The turnaround

Mathematicians are among other things, scientists. Their areas of investigation are numbers, shapes and various abstract structures. Like other scientests, they make observations, observe patterns and try to account for them, coming up with theories that tie together what may seem to be diverse things.

Here are two things for you to play with.

(1) Start with any number except 1. Form a new number found by dividing one more that the number you started with by one less than that number. For example, if you started with 7, you would form $8/6 = 4/3$. Repeat the process with this new number. Do you notice anything? If so, is it part of a general pattern and how would you know?

(2) Form a sequence of numbers as follows. Take any two nonzero numbers in order. The third number is the second number plus one, divided by the first. For example, suppose you start with the numbers 5 and 8. Then the third number is 9/5. To get the fourth number, add one to the third number and then divide by the second. For our example, you would get $(14/5) \div 8 = 7/20$.

Continue the sequence by adding one to the last number and dividing by the number before that. (If zero appears, we have to stop, so we forget about this situation.) What happens?

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In both sequences, you should come back to where you started. In the first, the second turn of the crank brings you back to the original number. In the case of the second, the sequence repeats itself after the first five entries. In both cases, as there are infinitely many choices of starting numbers, we cannot check every case, so if we are to be sure that our observations are generally true, we need a means of justification that gets around this barrier of infinity.

The tool we need is algebra. The idea is to have stand-ins for numbers and assume that they satisfy all the rules of ordinary arithmetic. (Anyone who has used a formula or an Excel spreadsheet will be familiar with this strategy.) We write a prescription for the numbers to follow.

For the first sequence, we suppose that the number is x. The process tells us to add one to this $(x+1)$ and then divide by one less than it $(x-1)$ to get the result $(x+1)/(x-1)$. We can substitute for x any starting number we want, and then read off from this expression what the second term is.

Now let us apply our rule to get the next term of the sequence. Since x goes to $(x+1)/(x-1)$, then $(x+1)/(x-1)$ goes to

$$\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{2x}{2} = x.$$

You can check that this works by plugging in any value for x. You will see that the arithmetic operations involved in "simplifying" the result correspond to exactly what you would do for actual numbers. We have here a template that we can use to verify the result that the sequence comes immediately back to its original term for any first number except 1.

For the second sequence, suppose we start with the numbers x and y in that order. Then the terms of the sequence are:

$$x \quad y \quad \frac{y+1}{x} \quad \frac{x+y+1}{xy} \quad \frac{x+1}{y} \quad x \quad y \quad \cdots.$$

A little bit of explanation is needed. When you work out examples with actual numbers, it appears that the fractions you have do deal with get increasingly fierce, but then things seem to collapse to a serene outcome. This is reflected in the algebra that you takes you from the term $(x+y+1)/xy$ to the next one:

$$\frac{\frac{x+y+1}{xy}+1}{\frac{y+1}{x}} = \frac{\frac{x+y+1+xy}{xy}}{\frac{y+1}{x}} = \frac{(x+1)\times(y+1)}{y\times(y+1)} = \frac{x+1}{y}.$$

Again you can substitute in particular numbers x and y and see that the algebra is taking you through ordinary numerical manipulations.

Algebra appears here in the guise of a proof technique, a way of validating general results that is special to mathematics. This is a wonderful situation to explore with a group of middle school students. At a technical level, it provides practice in dealing with fractions, an area many pupils have trouble with. If you have students start with their own numbers and present their findings at the blackboard, the students begin to get the idea that the sequences are supposed to repeat, and if they find that theirs does not, then they will spontaneously check their work (no need for a teacher with a red pencil here). This is also an occasion to develop their algebraic prowess. At the conceptual level, it teaches something very fundamental about algebra and its application.