MAT 212 Midterm. Spring 2008.

Solutions

(1) Since the rate of change of the volume $V(t)$ of water is proportional to the square root of the height $h(t)$ of the remaining water, the differential equation is

$$\dot{V} = \frac{dV}{dt} = \frac{d(\pi r^2 h(t))}{dt} = -\alpha\sqrt{h(t)},$$

or

$$\dot{h} = -\frac{\alpha}{\pi r^2}\sqrt{h}.$$

The variables separate, so we have

$$\frac{dh}{\sqrt{h}} = -\frac{\alpha}{\pi r^2}dt,$$

$$2\sqrt{h} = -\frac{\alpha}{\pi r^2}t + C$$

Plugging in the initial condition $h(0) = 144$cm and $r = 10$cm, we get that $C = 24$, so

$$h(t) = (12 - \frac{\alpha}{200\pi}t)^2$$

We also know that $h(20) = (12 - \frac{\alpha}{10\pi})^2 = 100$. Therefore $\alpha = 20\pi$ and the equation reads

$$h(t) = (12 - t/10)^2.$$

Clearly, $h(t) = 0$ when $t = 120$ min or 2 hours.

(2) The equation

$$y' - 2xy + y^2 = 5 - x^2$$

is an example of a Riccati equation. It is natural to expect it to have a linear function as a solution, because in this case both left and right hand side are polynomials of degree two and there is a hope to match the coefficients. Plugging in $y_1 = ax + b$ into the equation one gets:

$$a - 2ax^2 - 2bx + a^2x^2 + 2abx + b^2 =$$

$$= a(a-2)x^2 + 2b(a-1) + (a+b^2) = 5 - x^2$$

Therefore we should have $a(a-2) = -1$, $2b(a-1) = 0$, $a + b^2 = 5$. One possible solution is $a = 1$, $b = 2$, so $y_1 = x + 2$.

The Riccati equation can be reduced to a linear one by a pair of substitutions: $y = y_1 + z$ (the equation becomes Bernoulli with $\nu = 2$) and $w = z^{1-\nu} = 1/z$. For w we have,

$$w' - 4w = 1$$

The last equation can be solved, for example, by the variation of constants: $w = Ae^{4x}$, $A'e^{4x} = 1$, $A = -\frac{1}{4}e^{-4x} + C$, so $w = Ce^{4x} - 1/4$ and $y = x + 2 + (Ce^{4x} - 1/4)^{-1}$

(3) The right hand side of the equation is not the one for which we can "guess" a solution, therefore we need to use variation of constants.

The solution to the homogeneous equation is given by

$$y_h = C_1 e^x + C_2 x e^x,$$

so $y_1(x) = e^x$, $y_2(x) = xe^x$ and we have the following system for C_1' and C_2' :

$$\begin{cases} C_1' y_1 + C_2' y_2 = 0 \\ C_1' y_1' + C_2' y_2' = e^x \ln x, \end{cases} \quad \text{or} \quad \begin{cases} C_1' + C_2' x = 0 \\ C_1' + C_2'(1+x)' = \ln x, \end{cases}$$

From the last system we get $C_1' = -x\ln x$, $C_2' = \ln x$, therefore

$$C_1 = -\frac{1}{2}x^2\ln x + \frac{x^2}{4} + C,\; C_2 = x\ln x - x + D$$

and so

$$y(x) = (-\frac{1}{2}x^2\ln x + \frac{x^2}{4} + C)e^x + (x\ln x - x + D)xe^x$$

(4) The differential equation $y^3y'' = 1$ does not contain x, so we can assume that y is the independent variable, and that $y' = p(y)$. Then $y'' = p\frac{dp}{dy}$ and the equation becomes (derivatives are now with respect to y):

$$pp' = y^{-3}$$

Variables can be separated and we get:

$$pdp = y^{-3}dy,$$

$$p^2 = -y^{-2} + C$$

$$y' = p = \pm\sqrt{C - y^{-2}} = \pm\sqrt{\frac{Cy^2 - 1}{y^2}}$$

$$\frac{ydy}{\sqrt{Cy^2 - 1}} = \pm dx$$

$$\frac{1}{C}\sqrt{Cy^2 - 1} = \pm x + D$$

(5)

$$y' = 2\left(\frac{2x + y + 1}{4x + 2y - 3}\right) = \left(1 + \frac{5}{4x + 2y - 3}\right)$$

Let $z = 4x + 2y - 3$, then $z' = 4 + 2y'$, so $z'/2 - 2 = 1 + \frac{5}{z}$, or

$$z' = 2\frac{3z + 5}{z}$$

Here variables separate, and we have

$$\frac{zdz}{3z+5} = (\frac{1}{3} - \frac{5}{3}\frac{1}{3z+5})dz = dx$$

Therefore

$$\frac{z}{3} - \frac{5}{9}\ln|3z+5| = x + C$$

If $y(0) = 0$, then $z(0) = -3$, and so $C = -1$. All in all, we have

$$\frac{z}{3} - \frac{5}{9}\ln|3z+5| = x - 1$$

(6) If $y(x) = e^{-x} + 2e^{-2x}$ is a solution to $y'' + cy' + ky = 0$, then the roots of the characteristic equation are -1 and -2, and therefore $c = 3$, $k = 2$.

(7) The equation

$$x^2y'' + 7xy' + 5y = 10 - \frac{4}{x}$$

is an example of a Euler's equation. A substitution $x = e^t$ reduces it to a linear equation

$$\ddot{y} + 6\dot{y} + 5y = 10 - 4e^{-t}$$

The solution to the corresponding homogeneous equation is $y_h = ce^{-t} + de^{-5t}$.

A particular solution y_1 corresponding to the right hand side 10 should be a constant, so $y_1 = 2$.

A particular solution y_2 corresponding to the right hand side $-4e^{-t}$ should be of the form ate^{-t}, since -1 is a simple root of the characteristic equation. By plugging in ate^{-t} one finds that $a = -1$, so

$$y(t) = 2 - te^{-t} + ce^{-t} + de^{-5t},$$

or

$$y(x) = 2 - \frac{\ln x}{x} + \frac{c}{x} + \frac{d}{x^5}$$

(8) The solution to $y'' + by' + y = 0$ is critically damped, when the corresponding characteristic equation has one root of multiplicity two. Here it happens, when $b^2 - 4 = 0$. Formally, speaking only solution $b = 2$ corresponds to critical damping, since the damping constant is always non-negative, but the answer $b = \pm 2$ is also acceptable.