Don't be confounded by compounding

Suppose that I lend you \$10,000. Since I will not have the use of this money for some time, you should pay me a fee for my inconvenience. What principles should apply? The first is that the fee should be proportional to the amount loaned (called the *principal*). If the amount is doubled, then the fee should also be twice as much. So we can settle on a ratio of the fee to the principal, say 6 parts for each 100 (or 6 per cent, which says the same thing in Latin). (This may seem usurious in the present depressed times, but let's stick with it since the numbers are nice.) So for each \$100, I will expect a fee (or *interest*) of \$6. For \$10,000, the fee would be 6% of \$10,000, or \$600.

However, this does not take into account the amount of time I am deprived of my money. So the second principle kicks in, that the fee should also be proportional to the time that the loan is outstanding. If you have the money twice as long, then the fee will be doubled. So we may decide that the interest on the loan is to be 6% for each year (or *per annum*, Latin again!). So if you borrow the money for 6 months (half a year), I would expect to get the \$10,000 back plus half of \$600 or \$300 in interest.

However, from the point of the lender, things are not quite satisfactory. Because, as long as you have my money without any repayment, the amount of interest that is due is building up and I am not getting any reward for this outstanding interest. So every once in a while, we look at the amount of interest owing and add this to the principal.

At the end of six months, I decide to add the outstanding interest of \$300 to the principal, and so at the end of the year, you will owe me \$10,300 plus the 3% interest of \$309 accumlated over the second six months. So I should get \$10,609 back at the end of the year, an effective annual rate of interest of 6.09%. We say that the interest of 6% is *compounded semi-annually.*

But we could just as easily compound the interest quarterly, every three months. Over three months, the rate of interest is $\frac{1}{4} \times 6 = 1.5\%$ and the principal will accumulate to $\$10,000 + 0.015 \times 10,000 = 10,000 \times (1.015)$. In six months, the amount owing will be $\$(10,000) \times (1.015)^2$ and at the end of the year will be $\$(10,000) \times (1.015)^4 = (10,000) \times (1.061363550625)$ or \$10,613.63. As you see, as we reduce the compounding period, the amount owing at the end of the year goes up. But does it increase indefinitely, or is there a limit? For example, if we compount monthly, the interest rate is a half per cent per month, and the principal owing at the end of the year is the original principal multiplied by $(1.005)^{12} = 1.0616778118664$ for an effective annual interest rate of about 6.17%.

We can certainly compound more frequently – some banks have "daily interest" accounts with daily compounding, but it turns out that there is not much more room for expansion. If we let the compounding period go to zero, the effective annual interest rate goes up to about 6.1833%. This involves a very important mathematical constant, e, whose value is about 2.718281828459. If we denote the interest rate by i (0.06 in our example), the effective annual interest rate when we compound momentarily is $e^i - 1$.