CHAPTER SEVEN

DYNAMICAL SYSTEMS

§1. THE LOGISTIC FUNCTION

Suppose that $f(x)$ is a continuous function taking a locally compact Hausdorff space into itself. The point c is a *fixed point* for f if and only if $f(c) = c$. It is *attractive* (*stable*) if and only if there is a neighbourhood U of c such that, for each $x \in U$, $f^{(n)}(x) \rightarrow c$ as $n \rightarrow \infty$, where $f^{(n)}$ represents the nth iterate of f. It is *repellent* (*unstable*) if this sequence of iterates moves away from c as n increases. The set $\{f^{(n)}(x)\}$ is called the *orbit* of x. The point c is *periodic* if and only if for some positive integer r, $f^{(r)}(c) = c$; the smallest value of r for which this occurs is called the *prime period* of c. The set of iterates of such a point is a *periodic orbit*. A periodic orbit is attractive or repellent according as c is an attractive or repellent fixed point of $f^{(r)}$. The *basin of attraction* for a fixed point c is the set of points x for which $\lim_{n\rightarrow\infty} f^{(n)}(x) = c$.

When the function f is a real continuously differentiable function taking an open subset of $\mathbf{R}$ to itself, then a fixed point c is attractive whenever $|f'(c)| < 1$. There exists aa constant $\alpha < 1$ and an open interval U containing c for which $|f'(x)| \leq \alpha$ when $x \in U$. For each $x \in U$, there exists $\xi \in U$ for which $f(x) - f(c) = f'(\xi)(x - c)$. Then $|f(x) - c| = |f(x) - f(c)| \leq \alpha|x - c|$ and $f(x) \in U$. Thus, if $x_1 \in U$ and $x_{n+1} = f(x_n)$ for $n \geq 1$, then $|x_{n+1} - c| \leq \alpha|x_n - c|$ and $\lim x_n = c$. If $f'(c)$ is positive, then the sequence of convergents is eventually monotonic; if negative, then the sequence eventually oscillates about its limit.

A similar argument shows that the fixed point c is repellent when $|f'(c)| > 1$. The nature of periodic points of period r can be analyzed using the derivative of $f^{(r)}$.

Let λ be a positive parameter. Define the function $F_\lambda(x)$ for $0 \leq x \leq 1$ by

$$F_\lambda(x) = \lambda x(1 - x) \ .$$

For $0 < \lambda \leq 1$, this function has a unique fixed point $x = 0$ which is attractive, with every point of $[0, 1]$ in its basin of attraction.

For $1 < \lambda \leq 3$, the function has two fixed points, $x = 0$ which is repellent, and $x = (\lambda - 1)/\lambda$ which is attractive, with every point in $(0, 1)$ in its basin of attraction. For $1 < \lambda \leq 2$, the orbit of each point in $(0, 1)$ is eventually monotonic. When $2 < \lambda \leq 3$, the orbit of each point in $(0, 1)$ eventually oscillates about the fixed point.

When $\lambda > 3$, then the dynamics of this function become more interesting, as we begin to have periodic orbits. The fixed point in $(0, 1)$ becomes repellent and spawns an orbit of period 2, whose basin of attraction in $(0, 1)$. This means that the orbit of any point in this interval eventually oscillates between neighbourhoods of the period-2 points. Note that

$$\begin{aligned} F_\lambda(F_\lambda(x)) &= \lambda(\lambda x(1 - x))[1 - \lambda x(1 - x)] = \lambda^2 x[(1 - x) - \lambda x(1 - x)^2] \\ &= \lambda^2 x[1 - (\lambda + 1)x + 2\lambda x^2 - \lambda x^3] \ . \end{aligned}$$

To determine points of period 2, we need to solve the equation $x = F_\lambda(F_\lambda(x))$, which is equivalent, for $x \neq 0$ to

$$0 = \lambda^3 x^3 - 2\lambda^3 x^2 + \lambda^2(\lambda + 1)x - (\lambda^2 - 1) = [\lambda x - (\lambda - 1)][\lambda^2 x^2 - (\lambda^2 + \lambda)x + (\lambda + 1)] \ .$$

The first factor on the right corresponds to the fixed point. The points of period 2 are given by

$$x = \frac{(\lambda + 1) \pm \sqrt{(\lambda - 3)(\lambda + 1)}}{2\lambda} \ .$$

As λ further increases, the period-2 orbit becomes repellent and spawns an attractive orbit of period-4. This "period-doubling" property continues with increase of λ until there are period points for prime periods equal to all powers of 2. Then other periods appear, until finally there are points of period 3, at which stage there are points of all positive integral prime periods. The orderly accretion of periods with increase of λ is described by Sarkovskii's Theorem.

Sarkovskii's Theorem. *Let f be a continuous real function defined taking a real interval to itself. Suppose that f has a points of period 3; then f has a point of every other period exceeding 1.*

Moreover, if f has a point with period n exceeding 1, then it has a point of every period following n in the following sequence:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd \cdots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \cdots \cdots 2^3 \rhd 2^2 \rhd 2 \ .$$

($\rhd$ is a conventional notation in this situation that indicates the passage from each term to the next.)

We can analysis the dynamics of $F_4(x)$ explicitly. Making the substitution $x = \sin^2 \frac{\pi}{2} t$ with $0 \leq t \leq 1$, we find that

$$F_4(x) = 4x(1-x) = 4\sin^2 \frac{\pi}{2} t \cos^2 \frac{\pi}{2} t = (\sin^2 \frac{\pi}{2} 2t) \ .$$

Thus, the dynamics of F_4 are essentially those of the function defined by

$$g(t) = \begin{cases} 2t; & \text{if } 0 \leq \theta \leq \frac{1}{2} \ , \\ 2(1-t); & \text{if } \frac{1}{2} < t \leq 1 \end{cases}$$

Orbits $\{x_n\}$ of F_4 and $\{t_n\}$ of g are related by $x_n = \sin^2 \frac{\pi}{2} t_n$, so that, once we can identify periodic points of g, we can do so for F_4. For example, the fixed point $2/3$ for g corresponds to the fixed point $\sin^2 \pi/3 = 3/4$ for F_4. The orbit of period 2 for g, $\{2/5, 4/5\}$, corresponds to the orbit $\{\sin^2 \pi/5, \sin^2 2\pi/5\} = \{(5-\sqrt{5})/8, (5+\sqrt{5})/8\}$ for F_4. Similarly, we have corresponding orbits of period 3: $\{2/9, 4/9, 8/9\}$ and $\{\sin^2 \pi/9, \sin^2 2\pi/9, \sin^2 4\pi/9\}$.

If $\lambda > 4$, then F_λ takes some points of $[0,1]$ outside of $[0,1]$. We can restrict the domain of F_λ to

$$\Lambda \equiv \{x : F_\lambda^{(n)}(x) \in [0,1] \ \text{ for each nonnegative integer } n\} \ .$$

If we define

$$I_0 = \{x : x \in [0, 1/2), F_\lambda(x) \in [0,1]\}$$

and

$$I_1 = \{x : x \in (1/2, 1], F_\lambda(x) \in [0,1]\} \ ,$$

then Λ is a "Cantor set" contained in $I_0 \cup I_1$. For $x \in \Lambda$, we define its *itinerary* to be the sequence $S(x) = s_0 s_1 s_2 \cdots s_j \cdots$, where $s_j = 0$ when $F_\lambda^{(j)}(x) \in I_0$ and $s_j = 1$ when $F_\lambda^{(j)}(x) \in I_1$. When λ is sufficiently large, S is a homeomorphism between Λ and the space of $0-1$ sequences equipped with the metric

$$d[\mathbf{s}, \mathbf{t}] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

and F_λ is conjugate to the left shift

$$\sigma : s_0 s_1 s_2 \cdots \longrightarrow s_1 s_2 \cdots$$

in the sense that $S \circ f_\lambda = \sigma \circ S$. Thus, the dynamics of F_λ can be described in terms of the more tractable dynamics of σ, where it is easy, for example, to identify periodic points and characterize chaotic behaviour.

§2. THE MANDELBROT SET

To examine the dynamics of the logistic map, we can also look at a conjugate map, whose dynamics are generally considered in the complex plane. If $\lambda > 0$ and we define the linear mapping $h(x) = -(1/\lambda)x+(1/2)$ whose composition inverse is $h^{-1}(x) = -\lambda(x - (1/2))$, then we have

$$\begin{aligned} h^{-1} \circ F_\lambda \circ h(x) &= h^{-1}\left(\lambda\left(\frac{1}{2} - \frac{x}{\lambda}\right)\left(\frac{1}{2} + \frac{x}{\lambda}\right)\right) \\ &= -\lambda^2\left(\frac{1}{2} - \frac{x}{\lambda}\right)\left(\frac{1}{2} + \frac{x}{\lambda}\right) + \frac{\lambda}{2} = x^2 + \frac{\lambda}{2}\left(1 - \frac{\lambda}{2}\right) , \end{aligned}$$

so that $F_\lambda(z)$ is conjugate to a mapping of the form $Q_c(z) \equiv z^2 + c$. In particular, $c < -2$ corresponds to $\lambda > 4$. When $\lambda = 4$, we find that $F_4(x) = 4x(1 - x)$ gets transformed to $Q_{-2}(z) = z^2 - 2$.

The mapping $Q_0(z) = z^2$ is particularly easy to analyze. For $|z| > 1$, the points iterate to infinity, while for $|z| < 1$, they iterate to 0. On the circumference of the unit circle $\mathbf{T}$, the mapping is conjugate to $\theta \longrightarrow 2\theta$ modulo 2π.

The *Julia set* $J(f)$ of a continuous function f taking the complex numbers into themselves is the closure of the set of repellent periodic points of f. It is invariant under the action of f and under the taking of preimages of its points. In particular, the Julia set of Q_0 is $\mathbf{T}$. If f is a polynomial of degree exceeding 1, then the Julia set is nonvoid. This depends on the result that, for any polynomial f, either $f(z)$ has a fixed point q for which $f'(q) = 1$ or a fixed point q for which $|f'(q)| > 1$.

To see this, let $g(z) = f(z) - z$, so that $f'(z) = g'(z) + 1$. If $g(z)$ has a multiple root ω, then $g(\omega) = g'(\omega) = 0$ and $f(\omega) = \omega$ while $f'(\omega) = 1$. Suppose that $g(z)$ has distinct zeros ζ_i and, if possible, that $|f'(\zeta_i)| \leq 1$ and $f'(\zeta_i) \neq 1$ for all i. Then, using a result from the end of Section 4.2, we have that

$$\sum_i \frac{1}{f'(\zeta_i) - 1} = \sum_i \frac{1}{g'(\zeta_i)} = 0 .$$

Since $|f(\zeta_i)| \leq 1$ for all i, $f(\zeta_i) - 1$ lies in a circle of radius 1 and centre -1, so that each $(f'(\zeta) - 1)^{-1}$ is a nonzero number in the left half plane. Since the sum of such numbers must have a negative real part, we obtain a contradiction. The result follows. □

In the case that $|c| > 2$, then $\lim_{n\to\infty} Q_c^{(n)}(z) = \infty$ when $|z| \geq |c|$. Thus, $J(Q_c)$ lies inside $D(0, |c|)$. The set Λ of points whose forward orbits lie inside this disc is (like the set Λ defined for the logistic map) a Cantor set. The Julia set $J(Q_{-2})$ of $z^2 - 2$ is the real closed interval $[-2, 2]$. When $|c| < 1/4$, then $J(Q_c)$ is a simple closed curve.

The Julia set $J(Q_c)$ is the boundary of the *filled-in Julia set* $K(Q_c)$ of points whose orbits do not tend to ∞.

The point 0 has a special role to play for the mapping $Q_c(z)$ as it is the sole critical point (zero of its derivative). Since the basin of attraction of every attracting periodic orbit contains a critical point, if Q_c has an attractive periodic orbit, then 0 is in its basin of attraction. In this situation, $\lim_{n\to\infty} Q_c^{(n)}(0) \not\to \infty$. In fact this latter condition is equivalent to $K(Q_c)$ being connected.

The *Mandelbrot set* is defined to be the set of those parameter c for which $Q_c^{(n)}(0) \not\to \infty$ or, equivalently, $K(Q_c)$ is connected. We can identify portions of the Mandelbrot set that correspond to attractive periodic orbits of various orders.

A solid introduction to this area is provided in [**1**].

§3. NEWTON'S APPROXIMATION REVISITED

In Section 2.2, we discussed Newton's approximation method for solving the equation $p(x) = 0$, and its relation to the dynamical system defined by the function

$$f(x) = x - \frac{p(x)}{p'(x)} \ ,$$

where the simple zeros of p are fixed points of f.

Since

$$f'(x) = \frac{p(x)p''(x)}{p'(x)^2} \ ,$$

we see that c fixed point is attractive and Newton's approximating sequence will convergence to the solution c if its initiated on any interval of c for which

$$\left|\frac{p(x)p''(x)}{p'(x)^2}\right| \le \alpha < 1 \ .$$

When this condition does not hold for the initial approximating value, the Newton sequence can display chaotic behaviour and either settle down towards a more distant root or else not converge at all. In [**2**, pp. 28-33], it is shown how this behaviour can be analyzed through a "cobweb" analysis on the graph of $f(x)$.

§. PROBLEMS AND INVESTIGATIONS

1. Assume that the quadratic polynomial $f(x) = ax^2 + bx + c$ $(a \neq 0)$ has two distinct fixed points u and v, and that -1 and 1 are two fixed points of the function $f(f(x))$, neither of them equal to u or v. Determine u and v.

2. Suppose that $\{z_1, z_2, \cdots, z_n\}$ is a cycle of length n for the function $Q_c(z) = z^2 + c$, *i.e.*,

$$z_{i+1} = Q_c(z_i) \quad (1 \le i \le n-1) \quad \text{and} \quad z_1 = Q_c(z_n) \ .$$

Prove that

$$\prod_{i=1}^{n}(z_i + Q_c(z_i) = 1 \ .$$

3. Determine the values of the parameter c for which the mapping Q_c has (a) an attractive fixed point; (b) an attractive orbit of period 2. Sketch the loci of these values of c in the complex plan and relate them to the Mandelbrot set.

Hints and comments

1. The function has the form $ax^2 - x - a$. [CMJ #841: 38:1 (January, 2007), 60; 39:1 (January, 2008), 66=67]

2. Observe that $u + v = (Q_c(u) - Q_c(v))/(u - v)$. The result can be generalized to polynomial of higher degree. See [**1**].

References

1. Robert L. Benedetto, An elementary product identity in polynomial dynamics. *American Mathematical Monthly* 108:9 (November, 2001), 860-864

2. Robert L. Devaney, *An introduction to chaotic dynamical systems* 2nd edition. Addison-Wesley, 1989

3. Dan Kalman, *Uncommon mathematical excursions: Polynomia and related realms* Mathematical Association of America, 2009. Section 2.4.

4. Donald G. Saari & John B. Ulrenko, Newton's method, circle maps, and chaotic motion. *American Math. Monthly* 91:1 (January, 1984), 3-17