barbeau@math.utoronto.ca

The problems that follow are from the International Mathematics Talent Search, a regular program initiated about two decades ago in the United States by the mathematician George Berzsenyi, then at Rose-Hulman Institute of Technology. It has now become an international venture, with participants in many countries, including Canada. While the problems are designed for secondary students, they vary considerably in difficulty and background required, and it seems to me that many of them are suitable for more junior student. In some cases, they are posed in a language that would be strange to elementary children, especially in their use of algebraic notation, and so would need to be adapted.

Listed below, roughly in order of suitability, problems that could be used for students in the Grade 5-8 range. I have commented on some of them. Where the wording seems a little opaque, a different formulation has been suggested. I would like teachers to try these out on their students. Sometimes the appropriate setting might be as individual problems posed for students to look at after hours, sometimes as group problems in class, and perhaps occasionally incorporated into the curriculum as part of the regular program. I hope that you will try out the problems yourself before looking at the comments, as this will give you a better sense as to what might be involved and where the children might go with them.

It would be very helpful to hear from teachers who have used any of these problems. Please let me know how you have used them, where and with whom, how you presented the problem to make it intelligible to the pupils, and what the results were. Did points of difficulty or interesting mathematical or pedagogical issues arise? I would be very happy to engage in discussion with any teacher on any of these problems.

A mathematics undergraduate, who did some work with two grade 6, one grade seven and one grade eight class, presented some of these problems. I have appended her comments with the label LS.

One role that these problems might play is to help teachers determine which pupils have a talent for mathematics. Talented students may not always distinguish themselves in the regular program for a number of reasons. They may not be motivated to do the regular curriculum; their communication or reasoning patterns may not be standard and easily understood by the teacher; they may not be able to speak or write well; they may not work with sufficient care; they may not work systematically. It may be possible in school for children to get very good marks by working hard, memorizing well, practising and imitating examples, while not having much imagination or understanding of the deeper issues involved. Many of these problems require pupils to think strategically, to experiment, to think metaphorically and to use their imaginations, all markers of mathematical ability that classroom exercises might not reveal. So teachers might want to try these out with a view to seeing “what the traffic will bear”, drawing back tactically where necessary.

Integer arithmetic

Problem 1/25. Assume that we have 12 rods, each 13 units long. They are to be cut into pieces measuring 3, 4, and 5 units, so that the resulting pieces can be assembled into 13 triangles of sides 3, 4, and 5 units. How should the rods be cut?

Answer: Three of the rods are cut into lengths $(3,3,3,4)$, four into lengths $(3,5,5)$, and five into lengths $(4,4,5)$.

Comment. The wording of this problem seems to be pretty straightforward. It is not clear to me that the temptation to provide manipulatives should be yielded to (although this might be helpful in some cases), as it is more a matter of basic reasoning and canvassing possibilities.

We want to end up with 13 rods of each of the sides 3, 4 and 5, and we will need material of total length 13×12 to produce these rods. Available are rods of total length 12×13, so we must use all of the material available. This means that each of the thirteen rods must be cut into three pieces with no waste, each piece being of length 3, 4 or 5 units. So we need to see how we can produce the sum 13 using these three numbers. We have $13 = 3 + 3 + 3 + 4$, $13 = 3 + 5 + 5$ and $13 = 4 + 4 + 5$. A certain number of rods will need to be cut into lengths $(3, 3, 3, 4)$, a certain number into lengths $(3, 5, 5)$ and the remainder into lengths $(4, 4, 5)$.

There are to be 39 pieces altogether. Some of the rods of length 13 will be divided into three pieces and the remainder into four pieces. Dividing all the rods into three pieces would give us only 36 pieces to try to make up the desired triangles, so three of the rods of length 13 must be cut into four pieces, that is, each into one piece of length 4 and three pieces of length 3. This will give us nine rods of length 3 and we need exactly four more. So four of the rods of length 13 must be cut into pieces of length 3, 5, 5. Finally, the remaining five rods must be cut into pieces 4, 4, 5, and we see that this actually works.

There is a fair bit of reasoning involved in the foregoing, and it would be of interest to know how much of it is explicitly given by the pupils and how much is innate. The problem could be treated as an algebra problem, but there is a lot of merit in giving it to pre-algebra students and having it handled by numerical investigation, using some trial and error. For this reason, it lends itself particularly to group interaction. it may be too formidable for some students to do on their own. Note that the arithmetic demands are modest, just the ability to construct partitions of the number 13.

LS: (Gd 6-8) No class was interested in this question. It was deemed “not fun” and irrelevant. It was presented to the whole class, grades 6 and 7 separately, as a seat problem; both classes looked at it briefly and lost interest.

Problem 2/24. Let $N_k = 131313 \cdots 131$ be the $(2k+1)-$digit number (in base 10), formed from $k+1$ copies of 1 and k copies of 3. Prove that N_k is not divisible by 31 for any value of $k = 1, 2, 3, \cdots$

Comment. You can avoid the use of the symbol k by asking pupils to show that each of the numbers 131, 13131, 1313131, 131313131, and so on, is not divisible by 31. They could note that $3131 = 101$, $313131 = 10101 \times 31$, $31313131 = 1010101$, so that each of the numbers is a power of ten plus a multiple of 31. So each number would be divisible by 31 only when the corresponding power of 10 is divisible by 31. But what are the divisors of powers of 10? Pupils would need to recognize that 31 is a prime, and that the only primes that divide powers of 10 are 2 and 5.

LS: (Gd 6-8) This problem was given to a group of seven eager grade 6 students. The actual question was prefaced with an examination of the numbers 131, 13131, etc. The students did not like to be spoonfed in this manner and waited impatiently for a question they could attack on their own.

Problem 1/22. In 1996 nobody could claim that on their birthday their age was the sum of the digits of the year in which they were born. What was the last year prior to 1996 that had the same property?

Comment. As this may be hard for a pupil to negotiate on first reading, one might prepare for this problem by asking children to calculate the sum of the digits of the year in which they were born and ask them whether this coincides with their current age. The answer is likely to be a universal “no”. The same question could be asked with respect to older siblings or parents. For example, if some pupil had a brother or sister whose age was 19 this year (2000), then that person would have been born in 1981; the age this year, 19, would then be the same as the sum of digits of the year of birth.

Having got into the problem, pupils might then be asked to first look and see whether there are other ages that work for the year 2000, and then to look at what the situation might be in earlier years. They could be asked to tabulate all the years in the past century for which there could be a person whose age that year was the sum of the digits of the year in which they are born.

There are a number of strategies in which this problem might be approached and it would be interesting to see where the children take it. As this is an investigative problem where the output is a number of possibilities, group investigation may be the route to go.

Grade 7 students enjoyed this question for approximately fifteen minutes. They were given the opportunity to discuss their answers and strategies with their classmates. Within a few minutes, most of the students were resorting to trial and error. A few spent time trying to devise a better method; none was found.

Problem 1/33. The digits of the three-digit integers a, b, and c are the nine non-zero digits $1, 2, 3, \cdots, 9$, each of them appearing exactly once. Given that the ratio $a : b : c$ is $1 : 3 : 5$, determine a, b, and c.

Answer. 129, 387, 645.

Comment. If you want to avoid the use of the letters a, b and c, you might just indicate the three three-digit numbers as boxes that have to be filled, preferably on some erasable medium, such as a blackboard. Explain simply that the second number is three times the first and the third number is five times the first, so there is no need to talk about ratio.

How can pupils get into this problem? There are two entry points. Since 0 is not allowed and the third number is a multiple of 5, the last digit of the third number must be 5 (so 5 cannot appear elsewhere). Since the last number is less than 1000 and the first is one fifth of it, the first number must begin with a 1. This means that the third number cannot end with a 1, so that the first number cannot end with a 7. From this point on, there are various directions that might be taken, which, depending on the ingenuity of the pupils, will involve more or less trial and error.

The punch line for the curriculum is that it sensitizes the students to properties of multiples and forces them to make inequality estimates and determine what effects this has on the numeration. It turns out that there is exactly one possibility. Again, this could either be given for group work or for students to take home and perhaps puzzle over with their families. Although this problem is purely mathematical, I think it does have some appeal as a puzzle.

LS: (Gd 6-8) The question was given as a worksheet. One child attempted a single calculation and gave up. The worksheets were collected and redistributed to groups of six. One group needed only the following hint before attempting it on their own: if c is divisible by 5, what digit must be in the one's column? Other groups needed more hints: what is the hundred's digit of a? Are the numbers odd or even? Which digits do we have left to work with? Only two out of five groups needed to be given all the hints available. On average, each group took about 20 minutes. This question was also written on the blackboard in a grade 8 class. They were able to attempt it independently with only the first hint given. Nine of them got the correct answer within about ten minutes.

Problem 1/34. The number N consists of 1999 digits such that if each pair of consecutive digits in N were viewed as a two-digit number, then that number would either be a multiple of 17 or a multiple of 23. The sum of the digits of N is 9599. Determine the rightmost ten digits of N.

Answer. $\cdots 3469234685$.

Comments. This is one of those problems that seems worse than it actually is. It has the advantage that one can approach it systematically and deal with only a small number of possibilities. The only possible two-digit numbers to be considered are 17, 23, 34, 46, 51, 68, 69, 85 and 92. With the exception of 68 and 69, no pair of them have the same first digit; 7 does not appear as the first digit of any of the numbers, so that if two consecutive digits are 1 and 7, then the number must conclude with 17. We can look at pairs of digits to see what the possibilities are for following pairs. Possible chains of digits are ...23468517 and ...234692... . Since there are 1999 digits, there must be cyclic blocks of digits followed by an end run, so that N must consist of the block of five (23469) repeated over and over, with the left digit of N being one of the five possibilities 2, 3, 4, 6, 9.

Problem 5/17. What is the minimum number of 3×5 rectangles that will cover a 26×26 square? The rectangle may overlap each other and/or the edges of the square. You should demonstrate your conclusion with a sketch of the covering.

Answer. At least 46 rectangles are needed.

Comments. Note that $26 = 15 \times 45 + 1$, so that 45 rectangles will always leave at least one square uncovered. So we need at least 46 rectangles. However, the pupils need to go further to actually show that the job can be done with 46 rectangles; there are many ways of achieving this.

LS: This question was presented on a poster at a booth in the grade 6 math fair. There were two classes in attendance. Whoever submitted the correct answer got a candy. Everyone attempted the question, using methods that included: dividing the total number of squares by the number of squares in the rectangle and rounding up; dividing the total number of squares by the number of squares in the rectangle and rounding down; cutting out a piece of paper the size of the rectangle and physically moving around the grid; visualizing the movement of the above rectangle over the grid and using pencil dots to keep track of how many times it was moved; approximating. Over half the students got the right answer. Three students were able to determine why the division method might not always be accurate for questions of this type. After being given the same question involving an irregular shape that had the same area as the rectangle, everyone seemed to understand that the division method was not always accurate.

Problem 2/13. Erin is divising a game and wants to select four denominations out of the available denominations \$1, \$2, \$3, \$5, \$10, \$20, \$25, and \$50 for the play money. How should he choose them so that every value from \$1 to \$120 can be obtained by using at most seven bills?

Problem 2/21. Find the smallest positive integer that appears in each of the arithmetic progressions given below, and prove that there are infinitely many positive integers that appear in all three of the sequences:

$$5, 16, 27, 38, 49, 60, 71, 82, \cdots$$

$$7, 20, 33, 46, 59, 72, 85, 98, \cdots$$

$$8, 22, 36, 50, 64, 78, 92, 106, \cdots$$

Comment. In an arithmetic progression, the difference between any pair of consecutive terms is the same. In the problem, this difference is respectively 11, 13 and 14. The point of the question is that once any common number in the sequences can be found, any number that differs from it by a multiple of 11, 13 and 14, will also be in all three sequences. The pupils can find such a common number by trial and error; just extend the sequences far enough out to start getting some matches - this may give them a sense of the underlying mathematical mechanism. Actually, this problem involves a trick. We can extend the sequences backwards as well as forwards, with the same common difference. For the sequences in question, the term before the first one in each case would be -6, so that the number $(-6) + (11 \times 13 \times 14) = 1996$ will appear in all thre sequences. If the numbers in this question appear to be formidable, you can keep the same idea with smaller numbers:

$$1, 4, 7, 10, \cdots$$

$$2, 6, 10, 14, \cdots$$

$$3, 8, 13, 18, \cdots$$

for example. A variant is to pick three sequences which have no number in common, and ask the children to explain why. Or you could have the children construct their own sets of sequences to order, either to have or to have not common elements.

Problem 1/39. Find the smallest positive integer with the property that it has divisors ending in every decimal digit, *i.e.*, divisors ending in $0, 1, 2, \cdots, 9$.

Problem 1/32. Exhibit a 13-digit integer N that is an integer multiple of 2^{13} and whose digits consist of only 8s and 9s.

Comment. The high power of 2 might make this problem seem formidable, and one can reduce the sting but keep the essence of the problem by replacing 2^{13} by a lower power of 2, such as 32. The key to this problem is for the students to recognize that a number is divisible by 2 if and only if its last digit is divisible by 2; it is divisible by 4 if and only if the number consituted by its last two digitis is divisible by 4; it is divisible by 8 if and only if the number constituted by its last three digitis is divisible by 8; and so on for each successive higher power of 2. (To see why the result holds for 8, say, notice that any number can be written as a multiple of 1000 (which, being divisible by 10^3, is divisible by 2^3) added to a three-digit number.) So one simply starts working backward from the last digit on the right. Note that a number divisible by a power of 2 is divisible by each lower power of 2. Thus, the number must end in 8; it must end in 88; it must end in 888; it must end in 9888; and so on.

The great merit of this problem is that it focuses on the structure of numbers divisible by powers of 2 and provides scope for an explanation that is intelligible for at least some students in the Grade 5-8 level. One can extend this into a project by posing the following questions:

(a) Find all numbers whose digits are all the same divisible by a power of 2, with the exponent equal to the number of the digits.

(b) Determine all of the numbers divisible by 64 (you can replace this by other powers of 2) that have exactly two *distinct* digits.

LS: This question was given to grade 6 students as a lesson. The students were interested in learning about the various tests for divisibility by 2, 4 and 8. They were then given the problem. Instead, a few students decided to try numbers that were divisible by 8 and 16 and were comprised only of 8s and 9s. Anyone who tried the whole question lost interest.

Problem 2/33. Let $N = 111\cdots 1222\cdots 2$, where there are 1999 digits of 1 followed by 1999 digits of 2. Express N as the product of four integers, each of them greater than 1.

Comment. Again, one can remove a conceptual barrier without destroying the essence of the problem by reducing the number of digits in the situation. Write 111222 as the product of four distinct postive integers; write 1111122222 as the product of four distinct positive integers.

What would we like the students to observe? If we can find a factor, we can divide through by the factor to get a smaller quotient, whose factors can be found. Thus, an obvious factor is 2. By casting out 9s, we can see that another factor is 3. We can also take out a factor all of whose digits consist of 1s. We are focusing on the numeration structure of numbers that are likely to have certain factors.

LS: The problem was given in a group setting. The students were much happier working with numbers where they could see all the digits written.

Problem 1/5. The set S consists of five integers. If pairs of distinct elements of S are added, the folowing ten sums are obtained: 1967, 1972, 1973, 1974, 1975, 1980, 1983, 1984, 1989, 1991. What are the elements of S?

Answer. 983, 984, 989, 991, 1000.

Comment. To avoid large numbers, the problem could be posed with a smaller set of numbers: 7, 12, 13, 14, 15, 20, 23, 24, 29, 31. To get the pupils into the problem, you could begin by asking them for three positive whole numbers such that the sums of pairs of them are 5, 7, 8. This could be solved by trial and error. A more systematic approach is to note that the sum $5 + 7 + 8 = 20$ is equal to *twice* the sum of the three numbers, as each number figures in two pairs. Thus, the sum of the three numbers sought is 10. If two of them add up to 5, then 5 must be one of the numbers; if two of them add up to 7, then 3 is one of the numbers. So for the simpler problem, the three numbers are 2, 3, 5. Other sets of three pair sums can be given. Then one could move up to the six numbers which represent the sum of four numbers in pairs. To solve the problem at the beginning of this comment, note that

$$7 + 12 + 13 + 14 + 15 + 20 + 23 + 24 + 29 + 31 = 188$$

is four times the sum of the numbers we want, as each number figures in four different pairs. The the sum of the numbers that we are looking for is 47, one quarter of 188. The sum of the smallest two numbers is 7 while the sum of that largest two is 31. Thus, the sum of all the numbers except the middle one is 38, so the middle number in the set of five must be 9. The sum of the smallest and the middle number must be the second smallest pair sum, 12, so the smallest number is 3. Then the second smallest number is $4 = 7 - 3$. The sum of the largest and the middle number in the set of 5 must be 29, so the largest number must be 20. We find that the five numbers are 3, 4, 9, 11, 20. There will be many ways that the pupils might reason, and this would be a good exercise for group work.

Problem 3/31. The integers from 1 to 9 can be arranged into a 3×3 array so that the sum of the numbers in every row, column, and diagonal is a multiple of 9.

(a) Prove that the number in the centre of the array must be a multiple of 3.

(b) Give an example of such an array with 6 in the centre.

Comment. At first glance, this problem might seem to be entirely unsuitable for children at the presecondary level, because of the need for a proof. I have no doubt that there will be some children that would be capable of providing the proof desired in (a), or even that a class could be guided in a suitable lesson towards a proof. But one can avoid the issue altogether for the general run of students by simply asking them provide examples of such arrays (even when the centre element is not required to be a 6); there are several possibilities.

Just so the children are not completely looking for a needle in a haystack, you might want to give some guidance as to what the possible row, column and diagonal sums could be. The total sum of all the numbers involved is $1 + 2 + \cdots + 9 = 45$, and this is equal to the three row sums added together. Since each row sum is a multiple of 9, by looking at all possibilities, we see that two of the rows must add to 18 and the remaining row to 9. A similar comment holds for the columns. Then it is a matter of looking at all the ways in which three digits can be made to add up to 9 or 18 and trying to fit them in the square array.

One intriguing aspect of the situation is that given one square array, one can get a second square array by replacing in their places the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 by, respectively, 8, 7, 6, 5, 4, 3, 2, 1, 9. It would be interesting to see whether any kid picks this up.

LS: (Gd 6-8) All grades enjoyed this question, especially the grade six students. First they were given the problem of creating the square. A handful of students required assistance and were shown how to create rows by finding numbers that added up to 9 or 18 (both of which they discovered as the only possible sums on their own). Everyone who attempted the square completed it. Part (a) was more difficult, not because it was beyond their level of understanding, but becuase most students had difficulty expressing themselves in words as opposed to just giving examples (which is what most students did).

Problem 1/21. Determine the missing entries in the magic square shown below, so that the sum of the three numbers in each of the three rows, in each of the three columns, and along the two major diagonals is the same constant k. What is k?

?	?	33
?	?	?
31	28	?

Answer. $k = 96$.

Comment. For students with access to algebra, this could be done by using letters to denote the missing elements and setting up a system of equations to monkey with, a substantial task for pupils in Grade 8 or Grade 9. But there is a lot of merit in leaving this as a problem on the numerical level. While the situation will involve quite a bit of trial and error (pupils might want to make a substitution for the middle number in the array and see where it leads), the issue is what the students learn from their mistaken trials as to what is best to try next. This is one of those situations where the pupils might be quite at sea to start with, but as they get into the situation, the solution will start to emerge. There is the very long-shot risk that some kid will through superior intuition jump right away to the answer; you can try to get the kid to unpack his/her reasoning but this may not be possible. Anyhow, you can cross that bridge when you come to it.

LS: This was given as an overnight optional homework assignment with a prize for the right answer. No one got the right answer. Four grade six students attempted it. No one was interested in finding out what the right answer was.

Problem 3/29. It is possible to arrange eight of the nine numbers 2, 3, 4, 7, 10, 11, 12, 13, 15 in the vacant squares of the 3 by 4 array shown below so that the arithmetic average of the numbers in each row and in each column is the same integer.

1	?	?	?
?	9	?	5
?	?	14	?

Exhibit such an arrangement, and specify which one of the nine numbers must be left out when completing the array.

Comment. This problem carries quite a bit of mathematical baggage, but may well be worth the time taken to set it up properly. What the students need to know is that the average of a row multiplied by the number of elements in the row is equal to the sum of the elements in the row. Since the average is an integer, the sum of the elements in a given row must be divisible by 4 (it is four times the average). Similarly reasoning establishes that the sum of the elements in a given column is divisible by 3. The sum of all the numbers in the array is the sum of the three row sums and so is divisible by 4, and it is the sum of the three column sums and so is divsible by 3. In this way, we see that the sum of all the elements in the array must be a multiple of both 4 and 3, and so a multiple of 12.

Once this fact is established, the pupils now have to decide which of the given integers has to be left out. An efficient way to do this is to add the nine integers to the number already placed in the array and see what the remainder is when you divide by 12; this remainder is the number that you have to leave out. From this point on, it is a matter of modest trial and error to place the remaining eight numbers.

LS: Arithmetic average was part of the grade 6 testing that the students had recently completed. No one seemed interested in revisiting the concept even though the context was different. It might have gone over better with a grade 8 class since, to them, the question might not have been so daunting.

Problem 1/27. Are there integers M, N, K, such that $M + N = K$ and

(i) each of them contains each of the seven digits $1, 2, 3, \cdots, 7$ exactly once?

(ii) each of them contains each of the nine digits $1, 2, 3, \cdots, 9$ exactly once?

Comment. Part (ii) is the easier part, as there are many solutions and many students should be able to find some of them without too much trouble; if you give this, it might be better to start with Part (ii). Part (i) depends on the students knowing about the process of "casting out nines". It is impossible to do what (i) asks for. The sum of the digits of each number in part (i) is $1 + 2 + \cdots + 7 = 28$, which has a digital sum of 1. If such a sum works, the sum of the digital sums of the two addends should have as its digital sum the digital sum of the sum of the two numbers; this cannot happen. Thus, this problem should only be given after the students have had a lot of preparation with the use of casting out nines as a means of checking computations.

Problem 1/31. Determine the three leftmost digits of the number

$$1^1 + 2^2 + 3^3 + \cdots + 999^{999} + 1000^{1000} \ .$$

Comment. This problem is not really as bad as it looks, and it is hard to see how the size of the numbers can be reduced without destroying the problem. What makes it conceptually challenging is the need to make an estimate. We have that $1 < 1000$, $2^2 < 1000^2$, $3^3 < 1000^3$, and so on up to $999^{999} < 1000^{999}$. Thus, the number given is strictly less than

$$1000 + 1000^2 + 1000^3 + \cdots + 1000^{999} + 1000^{1000} = 100100100100100 \cdots 1001000 \ .$$

(This number has 1001 digits.) At the same time, the number is greater than

$$1000^{1000} \ ,$$

which also has 1001 digits. Thus, it must begin (on the left) with 100, which are its first three digits.

Ability to solve this problem depends on having a pretty secure sense of place value, and the teacher might want to work up to this by giving the students exercises on writing out the sum of various powers of 10, such as $10^3 + 10 + 1$, and then working up to sums of powers of powers of 10, such as $100^5 + 100^4 + 100$, and on to $1000^3 + 1000^2 + 1000$.

Problem 1/29. Several pairs of positive integers (m, n) satisfy the equation $19m + 90 + 8n = 1998$. Of these, $(100, 1)$ is the pair with the smallest value of n. Find the pair with the smallest value for m.

Answer. $(m, n) = (4, 229)$.

Comment. Rewrite the equation as $19m + 8n = 1908$. We know there is a solution with $n = 1$. What is the next largest value of n that gives an integer value of m? What is the next smaller value of m that gives an integer value of n? These are questions that can be investigated numerically, and the students will be able to see some patterns.

LS: Grade 8 students were given this question as a class. They were able to devise a table of values on their own and use trial and error. The entire class was involved for around 7 or 8 minutes.

Problem 3/26. Substitute different digits $(0, 1, 2, \cdots, 9)$ for different letters in the alphametics on the right, so that the corresponding addition is correct, and the resulting value of $M\ O\ N\ E\ Y$ is as large as possible. What is this value?

	S	H	O	W
			M	E
+		T	H	E
M	O	N	E	Y

Answer. 10376. There are actually two solutions to the sum:

$$9403 + 16 + 846 = 10265$$

$$9502 + 17 + 857 = 10376 \; .$$

Comment. This problem relies considerably on reasoning with a certain amount of trial and error; the pupils need to follow out a hypothetical situation until either it results in a contradiction or one achieves a successful substitution. Probably, the place to get into the situation is the letter M. Since the sum cannot exceed $9999 + 99 + 999 = 10097$, it is clear that $M = 1$ and $O = 0$. Since a carry of 2 cannot come from the hundreds column, it follows that S must be 9. Since H occurs in two places, one can work through the possible values of H and find ultimately that it must be either 4 or 5.

LS; Grade 6 students highly enjoyed this type of question. As an introduction, the class was given $ABC + AC = DEBE$ and $ABA + AB = BCBC$. The entire class worked together with one student recording on the board. The fact that more than one answer worked for the second question renewed some of the declining interest in a couple of students. They were then given the actual question to work on individually at their seats. Unfortunately, it took some students less than ten minutes to complete while others took over half an hour. This might have been better as a homework question.

Problem 1/40. Determine all positive integers with the property that they are one more than the sum of the squares of their digits in base 10.

Answer. 35 and 75.

Problem 2/31. There are infinitely many ordered pairs (m, n) of positive integers for which the sum

$$m + (m + 1) + (m + 2) + \cdots + (n - 1) + n$$

is equal to the product mn. The four pairs with the smallest values of m are $(1, 1)$, $(3, 6)$, $(15, 35)$ and $(85, 204)$. Find three more (m, n) pairs.

(a) Prove that the number in the center of the array must be a multiple of 3.

(b) Give an example of such an array with 6 in the centre.

Comment. You can remove the sting of the algebraic appearance of this by keeping everything in arithmetic terms. Point out to the children that $3 + 4 + 5 + 6 = 18$ and also 18 is the product of the first and last terms of the sum: $18 = 3 \times 6$. Then say that this happens with bigger numbers as well. If we add the numbers from 15 to 35 inclusive, then the sum turns out to be the product of 15 and 35, namely 225. (You might want to discuss with the children how one can efficiently sum a sequence of consecutive integers.) Then turn them loose on trying to find other examples of this phenomenon. With an exercise like this - anything goes. They can use calculators or computers, work in groups, or work with people at home.

LS: Since students often give up when confronted by large numbers, I gave a grade 7 class an example that did not work and asked them to check that the pair $(15, 35)$ did work; almost every student succeeded. The question was definitely within their reach. It took one group less than ten minutes to find an answer.

Problem 1/35. We define the *repetition number* of a positive integer n to be the number of distinct digits of n when written in base 10. Prove that each positive integer has a multiple which has a repetition number less than or equal to 2.

Problem 2/34. Let $\mathfrak{C}$ be the set of non-negative integers which can be expressed as $1999s + 2000t$ where s and t are also non-negative integers.

(a) Show that $3,994,001$ is not in $\mathfrak{C}$.

(b) Show that if $0 \leq n \leq 3,994,001$ and n is an integer not in $\mathfrak{C}$, then $3,994,001 - n$ is in $\mathfrak{C}$.

Problem 1/19. Is it possible to replace each of the $\pm$ signs below by either $+$ or $-$ so that

$$\pm 1 \pm 2 \pm 3 \pm 4 \pm \cdots \pm 96 = 1996 \; ?$$

At most how many of the ± signs can be replaced by a + sign?

Answer. We can have at most 81 plus signs. We find that

$$1 + 2 + \cdots + 80 - 81 - 82 - 83 - 84 - 85 + 86 - 87 - 88 - \cdots - 96 = 1996 .$$

Comments. This problem needs a strategic approach. If we want to maximize the number of plus signs and minimize the number of minus signs, then we should put the plus signs with smaller numbers and minus signs with larger numbers. If the pupils know the formula for the sum of the first n natural numbers, then they can take note that

$$\begin{aligned}(1 + 2 + \cdots + m) - (m+1) - (m+2) - \cdots - 96 \\ &= 2(1 + 2 + \cdots + m) - (1 + 2 + \cdots + 96) \\ &= m(m+1) - 4656 .\end{aligned}$$

If this is to exceed 1996, we must have $m \geq 82$. This gives us a value that is too large, while taking $m = 81$ gives us a value that is slightly too small, namely 1986. As this is 10 short, we can put things right by changing the signs of 81 and 86; note that these two numbers differ by 5, which is half 10 (what is the point of this observation?).

Problem 3/21. Rearrange the integers $1, 2, 3, \cdots, 96, 97$ into a sequence $a_1, a_2, \cdots, a_{97}$ such that the absolute value of the differences of a_{i+1} and a_i is either 7 or 9, for each $i = 1, 2, 3, 4, \cdots, 96$.

Comment. This is a nice problem for some exploration. Of course, for younger children, it can be formulated without the algebraic notation. Just ask the pupils to put the numbers in a different order, so that each number is either 7 or 9 more or less than its predecessor. You can oil the wheels by showing or asking some child to show how such a rearrangement might start off. One answer is to work in blocks of 16. Start with the numbers:

$$1, 10, 3, 12, 5, 14, 7, 16, 9, 2, 11, 4, 13, 6, 15, 8$$

This rearranges the first 16 numbers. The next block of 16 is found by adding 16 to each of these numbers in turn, so that the next number will be 17 (which differs from 8 by 9, and so does not transgress the condition). This will not be the only way to do this question, and it will be interesting to see how many pupils guess blindly and how many adopt some kind of strategy. If this problem seems to be difficult as it stands, it can be made easier by asking for a shorter sequence and by changing the differences to smaller numbers, say 3 and 5.

Problem 1/17. The 154-digit number, $19202122 \cdots 939495$, was obtained by listing the integers from 19 to 95 in succession. We are to remove 95 of its digits, so that the resulting number is as large as possible. What are the first 19 digits of this 59-digit number?

Answer. The number is

$$9999989707172737475 \cdots .$$

Comments. The strategy is to put as many nines at the beginning of the number as we can. For digits in the sequence up to 89, there are 19 digits between consecutive occurrences of nones. We begin by deleting all the digits that are not nones up to 59; this eliminates 77 digits. Our number will start with five nines. Can we manage another 9 for the sixth digit from the left? We would need to remove at least 19 more digits, which is too many. But we can remove 17 digits to get an eight, and then the digit following the eight to get a nine. At this point, we have shot our bolt and can remove no more digits.

LS: This question was introduced to a grade 6 class by giving them the following numbers and asking which digits should be removed to make them as big as possible: 27609 removing two digits; 909125843 removing three digits; 192021222324252627282930 removing 10 digits and producing an odd number. The students attempted this in groups, with all groups finding at least one answer. There were three groups of about five students in each, who tried to find more than one answer to at least one of the above questions. The class was captivated by the question for about 20 minutes, until they had to move on to another subject area.

Problem 2/17. Find all pairs of positive integers (m, n) for which $m^2 - n^2 = 1995$.

Answer. There are eight possibilities for (m, n):

$$(998, 997), (334, 331), (202, 197), (146, 139), (74, 59), (62, 43), (58, 37), (46, 11) \ .$$

Comments. This problem depends on the pupils knowing how to factor a difference of squares; this is a useful numerical device that they should have on tap in the middle school years, even if they have not had algebra. It is a useful device for doing mental arithemtic; one can compute products in ones head by expressing the two factors as a sum and a difference. One they have that, then all that has to be noted is that $1995 = (m + n)(m - n)$, so they need to check all the ways that 1995 can be written as the product of two positive integers and find the values of m and n that implement these.

LS: The problem was presented to a couple of groups of grade 8 students. They were all able to create a table of values for x and y, where $x = m + n$ and $y = m - n$. I had to introduce the factorization of a difference of squares. Once the values had been found the students were generally content to independently try to find m and n. Their attention was held for between two and fifteen minutes; some were not willing to find more than one right answer.

Problem 4/17. A man is 6 years older than his wife. He noticed 4 years ago that he has been married to her exactly half of his life. How old will he be on their 50-th anniversary if in 10 years she will have spent two-thirds of her life married to him?

Answer. The man will be 76 on his 50th wedding anniversary.

Comments. As usual with this type of problem, pupils could solve it numerically by trial and error. However, for the beginning algebra student, it is a nice example in which they can practise arguing for (this is important!) and setting up an equation. Part of the discussion should turn on a suitable choice for the variable, and the pupils may use one of a number of options.

For example, let x be the man's present age. Then

$$\frac{1}{2}(x - 4) + 4 = \frac{2}{3}(\overline{x - 6} + 10) - 10 \ .$$

Solving this yields $x = 56$. Taking a more pedestrian approach, we can argue that, as the man's age was $x-4$ four years ago, he was married at age $(x-4)/2$. His wife's age is now $x-6$ and was $\frac{1}{2}(x-4)-6 = \frac{1}{2}(x-16)$ when they were married. In ten years, her age will be $x + 4$, three times her age at marriage. Thus

$$x + 4 = 3\left(\frac{x - 16}{2}\right) \ ,$$

which leads to $x = 56$. Thus, the man is now 56 and was married at age 26. The answer now follows.

LS; The grade 8 class worked together to set up the equations (8 minutes) and independently to solve them (5 minutes). Setting up the equations needed little more than the odd "are you sure about that?" to prompt them in the right direction.

Problem 1/15. Is it possible to pair off the positive integers $1, 2, 3, \cdots, 50$ in such a manner that the sum of each pair of numbers is a different prime number?

Comments. This is a very difficult problem for the intermediate level as the task cannot be performed, so the pupils need to argue for the impossibility. The primes less than 100 are 2, 3, 5, 7, 11; 13, 17, 19, 23, 29; 31, 37, 41, 43, 47; 53, 59, 61, 67, 71; 73, 79, 83, 89, 97; there are twnety-five primes in all. The sum of any two numbers from the set $1, 2, \cdots, 50$ must exceed 2 but be less than 100. But then there are at most 24 primes available to be the sum of two numbers in the set. Since pairing off the numbers gives 25 pairs, it is not possible to make all 25 sums of pairs distinct primes.

Alternatively, one could note that the only possible representation for the prime is $3 = 1 + 2$. The only possible representations for 5 are $5 = 1 + 4 = 2 + 3$, but each of these requires a digit already taken to get the prime 3.

Problem 2/15. Substitute different digits $(0, 1, 2, \cdots, 9)$ for different letters in the following alphametics to ensure that the corresponding additions are correct. (Note that the two problems are independent of each other.)

	H	A	R	R	I	E	T
	M	A	R	R	I	E	D
+					H	E	R
	D	E	N	T	I	S	T

	D	I	A	N	A
			A	N	D
	S	A	R	A	H
+			A	R	E
R	E	B	E	L	S

LS: Both grade 6 classes were given this as a reward for finishing other work early.The students enjoyed it, while most finished it after school.

Problem 2/14. The price tages on three items in a store are as folows:

[\$0.75]   [\$2.00]   [\$5.50] .

Notice that the sum of these three prices is \$8.25, and that the product of these three numbers is also 8.25. Indentify four prices whose sum is \$8.25 and whose product is also 8.25.

Problem 1/13. Milo is a student at Mindbender High. After every test, he figures his cumulative average, which he always rounds to the nearest whole percent. (So 85.49 would round down to 85, but 85.50 would round up to 86.) Today he had two tests. First, he got 75 in French, which dropped his average by 1 point. Then he got 83 in history, which lowered his average by another 2 points. What is his average now?

Problem 1/12. A teacher writes a positive integer less than fifty thousand on the board. One student claims that it is an exact multiple of 2; a second student says it is an exact multiple of 3; and so on, until the twelfth student says that it is an exact multiple of 13. The teacher observes that all but two of the students were right, and that the two students making incorrect statements spoke one after the other. What was the number written on the board?

LS: (Gd 6-8) The students who attempted this optional problem enjoyed it immensely. They were given the carpeted area of the class to sit and discuss their answers. Many heated debates were started. The discussions (which included working with pencil-and-paper or calculators) lasted for about half an hour.

Problem 4/7. In an attempt to copy down from the board a sequence of six positive integers in arithmetic progression, a student wrote down the five numbers,

$$113, 137, 149, 155, 173,$$

accidentally omitting one. He later discovered that he had miscopied one of them. Can you help him and recover the original sequence?

Problem 2/6. In how many ways can 1992 be expressed as the sum of one or more consecutive integers?

Comment. This can be broken down by considering cases for different number of summands. For example, it should be straightforward for pupils to decide whether 1992 can be the sum of two consecutive integers (look at parity). A useful fact is that the sum of three consecutive integers is three times the middle one, of five consecutive integers is five times the middle one, and so on where the number of summands is odd. Thus, one is led to look for odd divisors of 1992 as candidates for the number of summands.

Problem 2/4. Find the smallest positive integer, n, which can be expressed as the sum of distinct positive integers a, b, and c, such that $a+b$, $a+c$ and $b+c$ are perfect squares.

Answer. $(n; a, b, c) = (55; 6, 19, 30)$.

Comment. Using algebra, we can let the three squares be w^2, v^2 and u^2 respectively, and solve the system to obtain $2a = v^2 + w^2 - u^2$, $2b = w^2 + u^2 - v^2$, $2c = u^2 + v^2 - w^2$. We need to select u^2, v^2 and w^2 sp that there are 0 or 2 odd squares and a, b, c are strictly positive.

LS: Two grade 6 students worked on this. It took about ten minutes to fully understand the question. They got the answer over lunch break. These students were generally regarded as exceptional in mathematics.

Problem 1/4. Use each of the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly twice to form distinct prime numbers whose sum is as small as possible. What must this minimal sum be? (Note: The five smallest primes are 2, 3, 5, 7, and 11.)

Solution. Since 2 is the only even prime, the only time an even number can be put in the units place is when the prime 2 is selected. Since 5 is the only prime divisible by 5, the only time 5 can appear in the units place is when the prime 5 is selected. Let us look at the situation where all the primes selected are less than 100. Then there must be two primes in the 40s, two primes in the 60s and two primes in the 80s, and at least one prime in the 20s. The only two primes in the 80s are 83 and 89, so both must be chosen. The only two primes in the 60s are 61 and 67, so both of these must be chosen. There are two primes in the 20s (23, 29) and two primes in the 50s (53, 59). Since we already have one 3 and one 9 appearing in our choices (83 and 89), the only possible choice from the 20s and 50s is either the pair 23 and 59 or the pair 29 and 53. We have now used up our complement of 3, 6, 8, 9, and we must now select 2 and 5, which uses up our complement of 2 and 5. Now 43 cannot be chosen, so we have to take 41 and 47, and this gives us our complete collection:

$$89 + 83 + 67 + 61 + 23 + 59 + 2 + 5 + 41 + 47 = 477$$

or

$$89 + 83 + 67 + 61 + 29 + 53 + 2 + 5 + 41 + 47 = 477.$$

Now can we get a set with a smaller sum with some of the primes exceeding 100? Pupils might be invited to explore this possibility.

Problem 1/23. In the addition problem below, each letter represents a different digit from 0 to 9. Determine them so that the resulting sum is as large as possible. What is the value of GB with the resulting assignment of the digits?

		A	R	L	O
		B	A	R	T
		B	R	A	D
	E	L	T	O	N
+	R	O	G	E	R

Problem 1/37. Determine the smallest five-digit positive integer N such that $2N$ is also a five-digit integer and all ten digits from 0 to 9 are found in N and $2N$.

Problem 1/2. What is the smallest integer multiple of 9997, other than 9997 itself, which contains only odd integers?

Answer. $3335 \times 9997 = 33339995$.

Comment. Observe that 9997 is close to $10000 = 10^4$, so that, for each integer n, $9997n = 10^4 n - 3n$. When n is sufficiently small, in particular, less than $10000/3$, $3n$ has at most four digits and so $9997n$ lies between $10^4(n-1)$ and $10^4 n$, and so will have an even digit if $10^4(n-1)$ has an even digit.

If n is even, then the last digit of $9997n$ is even, and so this multiple cannot have all of its digits odd. So let us suppose that n is odd and does not exceed 3333. Then $9997n$ has digits agreeing with those of $10^4(n-1)$ except for the last four digits. But $n-1$ is even, and so the fifth last digit (ten thousands place) must be even. Thus, for each n up to and including 3334, $9997n$ has an even digit. So the smallest multiplier we have to look at is 3335, and, fortunately, it delivers the goods.

Problem 1/36. Determine the unique 9-digit integer M that has the following properties: (1) its digits are all distinct and non-zero; and (2) for every positive integer $m = 2, 3, 4, \cdots, 9$, the integer formed by the leftmost m digits of M is divisible by m.

Problem 1/3. Note that if the product of any two distinct members of $\{1, 16, 27\}$ is increased by 9, the result is the perfect square of an integer. Find the unique positive integer n for which $n + 9$, $16n + 9$ and $27n + 9$ are also perfect squares.

Problem 1/38. A well-known test for divisibility by 19 follows: Remove the last digit of the number, add twice that digit to the truncated number, and keep repeating this procedure until a number less than 20 is obtained. Then, the original number is divisible by 19 if and only if the final number is 19. The method is exemplified below; it is easy to check that indeed 67944 is divisible by 19, while 44976 is not.

Find and prove a similar test for divisibility by 29.

	6	7	9	4	4		4	4	9	7	6
				8					1	2	
	6	8	0	2			4	5	0	9	
			4					1	8		
	6	8	4				4	6	8		
		8					1	6			
	7	6					6	2			
1	2						4				
1	9					1	0				

Answer. 280

Problem 2/36. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. It is well-known that the sum of any consecutive 10 Fibonacci numers is divisible by 11. Determine the smallest value of N so that the sum of any N consecutive Fibonacci numbers is divisible by 12.

Problem 3/38. Given the arithmetic progression of integers

$$308, 973, 1638, 2303, 2968, 3633, 4298,$$

determine the unique geometric progression of integers,

$$b_1, b_2, b_3, b_4, b_5, b_6,$$

so that

$$308 < b_1 < 973 < b_2 < 1638 < b_3 < 2303 < b_4 < 2968 < b_5 < 3633 < b_6 < 4298\ .$$

Problem 2/37. It was recently shown that $2^{2^{24}} + 1$ is not a prime number. Find the four rightmost digits of this number.

Problem 2/35. Let a be a positive real number, n a positive integer, and define the *power tower* $a \uparrow n$ recursively with $a \uparrow 1 = a$, $a \uparrow (i+1) = a^{a \uparrow i}$ for $i = 1, 2, cdots$. For example, we have $4 \uparrow 3 = 4^{4^4} = 4^{256}$, a number which has 155 digits. For each positive integer k, let x_k denote the unique positive real number solution of the equation $x \uparrow k = 10 \uparrow (k+1)$. Which is larger: x_{42} or x_{43}?

Problem 2/38. Compute $1776^{1492!} \pmod{2000}$, *i.e.*, the remainder when $1776^{1492!}$ is divided by 2000.

Fractional arithmetic

Problem 3/30. Nine cards can be numbered using positive half-integers $\{1/2, 1, 3/2, 2, 5/2, \cdots\}$ so that the sum of the numbers on a randomly chosen pair of cards gives an integer from 2 to 12 with the same frequency of occurrence as rolling that sum on two standard dice. What are the numbers on the nine cards and how often foes each number appear on the cards?

Comment. You can begin by looking at the number of distinct pairs that can be selected from nine cards, and verify that it is indeed equal to 36, the same number of possibilities when two dice are rolled.

Now you have to settle that either all of the cards must be numbered with integers or all with half integers, for otherwise you could find a pair consisting of cards of both types whose sum would not be an integer. Can they all be numbered with integers? Since we must get a sum of 2 in exactly one way, exactly two of the cards must be labelled with 1s. Since we need a sum of 3 in exactly two ways, there must be exactly one card labelled with a 2. Now we find that there is no way we can label cards with numbers at least as great as 3 to achieve exactly three pairs adding up to 4. So we must enter half-integers on all nine cards. By similar reasoning, working up from the low sums, we find the answer:

$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2} \ .$$

Note that there is a symmetry about the middle element; would you have expected this?

Problem 1/11. Express $\frac{19}{94}$ in the form $\frac{1}{m} + \frac{1}{n}$, where m and n are positive integers.

Answer. $\frac{19}{94} = \frac{1}{5} + \frac{1}{470}$.

Comment. This is a nice exercise on comparing vulgar fractions. Note that $19/94 > 19/95 = 1/5$. On the other hand, if both m and n exceed 10, the $1/m + 1/n \leq 2/11 < 19/94$. So the smaller of m and n is one of the numbers 5, 6, 7, 8, 9, 10. Now it is a matter of subtracting the reciprocal of each of these from 19/94 and reducing the resulting fraction to see which one of them works.

LS: This question caused much emotional distress. It was beyond the understanding of the grade 6 and 8 students who attempted it. Approximation with fractions did not seem tohave had much coverage with these students. The simple nature of the initial fraction led students to be confident that the actual answer would also be simple, despite hints to the contrary.

Problem 2/29. Determine the smallest rational number $\frac{r}{s}$ such that

$$\frac{1}{k} + \frac{1}{m} + \frac{1}{n} \leq \frac{r}{s}$$

whenever k, m, and n are positive integers that satisfy the inequality

$$\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1 \ .$$

Comment. As it stands, this problem needs a little interpretation, and one can get around the use of letters by talking about specific examples. Begin my having the students add together sums of specific reciprocals of positive integers, such as

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

with the result to be given as a vulgar fraction. Ask them to classify the sums as to being less than 1, equal to 1 or greater than 1. You might ask them to construct their own examples to give a sum of each of these three types. Do this long enough so that the kids get a sense of what they have to do to make the sum less than 1 (*i.e.*, the denominators have to be relatively large). Now you can ask them to try to find the largest sum strictly less than 1 that they can.

This will involve a bit of reasoning. If all of the denominators are at least equal to 4, then the sum of the three reciprocals cannot be greater than $3 \times (1/4)$ or $3/4$, so if we want to keep the sum less than 1 but beat 3/4, we will need to ensure that 2 or 3 have to appear among the denominators.

Suppose that 2 and 3 both appear as denominators. Then we have to add something to $1/2 + 1/3 = 5/6$ to keep the sum less than 1 but as large as possible. The best we can do is to have $1/2 + 1/3 + 1/7 = 41/42$. Suppose that 2 but not 3 appears among the denominators. Then the sum of the three reciprocals cannot be bigger than $1/2 + 1/4 + 1/5 = 19/20$, and this is less than $41/42$. If 3 but not 2 appear among the denominators, then the sum of the reciprocals cannot exceed $1/3 + 1/3 + 1/4 = 11/12$, which again is less than $41/42$.

This exercise requires a pretty good command of adding and comparing fractions, and might be pretty heady stuff for a lot of pupils. If so, you can work up to it with the following questions:

(a) What is the largest reciprocal that you can find which is strictly less than 1? (Answer: $1/2$. The reciprocal of 1 is equal to 1, while the reciprocal of any positive integer exceeding 2 is at most $1/3$.)

(b) Suppose you add together two positive integer reciprocals, and you require the sum to be strictly less than 1. What is the largest value that you can get? (Answer: $5/6 = 1/2 + 1/3$.)

If you find that some children are really getting the hang of things, you could then go to the analogous problem with 4 and 5 fractions, and elicit a conjecture as to what the answer is likely to be with 6, 7, 8, and so on, fractions. If some kid is into computing, you might have him/her check out the possibilities.

LS: This question was presented to a grade 6 class. Almost everyone got an answer. However, most students gave up after their first answer was reached. A handful of students kept working on it well into the next class and found an answer. Everyone used trial and error by adding three fractions together. This is probably because I used this method to demonstrate what the question was asking. The students who usually try to find alternative methods did not do so for this question.

Problem 4/24. Determine the positive integers $x < y < z$ for which

$$\frac{1}{x} - \frac{1}{xy} - \frac{1}{xyz} = \frac{19}{95} \ .$$

Answer. Without the inequality condition, there are two solutions: $(x, y, z) = (5, 97, 1), (5, 49, 97)$. The second of these is the desired answer.

Comment. When the answers are ultimately found, pupils might be invited to check them without actually carrying out a lot of multiplications; this would be an occasion to carry out the rules of algebra in an arithmetic setting:

$$\frac{1}{5} - \frac{1}{5 \times 49} - \frac{1}{5 \times 49 \times 97} = \frac{1}{5}\left(1 - \frac{97+1}{49 \times 97}\right)$$

$$= \frac{1}{5}\left(1 - \frac{2}{97}\right) = \frac{1}{5}\left(\frac{95}{97}\right) = \frac{19}{97} \ .$$

There are many different approaches that the pupils might take. It is probably useful for them to note at the outset that that the fraction $1/x$ must exceed $19/97$, so that $x \leq 5$. Probably the most efficient way in is to multiply the equation by $97xy$ to obtain

$$97(y-1) - \frac{97}{z} = 19xy \ .$$

Since all three terms must be integers, then $z = 1$ or $z = 97$. Taking $z = 1$ leads to $97 - (194/y) = 19x$ whence $y = 1, 2, 97, 194$. Taking $z = 97$ leads to $(97 - 19x)y = 98$. In either case, it is not very tedious to track down all the possibilities.

Problem 2/39. Assume that the irreducible fractions between 0 and 1, with denominators at most 99, are listed in ascending order. Determine which two fractions are adjacent to $\frac{17}{76}$ in this listing.

Answer. $\frac{19}{85} < \frac{17}{76} < \frac{15}{67}$.

Problem 1/1. For every positive integer n, form the number $n/s(n)$, where $s(n)$ is the sum of the digits of n in base 10. Determine the minimum value of $n/s(n)$ in each of the following cases:

(i) $10 \leq n \leq 99$
(ii) $100 \leq n \leq 999$
(iii) $1000 \leq n \leq 9999$
(iv) $10000 \leq n \leq 99999$.

Answer. (i) $19/10 = 1.9$; (ii) $199/19$; (iii) $1099/19$

Comments. (i) This can be done by trial-and-error; after some purposeful experimentation, checking what happens between adjacent numbers, some pupils will be led to the solution. If the pupils can do some algebraic manipulation, they can go directly to the sum as follows. Let $n = 10a + b$, where $1 \leq a \leq 9$ and $0 \leq b \leq 9$, so that $s(n) = a + b$. Then

$$\frac{n}{s(n)} = \frac{10a+b}{a+b} = 10 - \frac{9b}{a+b} = 10 - \frac{9}{(a/b)-1} \ .$$

To make $n/s(n)$ small, we need to make $9/[(a/b)+1]$ large and a/b small. Thus, we should take $a = 1$ and $b = 9$. This is a very nice question to have children dealing with inequalities, and, in particular, to examine how the value of a fraction changes when we monkey with its numerator or denominator. Notice the rather sophisticated use of algebraic manipulation. Some teachers may object to this level of sophistication, but any teacher who is willing to patiently help her better students through this will be rewarded by having them see the usefulness of such manipulations and greatly increase their algebraic literacy and judgment.

(ii) Let $n = 100a + 10b + c$. The strategy here is to write $n/s(n)$ in such a way that each digit appears only once and we can easily gauge how the fraction varies. We have that

$$\frac{n}{s(n)} = \frac{100a+10b+c}{a+b+c} = 100 - \frac{9(10b+11c)}{a+b+c} \qquad (1)$$

$$= \frac{99a+9b}{a+b+c} + 1 \qquad (2)$$

$$= \frac{90a-9c}{a+b+c} + 10 \qquad (3)$$

From (1), we see that for whatever values of b and c, we should make a as small as possible, so that $a = 1$. From (2), we see that for whatever values of a and b, we should make c as large as possible, so that $c = 9$. When $a = 1$ and $c = 9$, the numerator $90a - 9c$ in (3) is positive, so we should make b as large as possible, so $b = 9$.

Geometry

Problem 3/23. Exhibit in the plane 19 straight lines so that they intersect one another in exactly 97 points. Assume that it is permissible to have more than two lines intersect at some points. Be sure that your solution is accompanied by a carefully prepared sketch.

LS: This was left as an optional assignment for a Grade 7 class with a prize for the right answer. No one chose to do it.

Problem 4/29. Show that it is possible to arrange seven distinct points in the plane so that among any three of these seven points, two of the three points are a unit distance apart. (Your solution should include a carefully prepared sketch of the seven points, along with all segments that are of unit length.)

Comment. This problem has a flavour similar to that of 4/28, except that the setting is geometry rather than combinatorial. To get the students into the situation, you could ask them to do the problem with only three or four points (easy), and then advance to five or six points (somewhat harder, but should be possible for many students). Having prepared the ground in this way, the solution for seven points should be in reach for some of the students. This is a nice take-away problem.

LS: The problem was given on a worksheet to grade 7 students at the end of a class for those who had finished other work. It worked out very well. Three students in one of the grade 6 classes ended up attempting it. The main difficulty was internalizing the instructions. Understanding them was not difficult. However, the students produced many answers that did not completely follow the instructions (*i.e.*, there existed at least one group of dots where no two were a unit distance apart). One student asked what a "unit" was; once it was explained to her, she no longer wanted to attempt the question.

Problem 3/17. Show that it is possible to arrange in the plane 8 points so that no 5 of them will be the vertices of a convex pentagon. (A polygon is convex if all of its interior angles are less than or equal to 180°.

Answers. Here are some possibilities as sets of points in the cartesian plane:

$$(-4,0),(-3,1),(-2,3),(-1,6),(4,0),(3,1),(2,3),(1,6)$$

$$(\pm 3,3),(\pm 1,2),(\pm 3,-3),(\pm 1,-2)$$

$$(0,0),(8,0),(0,8),(8,8),(3,2),(6,3),(5,6),(2,5)$$

Comment. Part of the difficulty is to describe the position of the points that you want. So it would be advantageous to discuss this with the students and propose cartesian coordinates as one way to do this. Even if the pupils have not previously had cartesian coordinates, this application of them is sufficiently powerful as to serve as a motivator for their introduction. The teacher may also wish to spend some time having students look at examples of situations in which a finite number of points are and are not the vertices of a convex polygon, so that the pupils become comfortable with the concept.

Problem 3/40. The figures in **Figure 1** can be divided into two congruent halves that are related to each other by a glide reflection, as shown below it. A glide reflection reflects a figure about a line, but also moves the reflected figure in a direction parallel to that line. For a square-grid figure, the only lines of reflection that keep its reflection on the grid are horizontal, vertical, 45° diagonal and 135° diagonal. Of the two figures of **Figure 2**, divide one figure into two congruent halves related by a glide reflection, and tell why the other figure cannot be divided like that.

Problem 1/8. Prove that there is no triangle whose altitudes are of length 4, 7, and 10 units.

Solution. Suppose that such a triangle exists. Let A be the area of the triangle and a, b, c be the sides from which altitudes of length 4, 7 and 10 are erected. Then $A = \frac{1}{2}4a = \frac{1}{2}7b = \frac{1}{2}10c$, so that the three sides are in the ratio $\frac{1}{4} : \frac{1}{7} : \frac{1}{10}$, or equivalently in the ratio $35 : 20 : 14$. But in any triangle, the sum of the lengths of the shortest sides exceeds the length of the longest side. Since $20 + 14 < 35$, such a triangle is not viable in this case.

LS: (Gd 6-7) A few students had a verbal attempt at a proof, but gave up when asked to write it down. The explanations mostly consisted of the three altitudes drawn on the chalkboard and the students demonstrating that the perpendiculars of these lines cannot all connect at the same time to form a triangle.

Problem 1/7. In a trapezoid $ABCD$ with $AB \| DC$, the diagonals intersect at E, the area of ΔABE is 72, and the area of ΔCDE is 50. What is the area of trapezoid $ABCD$?

Problem 4/40. Let A and B be points on a circle which are not diametrically opposite, and let C be the midpoint of the smaller arc between A and B. Let D, E and F be the points determined by the tangent lines to the circle at A, B, and C. Prove that the area of ΔDEF is greater than half the area of ΔABC.

Problem 5/40. Hexagon $RSTUVW$ is constructed by starting with a right triangle of legs measuring p and q, constructing squares outwardly on the sides of this triangle, and then connecting the outer vertices of the squares as shown in the diagram. Given that p and q are integers with $p > q > 0$, and that the area of $RSTUVW$ is 1922, determine p and q.

Combinatorics

Problem 3/19. The numbers in the 7×8 rectangle shown below were obtained by putting together the 28 distinct dominoes of a standard set, recording the number of dots, ranging from 0 to 6 on each side of the dominoes, and then erasing the boundaries among them. Determine the original boundaries among the dominoes. (Note: Each domino consists of two adjacent squares, referred to as its sides.)

5	5	5	2	1	3	3	4
6	4	4	2	1	1	5	2
6	3	3	2	1	6	0	3
3	0	5	5	0	0	0	6
3	2	1	6	0	0	4	2
0	3	6	4	6	2	6	5
2	1	1	4	4	4	1	5

Comments. We look for adjacent pairs of numbers that occur only once; (1, 0) and (6, 6) are two such, and these pairs must be on the same dominos. The pair (6, 6) forces (5, 5) to be in the upper left corner, and eliminates one of the two possibilities for (5, 4). Thus the position of the domino (5, 4) is forced, as in the position of (4, 3). Continuing on in this vein, the positions of all the dominos can be determined.

LS: (Gd 6-8). The question was well-liked by all who tried it. The grade 6 class needed to be shown examples of how it might be tackled. After three dominos had been found, most of them were putting up their hands to suggest further dominos. The question was done on the blackboard with participation from almost everyone. The ones who were not participating actively were either trying it themselves or staring thoughtfully at the board. It took this class half an hour to complete the game. The grade 8 class were able to proceed once the 6-6 domino had been fond for them. It took them around 15 minutes, working at their seats, to finish.

Problem 3/22. Assume that there are 120 million telephones in current use in the United States. Is it possible to assign distinct 10-digit telephone numbers (with digits ranging from 0 to 9) to them so that any single error in dialing can be detected and corrected? (For example, if one of the assigned numbers is 812-877-2917 and if one mistakenly dials 812-872-2917, then none of the other numbers which differ from 812-872-2917 in a single digit should be an assigned telephone number.)

LS: This was presented to a grade 6 class to try in groups. I first asked how many digits would be required to give each person their own distinct telephone number. About half the class got an answer in five minutes. Then the actual question was given. One student devised a tree diagram to look at the question. The class unanimously decided on the right answer; however, the student with the tree diagram was the only one to attempt an explanation.

Problem 4/10. A bag contains 1993 red balls and 1993 black balls. We remove two balls at a time repeatedly and

(i) discard them if they are of the same colour,

(ii) discard the black ball and return to the bag the red ball if they are different colours.

What is the probability that this process will terminate with one red ball in the bag?

LS: This did not garner much enthusiasm from a grade 7 class on a Friday afternoon.

Problem 3/8. (i) Is it possible to rearrange the numbers $1, 2, 3, \cdots, 9$ as $a(1), a(2), a(3), \cdots, a(9)$ so that all the numbers listed below are different? Prove your assertion.

$$|a(1) - 1|, |a(2) - 2|, |a(3) - 3|, \cdots, |a(9) - 9| .$$

(ii) Is it possible to rearrange the numbers $1, 2, 3, \cdots, 9, 10$ as $a(1), a(2), a(3), \cdots, a(9), a(10)$ so that all the numbers listed below are different? Prove your assertion.

$$|a(1) - 1|, |a(2) - 2|, |a(3) - 3|, \cdots, |a(9) - 9|, |a(10) - 10| .$$

Problem 3/1. On an 8×8 board we place n dominoes, each covering two adjacent squres, so that no more dominoes can be placed on the remaining squares. What is the smallest value of n for which the above statement is true?

Problem 3/35. Suppose that the 32 computers in a certain network are numbered with the 5-bit integers 00000, 00001, $\cdots$, 11111 (bit is short for binary digit). Suppose that there is a one-way connection from computer A to computer B if and only if A and B share four of their bits with the remaining bit being a 0 at A and a 1 at B. (For example, 10101 can send messages to 11101 and to 10111.) We say that a computer is at level k in the network if it has exactly k 1's in its label ($k = 0, 1, cdots, 5$). Suppose further that we know that 12 computers, three at each of the levels 1, 2, 3 and 4, are malfunctioning, but we do not know which ones. Can we be sure that we can send a message from 00000 to 11111?

Problem 5/3. Two people, A and B, play the following game with a deck of 32 cards. With A starting, and thereafter the players alternating, each player takes either one card or a prime number of cards. Eventually, all the cards are chosen, and the person who has none to pick up is the loser. Who will win the game if they both follow optimal strategy?

LS: Two grade 8 students sat down with cards and tried a few different ways of doing this question. it eventually dissolved in laughter as they found that neither could devise an optimal strategy. A suggestion was made that they write down a few strategies first and try them. The half-hour period ended without significant progress.

Problem 3/34. The figure show the map of Squareville, where each city block is of the same length. Two friends, Alexandra and Brianna, live at the corners marked by A and B, respectively. They start walking towards each other's house, leaving at the same time, walking with the same speed, and independently choosing a path to the other's house with uniform distribution out of all possible minimum-distance paths (that is, all minumum-distance paths are equally likely). What is the probability that they will meet?

Problem 4/28. Let n be a positive integer and assume that for each integer k, $1 \leq k \leq n$, we have two disks numbered k. It is desired to arrange the $2n$ disks in a row so that for each k, $1 \leq k \leq n$, there are k disks between the two disks that are numbered k. Prove that

(i) if $n = 6$, then no such arrangement is possible;

(ii) if $n = 7$, then it is possible to arrange the discs as desired.

Comment. This problem needs a little reworking before the young can be turned loose upon it. Part (i) is too sophisticated for this level, but one can work up to (ii). One could begin by giving the illustration of the finite sequence

$$3 \quad 1 \quad 2 \quad 1 \quad 3 \quad 2 \, .$$

Tell the children that there are 6 numbers, two each of the numbers 1, 2 and 3; they are arranged in a line such that there is one number between the two occurrences of 1, two numbers between the two occurrences of 2, three numbers between the two occurrences of 3. You could ask them if there are any other arrangements of these numbers that have the same property; some may see that the backwards order 231213 also works.

Now you can begin to up the ante. Can they arrange eight numbers consisting of two each of 1, 2, 3, 4 in a line so that the previous conditions are satisfied and in addition there are four numbers between the two occurrences of 4. If you wish, you could ask about 5 and 6, or you could jump directly to the analogous problem for 7 (which is part (ii)) and 8. As there will be a number of solutions, you could allow the children to consult each other while they are solving this and challenge them to see what solutions they can come up with.