EXITS FROM A SEMI-INFINITE TUBE

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Abstract. We construct a semi-infinite domains which reverse the direction of the most of the incoming particles.

1. The Model

Consider a mechanical systems shown on Figure 1. It consists of a semi-infinite tube and a point particle flying into this tube. Without loss of generality assume that the width of the tube is equal to one and the distance between two successive verticals walls of height $\varepsilon/2$ is also equal to one. Inside the tube the particle is moving with the constant velocity and elastic reflections from the boundary. Since the kinetic energy of the particle is preserved, we can assume that the absolute value of the velocity of the particle is one.

The motion of the particle is therefore determined by the point $x_{\text{in}} \in [0, 1]$, where it enters the tube and the initial velocity $v_{\text{in}} = (\cos(2\pi\alpha), \sin(2\pi\alpha))$ at this point. The measure $\mathbb{P}$ on the initial conditions (α, x_{in}) considered below is the Lebesgue measure on $[0, 1] \times (-1/2, 1/2)$. If the particle leaves the tube its motion is no longer considered. The velocity with which it leaves the tube is denoted by v_{out}.

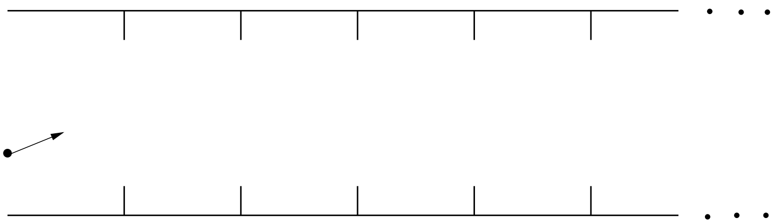

Figure 1. The model

Theorem 1. (1) *For almost every* $(\alpha, x_{\text{in}}, \varepsilon)$, *the particle eventually leaves the tube,*

(2) *As* $\varepsilon \to 0$, $\mathbb{P}\{(\alpha, x_{\text{in}}) \mid v_{\text{out}} = -v_{\text{in}}\} \to 1$

(3) *For every* $T > 0$, $\mathbb{P}\{(\alpha, x_{\text{in}})$, *such that corresponding trajectory leaves the tube after* K *reflections from vertical slits* $\}$ *has a limit as* $\varepsilon \to 0$.

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Notice that by part (2) of the theorem, the tube reverses the direction of most particle flying inside and so maximizes the total pressure of a flow of particles falling inside the tube. This property makes the model interesting in applications, such as sails, wind mills and quantum sails. The devices which require exact reversal of the direction of the flow also include road reflectors and mobile robots communicating by optical signals.

Partial results in this direction were obtained in [7], where a sequence of domains with approximate reversal of the direction was constructed. However, such domains do not reverse the direction exactly and therefore cannot be used, for example, in the last two applications mentioned above.

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2. The Reduction to Circle Rotations

In order to prove Theorem 1, we interpret it in terms of circle rotations.

We identify $[0,1)$ with S^1. For $\alpha \in \mathbb{R}$ let $T_\alpha : S^1 \to S^1$ be the circle rotation by angle $\alpha : T_\alpha x = x + \alpha \mod 1$. We shall assume that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Let $I_\varepsilon = [-\varepsilon/2, \varepsilon/2] \subset S^1$. For $x \in S^1$ let

$$n_\varepsilon(x) = \min\{n > 0 : \ T_\alpha^n(x) \in I_\varepsilon\}$$

be the return time of x to I_ε.

Let $n_{\varepsilon,1}(x) = n_\varepsilon(x)$, $n_{\varepsilon,k}(x) = n_\varepsilon(T^{\varepsilon, n_{k-1}} x)$ be relative return times to the interval I_ε. We are interested in the first moment $\mathcal{E} = \mathcal{E}(\alpha, x, \varepsilon) \in 2\mathbb{Z}$, when the alternating sum of the relative return times becomes non-positive:

$$\mathcal{E}(\alpha, x, \varepsilon) = \min\{j \in 2\mathbb{Z} \ : \ n_{\varepsilon,1}(x) - n_{\varepsilon,2}(x) + \ldots - n_{\varepsilon,j}(x) \le 0\}.$$

Theorem 1 follows from

Theorem 2. (1) *For almost every $x \in I_\varepsilon$, $\alpha \in [0,1)$ and $\varepsilon \in (0, 1/2)$ $\mathcal{E}(\alpha, x, \varepsilon)$ is finite,*

(2) *For every $k \in \mathbb{Z}$ the limit*

$$\lim_{\varepsilon \to 0^+} \mathbb{P}\{(\alpha, x) \ : \ \mathcal{E}(\alpha, x, \varepsilon) \ = \ k\}$$

exists.

Indeed, part 3 of the Theorem 1 clearly follows from part 2 of the Theorem 2. For part 1 of the Theorem 1 notice that if $x_{in} \in I_\varepsilon$, we can apply the theorem 2 directly. If $x_{in} \notin I_\varepsilon$, then we add vertical walls of the same height to the cross-section that corresponds to the entrance to the tube and add the symmetric image of the tube with respect to this cross-section to obtain bi-infinite tube. Now reverse the direction of the particle and consider its motion in the bi-infinite tube till it hits one of the vertical walls and remove the part of the tube which is to the left of the vertical wall hit by the particle. Reverse the direction again and apply theorem 2 to this new $(\hat{\alpha}, \hat{x_0})$, since

$\hat{x_0} \in I_\varepsilon$ by construction. Since the particle escapes this prolonged tube, it will escape the initial tube as well.

For part 2 of the Theorem 1 we need to count the parity of the number of the reflections of the particle from horizontal walls till it leaves the tube.

Without loss of generality assume that $x = x_{in} \in (0, 1/2)$ and $\alpha \in (0, 1/2)$.

Let $\omega = \tan(2\pi\alpha)$ and let $n'_{\varepsilon,\mathcal{E}}(x)$ be such that

$$n_{\varepsilon,1}(x) - n_{\varepsilon,2}(x) + \ldots - n'_{\varepsilon,\mathcal{E}}(x) = 0 \tag{2.1}$$

Obviously, $0 < n'_{\varepsilon,\mathcal{E}}(x) \leq n_{\varepsilon,\mathcal{E}}(x)$. Let $0 < x < 1$. The number of reflections from horizontal walls equals

$$L = \lfloor x + \omega(n_{\varepsilon,1}(x) + n_{\varepsilon,2}(x) + \ldots + n'_{\varepsilon,\mathcal{E}}(x)) \rfloor,$$

where $\lfloor \ldots \rfloor$ means integer part. Using (2.1), one gets

$$L = \lfloor -x + 2(x + \omega n_{\varepsilon,1}(x) + \omega n_{\varepsilon,3}(x) + \ldots + \omega n_{\varepsilon,\mathcal{E}-1}(x)) \rfloor;$$

where the sum is taken over the odd terms only.

Denote $\{\ldots\}$ the distance between the number and the closest integer; one has $\{x + \omega n_{\varepsilon,1}(x)\} < \varepsilon/2$, $\{\omega n_{\varepsilon,3}(x)\} < \varepsilon$, $\{\omega n_{\varepsilon,5}(x)\} < \varepsilon$, etc., therefore

$$\{x + \omega n_{\varepsilon,1}(x) + \omega n_{\varepsilon,3}(x) + \omega n_{\varepsilon,5}(x) + \ldots + \omega n_{\varepsilon,\mathcal{E}-1}(x)\} < \frac{\varepsilon}{2}(\mathcal{E} - 1).$$

If $\frac{\varepsilon}{2}(\mathcal{E} - 1) < x$ then L is odd and $v_{out} = -v_{in}$. According to part 2 of theorem 2, $\mathcal{E}$ has a limit distribution, therefore $\mathbb{P}\left(\frac{\varepsilon}{2}(\mathcal{E} - 1) < x\right) \to 1$ as $\varepsilon \to 0$.

It remains to prove Theorem 2.

3. Point-wise exits

We now prove part 1 of the Theorem 2.

Let $\hat{T}_\alpha : I_\varepsilon \to I_\varepsilon$ be the map induced on I_ε by circle rotation T_α. It is an exchange of three intervals. It follows from [1](paragraph 8), that for almost every (α, ε) the map $\hat{T}_\alpha$ is weakly-mixing and in particular the unitary operator corresponding to $\hat{T}_\alpha$ doesn't have eigenvalue $\lambda = -1$. Therefore, for almost every (α, ε) the mapping $\hat{T}_\alpha^2$ is ergodic.

Proposition 3. *Let T be an ergodic transformation on (X, μ), $\mu(X) = 1$, and let $f \in L^1(X, \mu)$, $\int f d\mu = 0$ and $S_n(f, x) = f(x) + f(Tx) + \ldots + f(T^{n-1}x)$ be its Birkhoff sums. Then either $S_n(f, x)$ is unbounded from below for almost every $x \in X$, or f is a co-boundary: there exists measurable $g(x)$, such that $f(x) = g(x) - g(Tx)$.*

Proof. Since T is ergodic, the set of points x for which $S_n(f, x)$ is bounded from below has measure zero or one. In the case of zero measure there is nothing to prove, so we can assume it has measure one. Therefore the function $g(x) = \inf_{n \geq 1} S_n(f, x)$ is finite almost everywhere.

Then $g(Tx)+f(x) = \inf_{n\geq 2} S_n(f,x)$, and so $h(x) = g(Tx)-g(x)+f(x) \geq 0$.

By Birkhoff ergodic theorem (even if $\int_X h d\mu = \infty$) for a.e. $x \in X$,

$$\lim_{n\to+\infty} \frac{S_n(h,x)}{n} = \lim_{n\to+\infty} \frac{S_n(f,x)}{n} + \frac{g(T^n x)-g(x)}{n} \to \int_X h d\mu$$

Therefore $\int_X h d\mu = \int_X (h(Tx) - h(x) + f(x)) d\mu = 0$, and so $f(x) = h(x) - h(Tx)$ almost everywhere. □

Part 1 of the Theorem 2 now follows from the proposition (3) applied to $\hat{T}_\alpha^2 : I_\varepsilon \to I_\varepsilon$ and $f(x) = n_{\varepsilon,1}(x) - n_{\varepsilon,1}(\hat{T}x)$ on I_ε.

Part 1 of the theorem 1 can be compared to the following random analogue. Fix $\varepsilon > 0$ and consider a random walk on $\mathbb{Z}$ with memory. A particle starts to move to the right from 0. At each lattice point with probability ε it changes the direction of the movement and with probability $1-\varepsilon$ continues to move in the same direction.

Proposition 4. [6]*(Theorem* 1.4.6*) For every $\varepsilon > 0$ the random walk is recurrent.*

4. Limiting distributions for the exit time

We now prove part 2 of the Theorem 2.

4.1. **The reduction to a limit distribution result.** For $T \geq 1$ let

$$S_T(\varepsilon) = \{\#n : \ 1 \leq n \leq T, \ \text{s.t.} \ \{\alpha n + x_0\} \in I_\varepsilon\}$$

Proposition 5. *For every m-tuple of non-negative integers $k_1, \ k_2, \ \dots \ k_m$ and every m-tuple of positive $\sigma_1, \sigma_2, \dots \sigma_m$*

$$\lim_{\varepsilon\to 0^+} \mathbb{P}\{S_{\frac{\sigma_1}{\varepsilon}} = k_1, \ S_{\frac{\sigma_2}{\varepsilon}} = k_2, \dots S_{\frac{\sigma_m}{\varepsilon}} = k_m\} = E(\sigma_1, \ \sigma_2, \ \dots \ \sigma_m, k_1, \ k_2, \ \dots \ k_m) \tag{4.2}$$

and

$$E(\sigma_1, \dots \sigma_m, k_1, \ k_2, \ \dots \ k_m,) \text{ is continuous in } \sigma_1, \dots \sigma_m. \tag{4.3}$$

For $m = 1$ the convergence (4.2) was first proved by Mazel and Sinai ([5]). They also obtained explicit formulas for $E(\sigma, k)$. It was later reproved and generalized by Marklof ([3], [4]) who used a different approach. We use his method in the next subsection to prove Proposition 5 and now show how part (2) of the Theorem 2 can be deduced from it.

It is convenient to use the set of entrance times $T_{m,\varepsilon}(x)$ for $x \in S^1, \ m = 0, 1, \dots$ defined similarly to return times:

$$T_{0,\varepsilon} = 0, \ T_{m,\varepsilon}(x) = \min\{l > T_{m-1,\varepsilon} : \ T_\alpha^l x \in I_\varepsilon\}$$

Then $n_{m,\varepsilon} = T_{m+1,\varepsilon} - T_{m,\varepsilon}$.

Let $F^\varepsilon_m(t_1,\dots t_m) = \mathbb{P}\{\varepsilon T_{1,\varepsilon} > t_1,\ \varepsilon T_{2,\varepsilon} > t_2,\ \dots\ \varepsilon T_{m,\varepsilon} > t_m\}$ be the joint distribution function for $(\varepsilon T_{1,\varepsilon}, \varepsilon T_{2,\varepsilon}, \dots \varepsilon T_{m,\varepsilon})$

Lemma 6. *For every m-tuple $(t_1, t_2 \dots t_m)$*

$$\lim_{\varepsilon\to 0^+} F^\varepsilon_m(t_1,\dots t_m) = F(t_1,\dots t_m) \tag{4.4}$$
$$\text{exists and is continuous in } (t_1, t_2 \dots t_m)$$

Proof. The lemma follows from the proposition 5, since for every m-tuple of positive $(t_1, t_2 \dots t_m)$ we have

$$\mathbb{P}\{\varepsilon T_{1,\varepsilon} > t_1,\ \varepsilon T_{2,\varepsilon} > t_2,\ \dots\ \varepsilon T_{m,\varepsilon} > t_m\} =$$

$$\sum_{\substack{0\le i_2\le\dots\le i_m\\ i_j<j}} \mathbb{P}\{S_{\frac{t_1}{\varepsilon}} = 0,\ S_{[\frac{t_2}{\varepsilon}]} = i_2,\ \dots S_{\frac{t_m}{\varepsilon}} = i_m\}.$$

□

For $j \ge 1$ let

$$\mathcal{T}_j = \mathcal{T}_j(\alpha, x, \varepsilon) = 2\sum_{i=1}^{j-1}(-1)^{i+1}(\varepsilon T_{i,\varepsilon}) + (-1)^{j+1}(\varepsilon T_{j,\varepsilon})$$

For every $\varepsilon > 0$ we have

$$\mathcal{E}(\alpha, x, \varepsilon) = \min\{j \in 2\mathbb{Z}_+ :\ \mathcal{N}(\alpha, x, \varepsilon) \le 0\} = \min\{j \in 2\mathbb{Z}_+ :\ \mathcal{T}_j \le 0\}$$

And so,

$$\mathbb{P}\{\mathcal{E}(\alpha, x, \varepsilon) = k\} = \mathbb{P}\{\mathcal{T}_1 > 0, \dots \mathcal{T}_{k-1} > 0,\ \mathcal{T}_k < 0\} = \int_{\mathbb{R}^k} \chi_{A_k} dF^\varepsilon, \tag{4.5}$$

where χ_{A_k} is the characteristic function of

$$\begin{aligned} A_k = \{(y_1, \dots y_k) \in \mathbb{R}^k\ &: \forall j = 1\dots k-1 \\ &\sum_{i=1}^{j-1}(-1)^{i+1}y_i + (-1)^{j+1}y_j < 0, \\ &\sum_{i=1}^{k-1}(-1)^{i+1}y_i + (-1)^{k+1}y_k > 0\} \end{aligned} \tag{4.6}$$

Part (2) of the Theorem 2 now follow from lemma 6 and the following version of Helly-Bray Theorem.

Proposition 7. [2]*(p.183) Suppose that joint distribution function F^ε of real-valued random variables converge in distribution to F as $\varepsilon \to 0^+$, and $F(\partial A) = 0$. Then*

$$\lim_{\varepsilon\to 0^+} \int \chi_A(x) dF^\varepsilon(x) = \int \chi_A(x) dF^(x)$$

4.2. **Proof of a limiting distribution result.** Let χ_I denote the characteristic function of the interval $I \subset \mathbb{R}$ and $\psi_\sigma(x, y) = \chi_{(0,1]}(x)\chi_{[-\sigma/2,\sigma/2]}(y)$ be the characteristic function of a rectangle $\mathcal{R}_\sigma \subset \mathbb{R}^2$.

Then

$$S_N(\varepsilon) = \sum_{m=1}^{N} \sum_{n\in\mathbb{Z}} \chi\left(\tfrac{N}{\sigma}(\alpha m + n + x_0)\right)$$

$$(4.7) \qquad \sum_{(m,n)\in\mathbb{Z}^2} \chi_{(0,1]}\left(\tfrac{m}{N}\right)\chi_{[-\sigma/2,\sigma/2]}(N(\alpha m + n + x_0))$$

$$= \sum_{(m,n)\in\mathbb{Z}^2} \psi_\sigma\left((m, n + x_0)\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} N^{-1} & 0 \\ 0 & N \end{pmatrix}\right)$$

Denote the Lie group $G = SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ with multiplication law

$$(M, \xi)(M', \xi') = (MM', \xi M' + \xi')$$

The function

$$F_\sigma(M, \xi) = \sum_{m\in\mathbb{Z}^2} \psi_\sigma(mM + \xi)$$

on G is left-invariant with respect to the discrete subgroup $\Gamma = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$, and so can be viewed as a function on the homogeneous space $\Gamma\backslash G$. Moreover,

$$S_N(\varepsilon) = F_\sigma(M, \xi),$$

$$(4.8) \qquad \textit{where } M = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} N^{-1} & 0 \\ 0 & N \end{pmatrix}, \ \xi = (0, x_0)M$$

Consider now the one-parameter subgroup $\{\Phi^t\}_{t\in\mathbb{R}}$, where

$$\Phi^t = \left(\begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}, (0,0)\right).$$

It defines a flow on $\Gamma\backslash G$ by right multiplication,

$$\Gamma g \to \Gamma g \Phi^t.$$

The crucial observation is that our object of interest, $S_N(\varepsilon)$, is related to a function F_σ of $\Gamma\backslash G$ evaluated along an orbit of this flow:

$$S_N(\varepsilon) = F_\sigma(g_0\Phi^t)$$

with $t = 2\log N$ and initial condition

$$g_0 = \left(\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, (0, x_0)\right)$$

Let us define

$$n_-(\alpha, y) = \left(\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, (0, x_0)\right).$$

Then the subgroup $H = \{n_-(\alpha, y)\}_{(\alpha,y)\in\mathbb{R}^2}$ parameterize the unstable manifold of the flow Φ^t on $\Gamma\backslash G$ (see [3] for details).

Let ν denote the Haar measure on $\Gamma\backslash G$.

Theorem 8. *([3], Theorem 3.3.)For any bounded $f : \Gamma\backslash G \to \mathbb{R}$, such that the discontinuities of f are contained in a set of ν-measure zero*

$$\lim_{t\to\infty} \int_0^1 \int_0^1 f(n_-(\alpha, y)\Phi^t) d\alpha dy = \frac{1}{\nu(\Gamma\diagdown G)} \int_{\Gamma\backslash G} f d\nu$$

Let

$$N_1 = \frac{\sigma_1}{\varepsilon},\ N_2 = \frac{\sigma_2}{\varepsilon}\ \ldots\ N_m = \frac{\sigma_1}{\varepsilon},\ t = 2\log\frac{1}{\varepsilon}$$

Then

$$\begin{aligned} S_{N_1} &= F_{\sigma_1}\left(g_0 \begin{pmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_1 \end{pmatrix} \Phi^t\right), \\ &\ldots \\ S_{N_1} &= F_{\sigma_1}\left(g_0 \begin{pmatrix} \sigma_m^{-1} & 0 \\ 0 & \sigma_m \end{pmatrix} \Phi^t\right), \end{aligned}$$

and

$$D(g) = \chi\left(F_{\sigma_1}\left(g\begin{pmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_1 \end{pmatrix}\right) = k_1, \ldots F_{\sigma_m}\left(g\begin{pmatrix} \sigma_m^{-1} & 0 \\ 0 & \sigma_m \end{pmatrix}\right) = k_m\right).$$

is a function on $\Gamma\backslash G$ satisfying the conditions of the theorem 8. The convergence (4.2) in proposition 5 now follows from theorem 8 applied to the function $D(g)$.

In order to prove continuity (4.3) we show that

$$\begin{aligned} E(\sigma_1, \ldots \sigma_m, k_1, \ldots k_m) = \nu\{g = (M, \xi) \in \Gamma\backslash G\ : \\ |\left(\mathbb{Z}^2 M \begin{pmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_1 \end{pmatrix} + \xi\right) \cap \mathcal{R}_{\sigma_1}|\ = k_1, \\ \ldots\ |\left(\mathbb{Z}^2 M \begin{pmatrix} \sigma_m^{-1} & 0 \\ 0 & \sigma_m \end{pmatrix} + \xi\right) \cap \mathcal{R}_{\sigma_m}|\ = k_m\} \end{aligned} \tag{4.9}$$

is continuous with respect to $(\sigma_1, \ldots, \sigma_m)$.

Let $\hat{\mathcal{R}}(\sigma, \xi) = \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{pmatrix} (\mathcal{R}_\sigma - \xi)$

We have

$$\begin{aligned} |E(\sigma_1, \ldots \sigma_m, k_1, \ldots k_m) - E(\sigma_1', \ldots \sigma_m', k_1, \ldots k_m)| \leq \\ \sum_{j=1}^m \nu\{g = (M, \xi) \in \Gamma\backslash G\ :\ |\mathbb{Z}^2 M \cap \left(\hat{\mathcal{R}}(\sigma_j, \xi) \vartriangle \hat{\mathcal{R}}(\sigma_j', \xi)\right)| \geq 1\} \end{aligned} \tag{4.10}$$

It is enough to estimate

$$\nu_j(\xi) = \nu\{M \in SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})\ :\ |\mathbb{Z}^2 M \cap \left(\hat{\mathcal{R}}(\sigma_j, \xi) \vartriangle \hat{\mathcal{R}}(\sigma_j', \xi)\right)| \geq 1\}$$

for a fixed $j = 1, \dots m$ and a fixed $\xi \in \mathbb{Z}^2 \backslash \mathbb{R}^2 \cong (0, 1]^2$.

By Siegel's formula applied to the characteristic function of

$$\hat{\mathcal{R}}(\sigma_j, \xi) \vartriangle \hat{\mathcal{R}}(\sigma'_j, \xi)$$

we have

$$\nu_j(\xi) \leq Vol\{(\hat{\mathcal{R}}(\sigma_j, \xi) \vartriangle \hat{\mathcal{R}}(\sigma'_j, \xi)\}.$$

Since $Vol\{\hat{\mathcal{R}}(\sigma_j, \xi)\}$ depends continuously on σ_j, the proof of the proposition 5 is now complete.

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