A problem for this year

We can write the number 2014 as the sum of whole numbers is a lot of ways. For example,

$$2014 = 382 + 401 + 419 + 812.$$

The product of the four summands $382 \times 401 \times 418 \times 812$ is a pretty big number. On the other hand, if we express 2014 as the sum of 2014 ones, then the product of the summands is as small as we can possibly make it: 1. Now here is the question: if we write 2014 as the sum of positive whole numbers, what is the largest possible product of the summands?

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Obviously, there are an awful lot of ways of writing 2014 as the sum of positive integers, and it would be incredibly tedious to check the product for each one to find the largest. So we need to use a little guile.

There is probably nothing in the problem that depends on 2014; the principle of its solution applies if we replace this number by any other. Before reading further, you should try to solve the problem replacing 2014 by the number 10, say. This makes it much more manageable and you can get a better feel for what is going on.

We can approach the problem by starting with a particular sum and asking whether we can make a small modification to make the product of the summands larger. We might notice that if, say, 7 is a summand, then we can replace it by $2 + 5$, so that in place of getting a contribution of 7 towards the product, we get a contribution of $10 = 2 \times 5$. In fact, if we have a term greater than 4 in the sum, we can replace it by a pair of smaller terms, one being 2, that add up to it and get a larger product. We should also note that it is no good having 1 as a summand, as we can incorporate it with another summand to make the product larger.

So to make the product of the summands as large as possible, none of the summands should exceed 4. As for 4 itself, we can replace it by $2 + 2$ while keeping the product the same. So we do this, and realize that we have to look only at representation of 2014 as the sum of twos and threes. If we have three twos among the summands, then we get a contribution of $8 = 2 \times 2 \times 2$ towards the product, which is less than the contribution $9 = 3 \times 3$ that two threes would make. So we can replace the three twos by two threes for a bigger product. This means that we will get the largest possible product by expressing 2014 as the sum of a lot of threes and no more than two twos. Since $2014 = 670 \times 3 + 2 \times 2$, we find that writing 2014 as the sum of 670 threes and two twos gives the largest product $3^{670} \times 2^2$. (By the same token, the largest product when you express 10 as the sum of whole number is $3^2 \times 2^2 = 36$.)