The angle in a semicircle

Consider the following result:

Proposition. *Let A and B be opposite ends of the diameter of a circle, and let P be a point on its circumference. Then angle APB is right.*

This geometric result is familiar to many high school students. Although it is simply stated, there are many dimensions to it and the mere statement of the result will inevitably fail to convey its richness. As with any geometric result, certain properties are highlighted for consideration and related; the posited relationship might seem quite mysterious and incomprehensible. In order to feel more at home and perceive that the result is somehow natural, it is desirable to probe deeply and sense how the mathematical structure is woven together.

This particular result can be approached from many directions (Barbeau, 1988), and the purpose of this note is to comment on the mathematical content of some of these.

The most straightforward proof is probably to join P to the centre O of the circle, observe that triangles AOP and BOP are isosceles and deduce that the angle APB is equal to the sum of the remaining two angles of the triangle APB.

What does such a proof do for us? The result is briefly stated; all we are given is that A, B and P are three points on a circle whose diameter is AB. Accordingly, we have to parse these facts closely for whatever meaning they may yield. What does it mean for A, B and P to lie on a circle? It means that they are the same distance from some point O. In other words, this proof forces students to move from an intuitive idea of what a circle is to a characterization of it, in this case, as a set of points a fixed distance from a certain point. The significance of the circle in the hypothesis is laid bare.

In fact, this is a common value of proofs. The mere statement of a theorem will involve terminology and (if we are lucky) contain some intuitive content for the student. At the point where try to explicitly link hypotheses and conclusion, we need to unpack the meaning of the terms and identify those particular characteristics that are particularly pertinent to the situation.

Other proofs raise different issues. Paraphrasing John Donne's dictum that no man is an island, we can assert that no result stands completely on its own. It fits into some kind of mathematical context, and different proofs indicate the places in the mathematical world that the result lives. There is a second proof, similar in some respects to the foregoing one, that situates the result in a larger context.

Suppose, keeping the same notation, that we produce the segment PO through O to a point X. Observe that XOB is the exterior angle of isosceles triangle POB and so is equal to twice angle OPB. Similarly, angle XOA is equal to twice angle OPA. Hence

$$\angle APB = \angle OPA + \angle OPB = \frac{1}{2}(\angle XOA + \angle XOB) = \frac{1}{2}(180^\circ) = 90^\circ \ .$$

However, by making a slight manoeuvre, we can see that the result is the special case of a more general one. Imagine that A and B are moved independently from their original positions, while the position of P is unchanged. Then exactly the same argument can be given to show that $\angle AOB$, the angle subtended by the chord AB at the centre of the circle, is equal to twice $\angle APB$, the angle subtended by AB at the circumference.

There are some areas of mathematics, such as algebra, calculus and trigonometry, that provide a general framework for proving results of a particular type. In using general techniques, we are situating the result among a category of those that can be handled in a specific way. This focuses attention on the particular characteristics that make the techniques applicable. For example, we can conceive of the situation of the proposition in the cartesian plane, the complex plane or two-dimensional vector space. The proposition contains elements that are capable of straightforward formulation in each of these areas.

In the cartesian plane, the circle can be described by a simple quadratic equation and the condition for perpendicularity of two lines involves their slopes. If we coordinatize A, B and P as $(-1, 0)$, $(1, 0)$ and (x, y) where $x^2 + y^2 = 1$, then we can check that 1 plus the product of the slopes of AP and BP is 0. In the complex plane, where multiplication by i corresponds to the geometric rotation through 90° about the origin, the proof becomes a matter of verifying that if A is taken to be -1, B as $+1$ and P as z where $z\bar{z} = 1$, then $(z-1)/(z+1)$ is a real multiple of $z - \bar{z}$ and so pure imaginary. Finally, the vector proof can be carried out with or without coordinates. In the latter case, the proof is particularly slick. Taking the centre of the circle as the origin of vectors, then

$$(P - B) \cdot (P - A) = P^2 - P \cdot (A + B) + A \cdot B = O \ ,$$

since $A = -B$ and $P^2 = B^2 = A^2$ is the square of the radius of the circle.

Some proofs reveal more than others; from some of the arguments, it can be quickly inferred that angle APB is right if and only if AB is the diameter of a circle that contains P, so that the converse really is also built into the proof.

A student's whose learning is robust is likely to have developed a multifaceted way of looking at mathematical facts. Her knowledge is hedged around with many connections and corroborations. The bald statement of results and practice of techniques may not achieve this; the process of having to construct or follows proofs has a much better opportunity to obtain this rich knowledge.