An interesting extraneous root

1. Problem. This problem was posed by Stanley Rabinowitz in the Spring, 1982 issue of the *AMATYC Review*:

Solve the system

$$x + xy + xyz = 12$$

$$y + yz + yzx = 21$$

$$z + zx + zxy = 30 \ .$$

Solution 1. Let $u = xyz$. Then $x+xy = 12-u$ so that $z+z(12-u) = 30$ and $z = 30/(13-u)$. Similarly, $y = 21/(31-u)$ and $x = 12/(22-u)$. Plugging these expressions into any one of the three equations yields that

$$0 = u^3 - 65u^2 + 1306u - 7560 = (u-10)(u-27)(u-28) \ .$$

We get the three solutions

$$(x, y, z) = (1, 1, 10), \left(-\frac{12}{5}, \frac{21}{4}, -\frac{15}{7}\right), (-2, 7, -2) \ ,$$

al of which satisfy the system.

Solution 2. Define u and obtain the expressions for x, y, z as in Solution 1. Substitute these into the equation $xyz = u$. This leads to the quartic equation

$$\begin{aligned} 0 &= u(u-13)(u-22)(u-31) + (12)(21)(30) = u^4 - 66u^3 + 1371u^2 - 8866u + 7560 \\ &= (u-1)(u^3 - 65u^2 + 1306u - 7560) \ . \end{aligned}$$

Apart from the three values of u already identified, we have $u = 1$. This leads to

$$(x, y, z) = \left(\frac{4}{7}, \frac{7}{10}, \frac{5}{2}\right) \ .$$

While, indeed, $xyz = 1$, we find that $x + xy + xyz = 69/35$, $y + yz + xyz = 69/20$, $z + zx + xyz = 69/14$, so the solution $u = 1$ is extraneous.

2. The extraneous solution. The second solution introduces an interesting extraneous root for u. What is its significance?

In the first solution, the values of x, y, z in terms of u were plugged back into the original equations in order to get an equation for u; so it is to be expected, that the values of x, y, z corresponding to u will satisfy the system.

However, in the second solution, the presence of the extraneous solution seems to indiucate a loss of information as we proceed through the solution and make some step that is not reversible. We notice that we set up the equation for u by using the equation $xyz = u$ rather than any one of the given equations.

We start with the substitution $u = xyz$, and then use the first and third equations to obtain an expression for z. We use the three equations $xyz = u$, $x + xy = 12 - u$ and $z(13 - u) = 30$ to obtain in turn $z = 30/(13 - u)$, $xy = u(13 - u)/30$,

$$x = (12-u) - \frac{u(13-u)}{30} = \frac{u^2 - 43u + 360}{30}$$

and

$$y = \frac{13u - u^2}{u^2 - 43u + 360} \ .$$

When $u = 10, 27, 28$, we obtain the solutions that we got before However, when $u = 1$, the $(x, y, z) = (53/5, 2/53, 5/2)$, and find that, indeed, $x + xy + xyz = 12$ and $z + zx + zxy = 30$, but that $y + yz + yzx = 60/53$. In fact, the equations

$$\frac{u^2 - 43u + 360}{30} = \frac{12}{22 - u}$$

and

$$\frac{13u - u^2}{u^2 - 43u + 360} = \frac{21}{31 - u}$$

both lead to the cubic equation with roots 10, 27, 28.

However, this still does not explain where the solution $u = 1$ comes from.

3. The general situation. Consider the more general system

$$x + xy + xyz = a$$

$$y + yz + yzx = b$$

$$z + zx + zxy = c \ .$$

Following the same strategy as in the foregoing problem, we let $xyz = u$ and get $x = a/(b + 1 - u)$, $y = b/(c + 1 - u)$, $z = c/(a + 1 - u)$. Plugging these values of x, y, z into any of the three equations (taking the xyz term as u or working it out as the product of x, y, z in terms of u) leads to the cubic equation

$$0 = u^3 - (a + b + c + 2)u^2 + (ab + bc + ca + a + b + c + 1)u - abc = 0 \ . \qquad (*)$$

On the other hand, substituting x, y, z in terms of u into the equation $xyz = u$ leads to the quartic equation

$$\begin{aligned} 0 &= u^4 - (a + b + c + 3)u^3 + (ab + bc + ca + 2a + 2b + 2c + 3)u^2 - (a + 1)(b + 1)(c + 1)u + abc \\ &= (u - 1)[u^3 - (a + b + c + 2)u^2 + (ab + bc + ca + a + b + c + 1)u - abc] \ . \end{aligned}$$

Another way of writing $xyz = u$ is as

$$abc = u(a + 1 - u)(b + 1 - u)(c + 1 - u)$$

where it is clear that $u = 1$ is a solution.

Let $u = 1$ and take $x = a/b$. $y = b/c$, $z = c/a$. Then $xyz = 1$ is satisfied, but

$$x + xy + xyz = \frac{ab + bc + ca}{bc} = va \ ,$$

$$y + yz + yzx = \frac{ab + bc + ca}{ca} = vb \ ,$$

$$z + zx + zxy = \frac{ab + bc + ca}{ab} = vc \ ,$$

where

$$v = \frac{ab + bc + ca}{abc} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ .$$

We observe that $(1/va) + (1/vb) + (1/vc) = 1$.

We can pursue the same sort of analysis in the general case as in Section 2. Solving the equations $xyz = u$, $x + xy = a - u$ and $z(a + 1 - u) = c$ in terms of the parameter u leads to

$$x = \frac{u^2 - (a + c + 1)u + ac}{c}$$

$$y = \frac{(a + 1)u - u^2}{u^2 - (a + c + 1)u + ac}$$

$$z = \frac{c}{a + 1 - u} \ .$$

These give the values a and c for $x + xy + xyz$ and $z + zx + zxy$, respectively, but in general a different value of $y + yz + yzx$. Indeed, the only values of u that will lead to a solution of all three equations simultaneously are the zeros of the cubic $(*)$.

4. Another example. Since the solution $u = 1$ will work when the sum of the reciprocals of a, b, c is 1, let us consider the following example:

$$x + xy + xyz = 2$$

$$y + yz + yzx = 3$$

$$z + zx + zxy = 6 \ .$$

In this case $(x, y, z) = (2(4 - u)^{-1}, 3(7 - u)^{-1}, 6(3 - u)^{-1})$, and we get the cubic equation

$$0 = u^3 - 13u^2 + 48u - 36 = (u - 1)(u - 6)^2 \ .$$

The two roots of this equation lead to the valid solutions $(x, y, z) = (\frac{2}{3}, \frac{1}{2}, 3)$ and $(x, y, z) = (-1, 3, -2)$.

The presence of the double root 6 is this example seems to be fortuitous; when $(a, b, c) = (2, 4, 4)$, the cubic has the real root 1 and a pair of imaginary roots. The only solution in this case is $(x, y, z) = (\frac{1}{2}, 1, 2)$.

For the general problem with $(1/a) + (1/b) + (1/c) = 1$, the cubic equation in u is

$$\begin{aligned} 0 &= u^3 - (a + b + c - 2)u^2 + (abc + a + b + c + 1)u - abc \\ &= (u - 1)[u^2 - (a + b + c + 1)u + abc] \ . \end{aligned}$$