

COMPUTING POWERS AND EXPONENTS OF THE MATRICES.

Consider a linear homogeneous system of 2 differential equations:

$$(1) \quad \begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

Let A be the coefficient matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $\mathbf{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ be the column vector of unknown functions $x(t)$ and $y(t)$. Then the system (1) can be rewritten as

$$\dot{\mathbf{u}} = A\mathbf{u}$$

Therefore, it is natural to expect (and it is, indeed, true), that the solution of the system (1), as in the case of one differential equation, is given by

$$\mathbf{u}(t) = e^{At}\mathbf{u}(0)$$

The only remaining question is:

What is e^{At} , or what is the exponent of a matrix?

The formal answer is simple:

By Taylor's formula,

$$e^x = 1 + x + x^2/2! + \dots + x^n/n! + \dots,$$

where $n! = n(n-1)(n-2)\dots 2 \cdot 1$ is the factorial function. So for any (square $n \times n$) matrix A (*why necessarily square?*), we have

$$e^A = I_n + A + A^2/2! + A^3/3! + \dots + A^n/n! + \dots,$$

where I_n is the identity matrix of size n .

However, it is not clear from this definition, how to compute e^A in practice. We need an *efficient* way to compute powers and exponents of a matrix. Below is an algorithm, which allows to do this for 2×2 matrices and is relatively easy to generalize to the general case of $n \times n$ matrices.

Notice, that if the matrix A is *diagonal*: $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then everything is simple:

$$A^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix},$$

and

$$e^A = \begin{pmatrix} 1 + \lambda_1 + \lambda_1^2 + \dots + \lambda_1^n/n! + \dots & 0 \\ 0 & 1 + \lambda_2 + \lambda_2^2 + \dots + \lambda_2^n/n! + \dots \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$$

Therefore, one should try to reduce any matrix to a diagonal one.

Definition 1. Matrix A is called *diagonalizable*, if there are a diagonal matrix Λ and an invertible matrix D , such that $A = D\Lambda D^{-1}$.

If $A = D\Lambda D^{-1}$, then $A^n = D\Lambda D^{-1}D\Lambda D^{-1}D\Lambda D^{-1} \dots D\Lambda D^{-1}$. Since $D^{-1}D = I$, we have $A^n = D\Lambda^n D^{-1}$, and $e^A = De^\Lambda D^{-1}$. Since Λ^n and e^Λ are easy to compute (see above), we are done.

Here is how you can find Λ, D for a given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$. By a formula for the inverse of a 2×2 matrix, we have

$$D^{-1} = \frac{1}{d_1d_4 - d_2d_3} \begin{pmatrix} d_4 & -d_2 \\ -d_3 & d_1 \end{pmatrix}$$

Therefore, we need to find λ_1, λ_2 and d_1, d_2, d_3, d_4 .

The entries of the diagonal matrix λ_1, λ_2 are called *eigenvalues*. They are the roots of the characteristic polynomial of the matrix A , which, in the case of 2×2 matrices is equal to

$$p_A(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc) (= \det(A - \lambda I))$$

If $\lambda_1 \neq \lambda_2$, then $\begin{pmatrix} d_1 \\ d_3 \end{pmatrix}$ is an *eigenvector* of A corresponding to λ_1 and $\begin{pmatrix} d_2 \\ d_4 \end{pmatrix}$ is the *eigenvector* of A corresponding to λ_2 . This means that

$$(2) \quad A \begin{pmatrix} d_1 \\ d_3 \end{pmatrix} = \lambda_1 \begin{pmatrix} d_1 \\ d_3 \end{pmatrix}, \text{ and } A \begin{pmatrix} d_2 \\ d_4 \end{pmatrix} = \lambda_2 \begin{pmatrix} d_2 \\ d_4 \end{pmatrix}$$

These (linear) equations on d_1, d_2, d_3, d_4 should be used to find d_1, d_2, d_3, d_4 and the matrix D .

If $\lambda_1 = \lambda_2 = \lambda$, the situation is more subtle. One still finds $\begin{pmatrix} d_1 \\ d_3 \end{pmatrix}$ as an eigenvector of A corresponding to λ :

$$(3) \quad A \begin{pmatrix} d_1 \\ d_3 \end{pmatrix} = \lambda_1 \begin{pmatrix} d_1 \\ d_3 \end{pmatrix}$$

If it is possible to find *another, not proportional to* $\begin{pmatrix} d_1 \\ d_3 \end{pmatrix}$ eigenvector, then one should take it as $\begin{pmatrix} d_2 \\ d_4 \end{pmatrix}$.

If, however, this is not possible, then the matrix A is *not diagonalizable*. But one shouldn't despair. In this case one can find $\begin{pmatrix} d_2 \\ d_4 \end{pmatrix}$ from the

equation

$$A \begin{pmatrix} d_2 \\ d_4 \end{pmatrix} = \lambda \begin{pmatrix} d_2 \\ d_4 \end{pmatrix} + \begin{pmatrix} d_1 \\ d_3 \end{pmatrix}$$

(Notice, that $\begin{pmatrix} d_1 \\ d_3 \end{pmatrix}$ is by now found from (3)). Then

$$A = D \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} D^{-1},$$

and so

$$A = D \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n D^{-1} = D \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda \end{pmatrix} D^{-1},$$

and

$$e^A = D e \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} D^{-1} = D \begin{pmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{pmatrix} D^{-1},$$

Example. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$ is given by the recursive relation $x_n = x_{n-1} + x_{n-2}$. Find a formula for the n 'th term of the sequence.

Notice, that by the recursive relation, we have

$$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = A \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix} = A^2 \begin{pmatrix} x_{n-2} \\ x_{n-3} \end{pmatrix} = \dots = A^{n-1} \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = A^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, we just need to compute

$$A^{n-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}.$$

This fits into the scheme above. The eigenvalues of the matrix A are the roots of the characteristic equation, which in this case is $p_A(\lambda) = \lambda^2 - \lambda - 1$.

They are different and equal to

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

Therefore $\begin{pmatrix} d_1 \\ d_3 \end{pmatrix}, \begin{pmatrix} d_2 \\ d_4 \end{pmatrix}$ are given by

$$(4) \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_3 \end{pmatrix} = \frac{1 + \sqrt{5}}{2} \begin{pmatrix} d_1 \\ d_3 \end{pmatrix},$$

and

$$(5) \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_2 \\ d_4 \end{pmatrix} = \frac{1 - \sqrt{5}}{2} \begin{pmatrix} d_2 \\ d_4 \end{pmatrix}$$

Notice, that the solution to (4), (5) and (2) in general is not unique. A particular solution here gives

$$D = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}.$$

The inverse is then equal to

$$D^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix}.$$

Therefore, we have

$$A^{n-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\frac{1+\sqrt{5}}{2})^{n-1} & 0 \\ 0 & (\frac{1-\sqrt{5}}{2})^{n-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix},$$

$$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1\frac{1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and so

$$x_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$