Algebraic Equations and Convex Bodies

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Dedicated to Oleg Yanovich Viro on the occasion of his sixtieth birthday

Abstract The well-known Bernstein–Kushnirenko theorem from the theory of Newton polyhedra relates algebraic geometry and the theory of mixed volumes. Recently, the authors have found a far-reaching generalization of this theorem to generic systems of algebraic equations on any algebraic variety. In the present note we review these results and their applications to algebraic geometry and convex geometry.

Keywords Bernstein–Kushnirenko theorem • Semigroup of integral points • Convex body • Mixed volume • Alexandrov–Fenchel inequality • Brunn–Minkowski inequality • Hodge index theorem • Intersection theory of Cartier divisors • Hilbert function

1 Introduction

The famous Bernstein–Kushnirenko theorem from the theory of Newton polyhedra relates algebraic geometry (mainly the theory of toric varieties) with the theory of mixed volumes in convex geometry. This relation is useful in both directions.

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On the one hand it allows one to prove the Alexandrov–Fenchel inequality (the most important and hardest result in the theory of mixed volumes) using the Hodge inequality from the theory of algebraic surfaces. On the other hand, it suggests new inequalities in the intersection theory of Cartier divisors analogous to the known inequalities for mixed volumes (see [Teissier, Khovanskii-1]).

Recently, the authors found a far-reaching generalization of the Kushnirenko theorem in which instead of the complex torus $(\mathbb{C}^*)^n$, we consider any algebraic variety X, and instead of a finite-dimensional space of functions spanned by monomials in $(\mathbb{C}^*)^n$, we consider any finite-dimensional space of rational functions on X.

To this end, first we develop an intersection theory for finite-dimensional subspaces of rational functions on a variety. It can be considered a generalization of the intersection theory of Cartier divisors to general (not necessarily complete) varieties. We show that this intersection theory enjoys all the properties of the mixed volume [Kaveh–Khovanskii-2]. Then we introduce the *Newton-Okounkov body*, which is a far generalization of the Newton polyhedron of a Laurent polynomial. Our construction of the Newton-Okounkov body depends on the choice of a $\mathbb{Z}^n$-valued valuation on the field of rational functions on X. It associates a Newton-Okounkov body to any finite-dimensional space L of rational functions on X. We obtain a direct generalization of the Kushnirenko theorem in this setting (see Theorem 11.1).

This construction then allows us to give a proof of the Hodge inequality using elementary geometry of planar convex domains and (as a corollary) an elementary proof of the Alexandrov–Fenchel inequality. In general, our construction does not imply a generalization of the Bernstein theorem, although we also obtain a generalization of this theorem for some cases in which the variety X is equipped with a reductive group action.

In this paper we present a review of the results mentioned above. We have omitted most of the proofs in this short note. A preliminary version together with proofs can be found in [Kaveh–Khovanskii-1]. Refined and generalized versions appear in the authors' more recent preprints: [Kaveh–Khovanskii-2] is a detailed version of the first half of [Kaveh–Khovanskii-1] (mainly about the intersection index), and [Kaveh–Khovanskii-3] is a refinement and generalization of the results in the second half of [Kaveh–Khovanskii-1] (mainly about Newton-Okounkov bodies).

After these results had been posted on arXiv, we learned that we were not the only ones working in this direction. Firstly, A. Okounkov (in his interesting papers [Okounkov1, Okounkov-2]) was a pioneer in defining (in passing) an analogue of the Newton polyhedron in the general situation (although his case of interest is that in which X has a reductive group action). Secondly, R. Lazarsfeld and M. Mustata, based on Okounkov's previous works, and independently of our preprints, have come up with closely related results [Lazarsfeld–Mustata]. Recently, following [Lazarsfeld–Mustata], similar results and constructions have been obtained for line bundles on arithmetic surfaces [Yuan].

2 Mixed Volume

By a *convex body* we mean a convex compact subset of $\mathbb{R}^n$. There are two operations of addition and scalar multiplication on convex bodies: Let Δ_1, Δ_2 be convex bodies. Then their sum

$$\Delta_1 + \Delta_2 = \{x + y \mid x \in \Delta_1,\ y \in \Delta_2\}$$

is also a convex body, called the *Minkowski sum* of Δ_1, Δ_2. Also, for a convex body Δ and a scalar $\lambda \geq 0$,

$$\lambda\Delta = \{\lambda x \mid x \in \Delta\}$$

is a convex body.

Let Vol_n denote the n-dimensional volume in $\mathbb{R}^n$ with respect to the standard Euclidean metric. The function Vol_n is a homogeneous polynomial of degree n on the cone of convex bodies, i.e., its restriction to each finite-dimensional section of the cone is a homogeneous polynomial of degree n. More precisely, for any $k > 0$, let $\mathbb{R}^k_+$ be the positive octant in $\mathbb{R}^k$ consisting of all $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geq 0, \dots, \lambda_k \geq 0$. The polynomiality of Vol_n means that for any choice of the convex bodies $\Delta_1, \dots, \Delta_k$, the function $P_{\Delta_1,\dots,\Delta_k}$ defined on $\mathbb{R}^k_+$ by

$$P_{\Delta_1,\dots,\Delta_k}(\lambda_1, \dots, \lambda_k) = \mathrm{Vol}_n(\lambda_1\Delta_1 + \dots + \lambda_k\Delta_k),$$

is a homogeneous polynomial of degree n.

The coefficients of this homogeneous polynomial are obtained from the *mixed volumes* of all the possible n-tuples $\Delta_{i_1}, \dots, \Delta_{i_n}$, of convex bodies for any choices of $i_1, \dots, i_n \in \{1, \dots, n\}$. By definition, the mixed volume of $V(\Delta_1, \dots, \Delta_n)$ of an n-tuple $(\Delta_1, \dots, \Delta_n)$ of convex bodies is the coefficient of the monomial $\lambda_1 \cdots \lambda_n$ in the polynomial $P_{\Delta_1,\dots,\Delta_n}$ divided by $n!$.[1] Several important geometric invariants can be recovered as mixed volumes. For example, the $(n-1)$-dimensional volume of the boundary of an n-dimensional convex body Δ is equal to $(1/n)V(\Delta, \dots, \Delta, B)$, where B is the n-dimensional unit ball. Indeed, it is easy to see that the $(n-1)$-dimensional volume of the boundary and the number $(1/n)V(\Delta, \dots, \Delta, B)$ are both equal to the derivative $\partial/\partial\varepsilon \mathrm{Vol}_n(\Delta + \varepsilon B)$ evaluated at $\varepsilon = 0$. Many applications of the theory of mixed volumes can be found in the book [Burago–Zalgaller].

The definition of mixed volume implies that it is the *polarization* of the volume polynomial, i.e., it is the unique function on the n-tuples of convex bodies satisfying the following:

(i) (Symmetry) V is symmetric with respect to permuting the bodies $\Delta_1, \dots, \Delta_n$.
(ii) (Multilinearity) It is linear in each argument with respect to the Minkowski sum. Linearity in the first argument means that for convex bodies Δ_1', Δ_1'', and $\Delta_2, \dots, \Delta_n$, we have

$$V(\Delta_1' + \Delta_1'', \dots, \Delta_n) = V(\Delta_1', \dots, \Delta_n) + V(\Delta_1'', \dots, \Delta_n).$$

[1] The notion of mixed volume was introduced by Hermann Minkowski (1864–1909).

(iii) (Relationship to volume) On the diagonal, it coincides with volume, i.e., if $\Delta_1 = \cdots = \Delta_n = \Delta$, then $V(\Delta_1, \ldots, \Delta_n) = \mathrm{Vol}_n(\Delta)$.

The above three properties characterize the mixed volume: it is the unique function satisfying (i)–(iii).

The following two inequalities are easy to verify:

1. Mixed volume is nonnegative. That is, for any n-tuple of convex bodies $\Delta_1, \ldots, \Delta_n$, we have
$$V(\Delta_1, \ldots, \Delta_n) \geq 0.$$
2. Mixed volume is monotone. That is, for two n-tuples of convex bodies $\Delta_1' \subset \Delta_1, \ldots, \Delta_n' \subset \Delta_n$, we have
$$V(\Delta_1, \ldots, \Delta_n) \geq V(\Delta_1', \ldots, \Delta_n').$$

The following inequality, attributed to Alexandrov and Fenchel, is important and very useful in convex geometry. All its previously known proofs are rather complicated. For a discussion of this inequality the reader can consult the book [Burago–Zalgaller] as well as the original three papers of A. D. Alexandrov cited therein.

Theorem 2.1 (Alexandrov–Fenchel). *Let $\Delta_1, \ldots, \Delta_n$ be convex bodies in $\mathbb{R}^n$. Then*

$$V(\Delta_1, \Delta_2, \ldots, \Delta_n)^2 \geq V(\Delta_1, \Delta_1, \Delta_3, \ldots, \Delta_n) V(\Delta_2, \Delta_2, \Delta_3, \ldots, \Delta_n).$$

Below, we mention a formal corollary of the Alexandrov–Fenchel inequality. First we need to introduce a notation for when we have repetition of convex bodies in the mixed volume. Let $2 \leq m \leq n$ be an integer and $k_1 + \cdots + k_r = m$ a partition of m with $k_i \in \mathbb{N}$. Denote by $V(k_1 * \Delta_1, \ldots, k_r * \Delta_r, \Delta_{m+1}, \ldots, \Delta_n)$ the mixed volume of the Δ_i, where Δ_1 is repeated k_1 times, Δ_2 is repeated k_2 times, etc., and $\Delta_{m+1}, \ldots, \Delta_n$ appear once.

Corollary 2.2. *With the notation as above, the following inequality holds:*

$$V^m(k_1 * \Delta_1, \ldots, k_r * \Delta_r, \Delta_{m+1}, \ldots, \Delta_n) \geq \prod_{1 \leq j \leq r} V^{k_j}(m * \Delta_j, \Delta_{m+1} \ldots, \Delta_n).$$

3 Brunn–Minkowski Inequality

The celebrated *Brunn–Minkowski inequality* concerns volumes of convex bodies in $\mathbb{R}^n$.

Theorem 3.1 (Brunn–Minkowski). *Let Δ_1, Δ_2 be convex bodies in $\mathbb{R}^n$. Then*

$$\mathrm{Vol}_n^{1/n}(\Delta_1) + \mathrm{Vol}_n^{1/n}(\Delta_2) \leq \mathrm{Vol}_n^{1/n}(\Delta_1 + \Delta_2).$$

The inequality was first found and proved by Brunn toward the end of nineteenth century in the following form.

Theorem 3.2. *Let $V_\Delta(h)$ be the n-dimensional volume of the section $x_{n+1} = h$ of a convex body $\Delta \subset \mathbb{R}^{n+1}$. Then $V_\Delta^{1/n}(h)$ is a concave function in h.*

To obtain Theorem 3.1 from Theorem 3.2, one takes $\Delta \subset \mathbb{R}^{n+1}$ to be the convex combination of Δ_1 and Δ_2, i.e.,

$$\Delta = \{(x,h) \mid 0 \le h \le 1,\, x \in h\Delta_1 + (1-h)\Delta_2\}.$$

The concavity of the function

$$V_\Delta(h)^{1/n} = \mathrm{Vol}_n^{1/n}(h\Delta_1 + (1-h)\Delta_2)$$

then readily implies Theorem 3.1.

For $n = 2$, Theorem 3.2 is equivalent to the Alexandrov–Fenchel inequality (see Theorem 4.1). Below we give a sketch of its proof in the general case.

Proof (Sketch of proof of Theorem 3.2).

(1) When the convex body $\Delta \subset \mathbb{R}^{n+1}$ is rotationally symmetric with respect to the x_{n+1}-axis, Theorem 3.2 is obvious: the section $x_{n+1} = h$ of the body Δ at level h is a ball (or empty), and $V_\Delta^{1/n}(h)$ is a constant times the radius, which is a concave function of h, since Δ is a convex body.

(2) Now suppose Δ is not rotationally symmetric. Fix a hyperplane H containing the x_{n+1}-axis. Then one can construct a new convex body Δ' that is symmetric with respect to the hyperplane H and such that the volume of sections of Δ' is the same as that of Δ. To do this, just think of Δ as the union of line segments perpendicular to the plane H. Then shift each segment along its line in such a way that its center lies on H. The resulting body is then symmetric with respect to H and has the same volume of sections as Δ. The above construction is called the *Steiner symmetrization process*.

(3) It will now be enough to show that by repeated application of Steiner symmetrization, we can make Δ as close as we wish to a rotationally symmetric body. This can be proved as follows: First, we show that given a non-rotationally symmetric body, there is always a Steiner symmetrization making it "more symmetric." Then we use a compactness argument on the collection of convex bodies inside a bounded closed domain to conclude the proof. □

4 Brunn–Minkowski and Alexandrov–Fenchel Inequalities

We recall the classical *isoperimetric inequality*, whose origins date back to antiquity. According to this inequality, if P is the perimeter of a simple closed curve in the plane and A is the area enclosed by that curve, then

$$4\pi A \le P^2. \tag{1}$$

Equality is obtained when the curve is a circle. To prove (1), it is enough to prove it for convex regions. The Alexandrov–Fenchel inequality for $n = 2$ implies the isoperimetric inequality (1) as a particular case and hence has inherited the name.

Theorem 4.1 (Isoperimetric inequality). *If Δ_1 and Δ_2 are convex regions in the plane, then*

$$\mathrm{Area}(\Delta_1)\mathrm{Area}(\Delta_2) \le A(\Delta_1,\Delta_2)^2,$$

where $A(\Delta_1,\Delta_2)$ is the mixed area.

When Δ_2 is the unit disk in the plane, $A(\Delta_1,\Delta_2)$ is one-half the perimeter of Δ_1. Thus the classical form (1) of the inequality (for convex regions) follows from Theorem 4.1.

Proof (Proof of Theorem 4.1). It is easy to verify that the isoperimetric inequality is equivalent to the Brunn–Minkowski inequality for $n = 2$. Let us check this in one direction, i.e., that the isoperimetric inequality follows from Brunn–Minkowski for $n = 2$:

$$\begin{aligned}
&\mathrm{Area}(\Delta_1) + 2A(\Delta_1,\Delta_2) + \mathrm{Area}(\Delta_2)\\
&\quad = \mathrm{Area}(\Delta_1 + \Delta_2)\\
&\quad \ge (\mathrm{Area}^{1/2}(\Delta_1) + \mathrm{Area}^{1/2}(\Delta_2))^2\\
&\quad = \mathrm{Area}(\Delta_1) + 2\mathrm{Area}(\Delta_1)^{1/2}\mathrm{Area}(\Delta_2)^{1/2} + \mathrm{Area}(\Delta_2),
\end{aligned}$$

which readily implies the isoperimetric inequality. □

The following generalization of the Brunn–Minkowski inequality is a corollary of the Alexandrov–Fenchel inequality.

Corollary 4.2. *(Generalized Brunn–Minkowski inequality) For any $0 < m \le n$ and for any fixed convex bodies $\Delta_{m+1},\dots,\Delta_n$, the function F that assigns to a body Δ the number $F(\Delta) = V^{1/m}(m * \Delta,\Delta_{m+1},\dots,\Delta_n)$ is concave, i.e., for any two convex bodies Δ_1,Δ_2, we have*

$$F(\Delta_1) + F(\Delta_2) \le F(\Delta_1 + \Delta_2).$$

On the other hand, the usual proof of the Alexandrov–Fenchel inequality deduces it from the Brunn–Minkowski inequality. But this deduction is the main part (and the most complicated part) of the proof (see [Burago–Zalgaller]). Interestingly, the main construction in the present paper (using algebraic geometry) allows us to obtain the Alexandrov–Fenchel inequality as an immediate corollary of the simplest case of the Brunn–Minkowski inequality, i.e., the isoperimetric inequality.

5 Generic Systems of Laurent Polynomial Equations in $(\mathbb{C}^*)^n$

In this section we recall the famous results due to Kushnirenko and Bernstein on the number of solutions of a generic system of Laurent polynomials in $(\mathbb{C}^*)^n$.

Let us identify the lattice $\mathbb{Z}^n$ with *Laurent monomials* in $(\mathbb{C}^*)^n$: to each integral point $k \in \mathbb{Z}^n$, $k = (k_1, \dots, k_n)$, we associate the monomial $z^k = z_1^{k_1} \cdots z_n^{k_n}$, where $z = (z_1, \dots, z_n)$. A *Laurent polynomial* $P = \sum_k c_k z^k$ is a finite linear combination of Laurent monomials with complex coefficients. The *support* $\text{supp}(P)$ of a Laurent polynomial P is the set of exponents k for which $c_k \neq 0$. We denote the convex hull of a finite set $A \subset \mathbb{Z}^n$ by $\Delta_A \subset \mathbb{R}^n$. The *Newton polyhedron* $\Delta(P)$ of a Laurent polynomial P is the convex hull $\Delta_{\text{supp}(P)}$ of its support. With each finite set $A \subset \mathbb{Z}^n$ one associates a vector space L_A of Laurent polynomials P with $\text{supp}(P) \subset A$.

Definition 5.1. We say that a property holds for a *generic element* of a vector space L if there is a proper algebraic set Σ such that the property holds for all the elements in $L \setminus \Sigma$.

Definition 5.2. For a given n-tuple of finite sets $A_1, \dots, A_n \subset \mathbb{Z}^n$, the *intersection index* of the n-tuple of spaces $[L_{A_1}, \dots, L_{A_n}]$ is the number of solutions in $(\mathbb{C}^*)^n$ of a generic system of equations $P_1 = \cdots = P_n = 0$, where $P_1 \in L_{A_1}, \dots, P_n \in L_{A_n}$.

Problem: *Find the intersection index* $[L_{A_1}, \dots, L_{A_n}]$. That is, for a generic element $(P_1, \dots, P_n) \in L_{A_1} \times \cdots \times L_{A_n}$, find a formula for the number of solutions in $(\mathbb{C}^*)^n$ of the system of equations $P_1 = \cdots = P_n = 0$.

Kushnirenko found the following important result, which answers a particular case of the above problem [Kushnirenko].

Theorem 5.3. *When the convex hulls of the sets A_i are the same and equal to a polyhedron Δ, we have*

$$[L_{A_1}, \dots, L_{A_n}] = n!\text{Vol}_n(\Delta),$$

where Vol_n *is the standard n-dimensional volume in* $\mathbb{R}^n$.

According to Theorem 5.3, if $P_1, \dots, P_n$ are sufficiently general Laurent polynomials with given Newton polyhedron Δ, the number of solutions in $(\mathbb{C}^*)^n$ of the system $P_1 = \cdots = P_n = 0$ is equal to $n!\text{Vol}_n(\Delta)$.

The problem was solved by Bernstein in full generality [Bernstein]:

Theorem 5.4. *In the general case, i.e., for arbitrary finite subsets $A_1, \dots, A_n \subset \mathbb{Z}^n$, we have*

$$[L_{A_1}, \dots, L_{A_n}] = n!V(\Delta_{A_1}, \dots, \Delta_{A_n}),$$

where V is the mixed volume of convex bodies in $\mathbb{R}^n$.

According to Theorem 5.4, if $P_1,\dots,P_n$ are sufficiently general Laurent polynomials with Newton polyhedra $\Delta_1,\dots,\Delta_n$ respectively, then the number of solutions in $(\mathbb{C}^*)^n$ of the system $P_1=\cdots=P_n=0$ is equal to $n!V(\Delta_1,\dots,\Delta_n)$.

6 Convex Geometry and the Bernstein–Kushnirenko Theorem

Let us examine Theorem 5.4 (which we will call the Bernstein–Kushnirenko theorem) more closely. In the space of regular functions on $(\mathbb{C}^*)^n$, there is a natural class of finite-dimensional subspaces, namely the subspaces that are stable under the action of the multiplicative group $(\mathbb{C}^*)^n$. Each such subspace is of the form L_A for some finite set $A\subset\mathbb{Z}^n$ of monomials.

For two finite-dimensional subspaces L_1, L_2 of regular functions in $(\mathbb{C}^*)^n$, let us define the product L_1L_2 as the subspace spanned by the products fg, where $f\in L_1$, $g\in L_2$. Clearly, multiplication of monomials corresponds to the addition of their exponents, i.e., $z^{k_1}z^{k_2}=z^{k_1+k_2}$. This implies that $L_{A_1}L_{A_2}=L_{A_1+A_2}$.

The Bernstein–Kushnirenko theorem defines and computes the intersection index $[L_{A_1},L_{A_2},\dots,L_{A_n}]$ of the n-tuples of subspaces L_{A_i} for finite subsets $A_i\subset\mathbb{Z}^n$. Since this intersection index is equal to the mixed volume, it enjoys the same properties, namely (1) positivity, (2) monotonicity, (3) multilinearity, and (4) the Alexandrov–Fenchel inequality and its corollaries. Moreover, if for a finite set $A\subset\mathbb{Z}^n$ we let $\overline{A}=\Delta_A\cap\mathbb{Z}^n$, then (5) the spaces L_A and $L_{\overline{A}}$ have the same intersection indices. That is, for any $(n-1)$-tuple of finite subsets $A_2,\dots,A_n\in\mathbb{Z}^n$, we have

$$[L_A,L_{A_2},\dots,L_{A_n}]=[L_{\overline{A}},L_{A_2},\dots,L_{A_n}].$$

This means (surprisingly!) that enlarging $L_A\mapsto L_{\overline{A}}$ does not change any of the intersection indices we have considered. Hence in counting the number of solutions of a system, instead of support of a polynomial, its convex hull plays the main role. Let us denote the subspace $L_{\overline{A}}$ by $\overline{L}_A$ and call it the *completion of* L_A.

Since the semigroup of convex bodies with Minkowski sum has the cancellation property, we get the following cancellation property for the finite subsets of $\mathbb{Z}^n$: if for finite subsets $A,B,C\in\mathbb{Z}^n$ we have $\overline{\overline{A}+\overline{C}}=\overline{\overline{B}+\overline{C}}$, then $\overline{A}=\overline{B}$. And we have the same cancellation property for the corresponding semigroup of subspaces L_A. That is, if $\overline{\overline{L}_A\,\overline{L}_C}=\overline{\overline{L}_B\,\overline{L}_C}$, then $\overline{L}_A=\overline{L}_B$.

The Bernstein–Kushnirenko theorem relates the notion of mixed volume in convex geometry with that of intersection index in algebraic geometry. In algebraic geometry, the following inequality about intersection indices on a surface is well known:

Theorem 6.1 (Hodge inequality). *Let Γ_1,Γ_2 be algebraic curves on a smooth irreducible projective surface. Assume that Γ_1,Γ_2 have positive self-intersection indices. Then*

$$(\Gamma_1,\Gamma_2)^2\geq(\Gamma_1,\Gamma_1)(\Gamma_2,\Gamma_2),$$

where (Γ_i,Γ_j) denotes the intersection index of the curves Γ_i and Γ_j.

On the one hand, Theorem 5.4 allows one to prove the Alexandrov–Fenchel inequality algebraically using Theorem 6.1 (see [Khovanskii-1, Teissier]). On the other hand, Theorem 5.4 suggests an analogy between the theory of mixed volumes and the intersection theory of Cartier divisors on a projective algebraic variety.

We will return to this discussion after stating our main theorem (Theorem 11.1) and its corollary, which is a version of the Hodge inequality.

7 An Extension of the Intersection Theory of Cartier Divisors

Now we discuss general results, inspired by the Bernstein–Kushnirenko theorem, that can be considered an analogue of the intersection theory of Cartier divisors for general (not necessarily complete) varieties [Kaveh–Khovanskii-2]. Instead of $(\mathbb{C}^*)^n$, we take any irreducible n-dimensional variety X, and instead of a finite-dimensional space of functions spanned by monomials, we take any finite-dimensional space of rational functions. For these spaces we define an intersection index and prove that it enjoys all the properties of the mixed volume of convex bodies.

Consider the collection $\mathbf{K}_{\text{rat}}(X)$ of all nonzero finite-dimensional subspaces of rational functions on X. The set $\mathbf{K}_{\text{rat}}(X)$ has a natural multiplication: the product L_1L_2 of two subspaces $L_1, L_2 \in \mathbf{K}_{\text{rat}}(X)$ is the subspace spanned by all the products fg, where $f \in L_1$, $g \in L_2$. With respect to this multiplication, $\mathbf{K}_{\text{rat}}(X)$ is a commutative semigroup.

Definition 7.1. The *intersection index* $[L_1, \dots, L_n]$ of $L_1, \dots, L_n \in \mathbf{K}_{\text{rat}}(X)$ is the number of solutions in X of a generic system of equations $f_1 = \cdots = f_n = 0$, where $f_1 \in L_1, \dots, f_n \in L_n$. In counting the solutions, we neglect the solutions x for which all the functions in some space L_i vanish as well as the solutions for which at least one function from some space L_i has a pole.

More precisely, let $\Sigma \subset X$ be a hypersurface that contains (1) all the singular points of X, (2) all the poles of functions from any of the L_i, (3) for any i, the set of common zeros of all the $f \in L_i$. Then for a generic choice of $(f_1, \dots, f_n) \in L_1 \times \cdots \times L_n$, the intersection index $[L_1, \dots, L_n]$ is equal to the number of solutions $\{x \in X \setminus \Sigma \mid f_1(x) = \cdots = f_n(x) = 0\}$.

Theorem 7.2. *The intersection index $[L_1, \dots, L_n]$ is well defined. That is, there is a Zariski-open subset U in the vector space $L_1 \times \cdots \times L_n$ such that for any $(f_1, \dots, f_n) \in U$, the number of solutions $x \in X \setminus \Sigma$ of the system $f_1(x) = \cdots = f_n(x) = 0$ is the same (and hence equal to $[L_1, \dots, L_n]$). Moreover, the above number of solutions is independent of the choice of Σ containing (1)–(3) above.*

The following properties of the intersection index are easy consequences of the definition:

Proposition 7.3. (1) $[L_1,\dots,L_n]$ *is a symmetric function of the n-tuples* $L_1,\dots,L_n \in \mathbf{K}_{\mathrm{rat}}(X)$ *(i.e., it takes the same value under a permutation of* $L_1,\dots,L_n$*).*

(2) *The intersection index is monotone (i.e., if* $L'_1 \subseteq L_1,\dots,L'_n \subseteq L_n$*, then* $[L_1,\dots,L_n] \geq [L'_1,\dots,L'_n]$*.*

(3) *The intersection index is nonnegative (i.e.,* $[L_1,\dots,L_n] \geq 0$*).*

The next two theorems contain the main properties of the intersection index.

Theorem 7.4 (Multilinearity). (1) *Let* $L'_1, L''_1, L_2,\dots,L_n \in \mathbf{K}_{\mathrm{rat}}(X)$ *and put* $L_1 = L'_1 L''_1$*. Then*

$$[L_1,\dots,L_n] = [L'_1,\dots,L_n] + [L''_1,\dots,L_n].$$

(2) *Let* $L_1,\dots,L_n \in \mathbf{K}_{\mathrm{rat}}(X)$ *and take 1-dimensional subspaces* $L'_1,\dots,L'_n \in \mathbf{K}_{\mathrm{rat}}(X)$*. Then*

$$[L_1,\dots,L_n] = [L'_1 L_1,\dots,L'_n L_n].$$

Let us say that $f \in \mathbb{C}(X)$ is *integral over a subspace* $L \in \mathbf{K}_{\mathrm{rat}}(X)$ if f satisfies an equation

$$f^m + a_1 f^{m-1} + \cdots + a_m = 0,$$

where $m > 0$ and $a_i \in L^i$ for each $i = 1,\dots,m$. It is well known that the collection $\overline{L}$ of all integral elements over L is a vector subspace containing L. Moreover, if L is finite-dimensional, then $\overline{L}$ is also finite-dimensional (see [Zariski–Samuel, Appendix 4]). It is called the *completion of* L. For two subspaces $L, M \in \mathbf{K}_{\mathrm{rat}}(X)$ we say that L is equivalent to M (written $L \sim M$) if there is $N \in \mathbf{K}_{\mathrm{rat}}(X)$ with $LN = MN$. One shows that the completion $\overline{L}$ is in fact the largest subspace in $\mathbf{K}_{\mathrm{rat}}(X)$ that is equivalent to L. The enlarging $L \to \overline{L}$ is analogous to the geometric operation $A \mapsto \Delta(A)$ that associates to a finite set A its convex hull $\Delta(A)$.

Theorem 7.5. (1) *Let* $L_1 \in K_{\mathrm{rat}}(X)$ *and let* $G_1 \in K_{\mathrm{rat}}(X)$ *be the subspace spanned by* L_1 *and a rational function g integral over* L_1*. Then for any* $(n-1)$*-tuple* $L_2,\dots,L_n \in K_{\mathrm{rat}}(X)$ *we have*

$$[L_1, L_2,\dots,L_n] = [G_1, L_2,\dots,L_n].$$

(2) *Let* $L_1 \in \mathbf{K}_{\mathrm{rat}}(X)$ *and let* $\overline{L}_1$ *be its completion as defined above. Then for any* $(n-1)$*-tuple* $L_2,\dots,L_n \in \mathbf{K}_{\mathrm{rat}}(X)$ *we have*

$$[L_1, L_2,\dots,L_n] = [\overline{L}_1, L_2,\dots,L_n].$$

The proof of Theorem 7.5 is not complicated. If X is a curve, statement (1) is obvious, and one can easily obtain the general case from the curve case (see [Kaveh–Khovanskii-2, Theorem 4.25]). Statement (2) follows from (1). Alternatively, statement (2) follows from the multilinearity of the intersection index and the fact that L and $\overline{L}$ are equivalent.

As with any other commutative semigroup, there corresponds a Grothendieck group to the semigroup $\mathbf{K}_{\text{rat}}(X)$. Let K be a commutative semigroup. The Grothendieck group $G(K)$ of K is defined as follows: two elements $x, y \in K$ are called *equivalent*, written $x \sim y$, if there is $z \in K$ with $xz = yz$. The Grothendieck group $G(K)$ is the collection of all formal fractions x_1/x_2, $x_1, x_2 \in K$, where two fractions x_1/x_2 and y_1/y_2 are considered equal if $x_1y_2 \sim y_1x_2$. There is a natural homomorphism $\phi : K \to G(K)$. The Grothendieck group has the following universal property: for any group G' and a homomorphism $\phi' : K \to G'$, there exists a unique homomorphism $\psi : G(K) \to G'$ such that $\phi' = \psi \circ \phi$.

From the multilinearity of the intersection index it follows that the intersection index extends to the the Grothendieck group of $\mathbf{K}_{\text{rat}}(X)$. The Grothendieck group of $\mathbf{K}_{\text{rat}}(X)$ can be considered an analogue (for a not necessarily complete variety X) of the group of Cartier divisors on a projective variety, and the intersection index on this Grothendieck group an analogue of the intersection index of Cartier divisors.

The intersection theory on the Grothendieck group of $\mathbf{K}_{\text{rat}}(X)$ enjoys all the properties of mixed volume. Some of those properties have already been discussed in the present section. The others will be discussed later (see Theorem 12.3 and Corollary 12.4 below).

8 Proof of the Bernstein–Kushnirenko Theorem Via the Hilbert Theorem

Let us recall the proof of the Bernstein–Kushnirenko theorem from [Khovanskii-2], which will be important for our generalization.

For each space $L \in \mathbf{K}_{\text{rat}}(X)$, let us define the Hilbert function H_L by $H_L(k) = \dim(L^k)$. For sufficiently large values of k, the function $H_L(k)$ is a polynomial in k, called the *Hilbert polynomial* of L.

With each space $L \in \mathbf{K}_{\text{rat}}(X)$, one associates a rational *Kodaira map* from X to $\mathbb{P}(L^*)$, the projectivization of the dual space L^*: to any $x \in X$ at which all the $f \in L$ are defined, there corresponds a functional in L^* that evaluates $f \in L$ at x. The Kodaira map sends x to the image of this functional in $\mathbb{P}(L^*)$. It is a rational map, i.e., defined on a Zariski-open subset in X. We denote by Y_L the closure of the image of X under the Kodaira map in $\mathbb{P}(L^*)$.

The following theorem is a version of the classical Hilbert theorem on the degree of a subvariety of the projective space.

Theorem 8.1 (Hilbert). *The degree m of the Hilbert polynomial of the space L is equal to the dimension of the variety Y_L, and its leading coefficient c is the degree of $Y_L \subset \mathbb{P}(L^*)$ divided by $m!$.*

Let A be a finite subset in $\mathbb{Z}^n$ with $\Delta(A)$ its convex hull. Denote by $k * A$ the sum $A + \cdots + A$ of k copies of the set A, and by $(k\Delta(A))_C$ the subset of $k\Delta(A)$ containing points whose distance to the boundary $\partial(k\Delta(A))$ is bigger than C. The following combinatorial theorem gives an estimate for the set $k * A$ in terms of the set of integral points in $k\Delta(A)$.

Theorem 8.2 ([Khovanskii-2]). (1) *One has* $k * A \subset k\Delta(A) \cap \mathbb{Z}^n$.
(2) *Assume that the differences* $a - b$ *for* $a, b \in A$ *generate the group* $\mathbb{Z}^n$. *Then there exists a constant* C *such that for any* $k \in \mathbb{N}$, *we have*

$$(k\Delta(A))_C \cap \mathbb{Z}^n \subset k * A.$$

Corollary 8.3. *Let* $A \subset \mathbb{Z}^n$ *be a finite subset satisfying the condition in Theorem 8.2(2). Then*

$$\lim_{k \to \infty} \frac{\#(k * A)}{k^n} = \mathrm{Vol}_n(\Delta(A)).$$

Corollary 8.3 together with the Hilbert theorem (Theorem 8.1) proves the Kushnirenko theorem for sets A such that the differences $a - b$ for $a, b \in A$ generate the group $\mathbb{Z}^n$. The Kushnirenko theorem for the general case easily follows from this. The Bernstein theorem, Theorem 5.4, follows from the Kushnirenko theorem (Theorem 5.3) and the identity $L_{A+B} = L_A L_B$.

9 Graded Semigroups in $\mathbb{N} \oplus \mathbb{Z}^n$ and the Newton-Okounkov Body

Let S be a subsemigroup of $\mathbb{N} \oplus \mathbb{Z}^n$. For any integer $k > 0$ we denote by S_k the section of S at level k, i.e., the set of elements $x \in \mathbb{Z}^n$ such that $(k, x) \in S$.

Definition 9.1. (1) A subsemigroup S of $\mathbb{N} \oplus \mathbb{Z}^n$ is called a *graded semigroup* if for any $k > 0$, S_k is finite and nonempty.
(2) Such a subsemigroup is called an *ample semigroup* if there is a natural m such that the set of all the differences $a - b$ for $a, b \in S_m$ generates the group $\mathbb{Z}^n$.
(3) Such a subsemigroup is called a *semigroup with restricted growth* if there is a constant C such that for any $k > 0$, we have $\#(S_k) \leq Ck^n$.

For a graded semigroup S, let $\mathrm{Con}(S)$ denote the closure of the convex hull of $S \cup \{0\}$. It is a cone in $\mathbb{R}^{n+1}$. Denote by $\tilde{S}$ the semigroup $\mathrm{Con}(S) \cap (\mathbb{N} \oplus \mathbb{Z}^n)$. The semigroup $\tilde{S}$ contains the semigroup S.

Definition 9.2. For a graded semigroup S, define the *Newton-Okounkov set* $\Delta(S)$ to be the section of the cone $\mathrm{Con}(S)$ at $k = 1$, i.e.,

$$\Delta(S) = \{x \mid (1, x) \in \mathrm{Con}(S)\}.$$

Theorem 9.3 (Asymptotics of graded semigroups). *Let S be an ample graded semigroup with restricted growth in $\mathbb{N}\oplus\mathbb{Z}^n$. Then:*

(1) *The cone* $\mathrm{Con}(S)$ *is strictly convex, i.e., the Newton-Okounkov set $\Delta(S)$ is bounded.*
(2) *Let $d(k)$ denote the maximum distance of the points (k,x) from the boundary of* $\mathrm{Con}(S)$ *for $x \in \tilde{S}_k \setminus S_k$. Then*

$$\lim_{k\to\infty} \frac{d(k)}{k} = 0.$$

Theorem 9.3 basically follows from Theorem 8.2. For a proof and generalizations, see [Kaveh–Khovanskii-3, Sect. 1.3].

Corollary 9.4. *Let S be an ample graded semigroup with restricted growth in $\mathbb{N}\oplus\mathbb{Z}^n$. Then*

$$\lim_{k\to\infty} \frac{\#(S_k)}{k^n} = \mathrm{Vol}_n(\Delta(S)).$$

10 Valuations on the Field of Rational Functions

We start with the definition of a prevaluation. Let V be a vector space and let I be a set totally ordered with respect to some ordering $<$.

Definition 10.1. A *prevaluation* on V with values in I is a function $v : V \setminus \{0\} \to I$ satisfying the following:

(1) For all $f,g \in V\setminus\{0\}$, $v(f+g) \geq \min(v(f), v(g))$
(2) For all $f \in V\setminus\{0\}$ and $\lambda \neq 0$, $v(\lambda f) = v(f)$
(3) If for $f,g \in V\setminus\{0\}$ we have $v(f) = v(g)$ then there is $\lambda \neq 0$ such that $v(g - \lambda f) > v(g)$.

It is easy to verify that if $L \subset V$ is a finite-dimensional subspace, then $\dim(L)$ is equal to $\#v(L\setminus\{0\})$.

Example 10.2. Let V be a finite-dimensional vector space with basis $\{e_1,\ldots,e_n\}$, and let $I = \{1,\ldots,n\}$ with the usual ordering of numbers. For $f = \sum_i \lambda_i e_i$, define

$$v(f) = \min\{i \mid \lambda_i \neq 0\}.$$

Example 10.3 (Schubert cells in the Grassmannian). Let $\mathrm{Gr}(n,k)$ be the Grassmannian of k-dimensional planes in $\mathbb{C}^n$. In Example 10.2, take $V = \mathbb{C}^n$ with the standard basis. Under the prevaluation v above, each k-dimensional subspace $L \subset \mathbb{C}^n$ goes to a subset $M \subset I$ containing k elements. The set of all the k-dimensional subspaces that are mapped onto M forms the *Schubert cell* X_M in the Grassmannian $\mathrm{Gr}(n,k)$.

Similar to Example 10.3, the Schubert cells in the variety of complete flags can also be recovered from the prevaluation v above on $\mathbb{C}^n$.

Next we define the notion of a valuation with values in a totally ordered abelian group.

Definition 10.4. Let K be a field and Γ a totally ordered abelian group. A prevaluation $v : K \setminus \{0\} \to \Gamma$ is a *valuation* if it further satisfies the following: for any $f, g \in K \setminus \{0\}$ we have

$$v(fg) = v(f) + v(g).$$

The valuation v is called *faithful* if its image is all of Γ.

We will be concerned only with the field $\mathbb{C}(X)$ of rational functions on an n-dimensional irreducible variety X and $\mathbb{Z}^n$-valued valuations on it (with respect to some total order on $\mathbb{Z}^n$).

Example 10.5. Let X be an irreducible curve. Take the field of rational functions $\mathbb{C}(X)$ and $\Gamma = \mathbb{Z}$. Take a smooth point a on X. Then the map

$$v(f) = \mathrm{ord}_a(f)$$

defines a faithful valuation on $\mathbb{C}(X)$.

Example 10.6. Let X be an irreducible n-dimensional variety. Take a smooth point $a \in X$. Consider a local system of coordinates at a with analytic coordinate functions $x_1, \ldots, x_n$. Let $\Gamma = \mathbb{Z}^n_+$ be the semigroup in $\mathbb{Z}^n$ of points with nonnegative coordinates. Take any well-ordering $\prec$ that respects the addition, i.e., if $a \prec b$, then $a + c \prec b + c$. For a germ f at the point a of an analytic function in $x_1, \ldots, x_n$, let $cx^{\alpha(f)} = cx_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be the term in the Taylor expansion of f with minimum exponent $\alpha(f) = (\alpha_1, \ldots, \alpha_n)$ with respect to the ordering $\prec$. For a germ F at the point a of a meromorphic function $F = f/g$, define $v(F)$ as $\alpha(f) - \alpha(g)$. This function v induces a faithful valuation on the field of rational functions $\mathbb{C}(X)$.

Example 10.7. Let X be an irreducible n-dimensional variety and Y any variety birationally isomorphic to X. Then the fields $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ are isomorphic, and thus any faithful valuation on $\mathbb{C}(Y)$ gives a faithful valuation on $\mathbb{C}(X)$ as well.

11 Main Construction and Theorem

Let X be an irreducible n-dimensional variety. Fix a faithful valuation $v : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n$, where $\mathbb{Z}^n$ is equipped with a total ordering respecting addition.

Let $L \in \mathbf{K}_{\mathrm{rat}}(X)$ be a finite-dimensional subspace of rational functions. Consider the semigroup $S(L)$ in $\mathbb{N} \oplus \mathbb{Z}^n$ defined by

$$S(L) = \bigcup_{k>0} \{(k, v(f)) \mid f \in L^k \setminus \{0\}\}.$$

It is easy to see that $S(L)$ is a graded semigroup. Moreover, by Hilbert's theorem, $S(L)$ is contained in a semigroup of restricted growth.

Definition 11.1 (Newton-Okounkov body for a subspace of rational functions). We define the *Newton-Okounkov body* for a subspace L to be the convex body $\Delta(S(L))$ associated to the semigroup $S(L)$.

Denote by $s(L)$ the index of the subgroup in $\mathbb{Z}^n$ generated by all the differences $a-b$ such that a, b belong to the same set $S_m(L)$ for some $m > 0$. Also let Y_L be the closure of the image of the variety X (in fact, the image of a Zariski-open subset of X) under the Kodaira rational map $\Phi_L : X \to \mathbb{P}(L^*)$. If $\dim(Y_L)$ is equal to $\dim(X)$, then the Kodaira map from X to Y_L has finite mapping degree. Denote this mapping degree by $d(L)$.

Theorem 11.1 (Main theorem). *Let X be an irreducible n-dimensional variety. Then:*

(1) *The complex dimension of the variety Y_L is equal to the real dimension of the Newton-Okounkov body $\Delta(S(L))$.*
(2) *If $\dim(Y_L) = n$, then*

$$[L, \dots, L] = \frac{n!d(L)}{s(L)} \mathrm{Vol}_n(\Delta(S(L))).$$

(3) *In particular, if $\Phi_L : X \to Y_L$ is a birational isomorphism, then*

$$[L, \dots, L] = n!\mathrm{Vol}_n(\Delta(S(L))).$$

(4) *For any two subspaces $L_1, L_2 \in \mathbf{K}_{\mathrm{rat}}(X)$ we have*

$$\Delta(S(L_1)) + \Delta(S(L_2)) \subseteq \Delta(S(L_1 L_2)).$$

The proof of the main theorem is based on Theorem 9.3 (which describes the asymptotic behavior of an ample graded semigroup with restricted growth) and Hilbert's theorem (Theorem 8.1). A sketch of a proof can be found in [Kaveh–Khovanskii-1]. For a complete proof as well as generalizations, see [Kaveh–Khovanskii-3, Sect. 4.5]. Also see [Kaveh–Khovanskii-3, Sect. 2.2] for an example of two spaces $L_1, L_2 \in \mathbf{K}_{\mathrm{rat}}(X)$ and a valuation on X such that the inclusion in (4) is not the identity.

12 Algebraic Analogue of the Alexandrov–Fenchel Inequality

Part (2) of the main theorem (Theorem 11.1) can be considered a far-reaching generalization of the Kushnirenko theorem, in which instead of $(\mathbb{C}^*)^n$, one takes any n-dimensional irreducible variety X, and instead of a finite-dimensional space generated by monomials, one takes any finite-dimensional space L of rational functions. The proof of Theorem 11.1 is an extension of the arguments used in [Khovanskii-2] to prove the Kushnirenko theorem (see also Sect. 8). As we mentioned, the Bernstein theorem (Theorem 5.4) follows immediately from the Kushnirenko theorem and the identity

$$L_{A+B} = L_A L_B.$$

Thus the Bernstein–Kushnirenko theorem is a corollary of our Theorem 11.1.

Note that although the Newton-Okounkov body $\Delta(S(L))$ depends on a choice of a faithful valuation, its volume depends on L only: after multiplication by $n!$, it equals the self-intersection index $[L,\dots,L]$.

Our generalization of the Kushnirenko theorem does not imply the generalization of the Bernstein theorem. The point is that in general, we do not always have an equality $\Delta(S(L_1)) + \Delta(S(L_2)) = \Delta(S(L_1L_2))$. In fact, by Theorem 11.1(4), what is always true is the inclusion

$$\Delta(S(L_1)) + \Delta(S(L_2)) \subseteq \Delta(S(L_1L_2)).$$

This inclusion is sufficient for us to prove the following interesting corollary.

Let us call a subspace $L \in \mathbf{K}_{\mathrm{rat}}(X)$ a *big subspace* if for some $m > 0$, the Kodaira rational map of the completion $\overline{L^m}$ is a birational isomorphism between X and its image. It is not hard to show that the product of two big subspaces is again a big subspace and that thus the big subspaces form a subsemigroup of $\mathbf{K}_{\mathrm{rat}}(X)$.

Corollary 12.1 (Algebraic analogue of Brunn–Minkowski). *Assume that $L, G \in \mathbf{K}_{\mathrm{rat}}(X)$ are big subspaces. Then*

$$[L,\dots,L]^{1/n} + [G,\dots,G]^{1/n} \leq [LG,\dots,LG]^{1/n}.$$

Proof. Replacing L and G by $\overline{L^m}$ and $\overline{G^m}$, for any $m > 0$, does not change the inequality (see Theorems 7.4 and 7.5). Thus, without loss of generality, we can assume that the Kodaira maps of L and G are birational isomorphisms onto their images. From statement (4) in Theorem 11.1 we have $\Delta(S(L)) + \Delta(S(G)) \subseteq \Delta(S(LG))$. So $\mathrm{Vol}_n(\Delta(S(L)) + \Delta(S(G))) \leq \mathrm{Vol}_n(\Delta(S(LG)))$. Also, from statement (3) in the same theorem we have

$$[L,\dots,L] = n!\mathrm{Vol}_n(\Delta(S(L)),$$

$$[G,\dots,G] = n!\mathrm{Vol}_n(\Delta(S(G)),$$

$$[LG,\dots,LG] = n!\mathrm{Vol}_n(\Delta(S(LG)).$$

To complete the proof, it is enough to use the Brunn–Minkowski inequality. □

In fact, it is shown in [Kaveh–Khovanskii-3, Theorem 4.23] that the above Brunn–Minkowski inequality holds without the assumption that L, G are big.

Corollary 12.2 (A version of the Hodge inequality). *If X is an algebraic surface and $L, G \in \mathbf{K}_{\mathrm{rat}}(X)$ are big, then*

$$[L,L][G,G] \le [L,G]^2.$$

Proof. From Corollary 12.1, for $n = 2$, we have

$$\begin{aligned}[L,L] + 2[L,G] + [G,G] = [LG,LG] &\ge ([L,L]^{1/2} + [G,G]^{1/2})^2 \\ &= [L,L] + 2[L,L]^{1/2}[G,G]^{1/2} + [G,G],\end{aligned}$$

which readily implies the Hodge inequality. □

Thus Theorem 11.1 immediately enables us to reduce the Hodge inequality to the isoperimetric inequality. In this way, we can easily prove an analogue of the Alexandrov–Fenchel inequality and its corollaries for intersection index:

Theorem 12.3 (Algebraic analogue of the Alexandrov–Fenchel inequality). *Let X be an irreducible n-dimensional variety and let $L_1,\dots,L_n \in \mathbf{K}_{\mathrm{rat}}(X)$ be big subspaces. Then the following inequality holds:*

$$[L_1,L_2,L_3,\dots,L_n]^2 \ge [L_1,L_1,L_3,\dots,L_n][L_2,L_2,L_3,\dots,L_n].$$

In fact, we can show that in Theorem 12.3, it is enough to assume only that $L_3,\dots,L_n$ are big subspaces (see [Kaveh–Khovanskii-3, Theorem 4.27]).

Corollary 12.4 (Corollaries of the algebraic analogue of the Alexandrov–Fenchel inequality). *Let X be an irreducible n-dimensional variety.*

(1) *Let $2 \le m \le n$ and $k_1 + \dots + k_r = m$ with $k_i \in \mathbb{N}$. Take big subspaces of rational functions $L_1,\dots,L_n \in \mathbf{K}_{\mathrm{rat}}(X)$. Then*

$$[k_1 * L_1,\dots,k_r * L_r, L_{m+1},\dots,L_n]^m \ge \prod_{1\le j\le r} [m * L_j, L_{m+1},\dots,L_n]^{k_j}.$$

(1) *(Generalized Brunn–Minkowski inequality) For any fixed big subspaces $L_{m+1},\dots,L_n \in \mathbf{K}_{\mathrm{rat}}(X)$, the function*

$$F : L \mapsto [m * L, L_{m+1},\dots,L_n]^{1/m}$$

is a concave function on the semigroup of big subspaces.

As we saw above, the Bernstein–Kushnirenko theorem follows from the main theorem. Applying the algebraic analogue of the Alexandrov–Fenchel inequality to the situation considered in the Bernstein–Kushnirenko theorem, one can prove the Alexandrov–Fenchel inequality for convex polyhedra with integral vertices. Out of this one can easily complete the proof of Alexandrov–Fenchel for general convex bodies: the homogeneity gives the inequality for convex polyhedra with rational vertices. But each convex body can be approximated arbitrarily well with polyhedra with rational vertices. The statement now follows from the continuity of mixed volume.

Thus the Bernstein–Kushnirenko theorem and the Alexandrov–Fenchel inequality in algebra and in geometry can be considered corollaries of the main theorem (Theorem 11.1).

13 Additivity of the Newton-Okounkov Body for Varieties with a Reductive Group Action

In this section we announce some results that the authors are currently writing up. They will be presented in a forthcoming paper.[2]

While the additivity of the Newton-Okounkov body does not hold in general, we recall, as mentioned in Sect. 8, that it does hold for the subspaces L_A of Laurent polynomials on $(\mathbb{C}^*)^n$ spanned by monomials. The subspaces L_A are exactly the subspaces that are stable under the natural action of the multiplicative group $(\mathbb{C}^*)^n$ on Laurent polynomials (induced by the natural action of $(\mathbb{C}^*)^n$ on itself). It turns out that the additivity generalizes to some classes of varieties with a reductive group action.

Let G be a connected reductive algebraic group over $\mathbb{C}$ (in other words, the complexification of a connected compact real Lie group). Also let X be a G-variety, that is, a variety equipped with an algebraic action of G.

The group G naturally acts on $\mathbb{C}(X)$ by $(g \cdot f)(x) = f(g^{-1} \cdot x)$. A subspace $L \in \mathbf{K}_{\mathrm{rat}}(X)$ is G-stable if for any $f \in L$ and $g \in G$ we have $g \cdot f \in L$.

A G-variety X is called *spherical* if a Borel subgroup of G has a dense orbit. Toric varieties, flag varieties, and group compactifications are well-known examples of spherical varieties.

Theorem 13.1. *Let X be an n-dimensional spherical G-variety. Then there is a naturally defined faithful valuation $v : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n$ such that for any G-stable subspace $L \in \mathbf{K}_{\mathrm{rat}}(X)$, the Newton-Okounkov body $\Delta(S(L))$ is in fact a polyhedron.*

Definition 13.2. Let V be a finite-dimensional representation of G. Let $v = v_1 + \cdots + v_k$ be a sum of highest-weight vectors in V. The closure of the G-orbit of v in V is called an *S-variety*.

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[2] While the present volume was under preparation the related papers [Kaveh-Khovanskii-4] and [Kaveh-Khovanskii-5] appeared.

Affine toric varieties are S-varieties for $G = (\mathbb{C}^*)^n$. One can show that any S-variety is spherical.

Theorem 13.3. *Let X be an S-variety for one of the groups* $G = \mathrm{SL}(n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{C})$, $\mathrm{SP}(2n,\mathbb{C})$, $(\mathbb{C}^*)^n$, *or a direct product of them. Then for the valuation in Theorem 13.1 and for any choice of G-stable subspaces* L_1, L_2 *in* $\mathbf{K}_{\mathrm{rat}}(X)$, *we have*

$$\Delta(S(L_1L_2)) = \Delta(S(L_1)) + \Delta(S(L_2)).$$

Corollary 13.4 (Bernstein theorem for S-varieties). *Let X be an S-variety for one of the groups* $G = \mathrm{SL}(n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{C})$, $\mathrm{SP}(2n,\mathbb{C})$, $(\mathbb{C}^*)^n$, *or a direct product of them. Let* $L_1,\dots,L_n \in \mathbf{K}_{\mathrm{rat}}(X)$ *be G-stable subspaces. Then for the valuation in Theorem 13.1, we have*

$$[L_1,\dots,L_n] = n!V(\Delta(S(L_1)),\dots,\Delta(S(L_n))),$$

where V is the mixed volume.

Another class of G-varieties for which the additivity of the Newton polyhedron holds is the class of symmetric homogeneous spaces.

Definition 13.5. Let σ be an involution of G, i.e., an order-2 algebraic automorphism. Let $H = G^\sigma$ be the fixed-point subgroup of σ. The homogeneous space G/H is called a *symmetric homogeneous space*.

Example 13.6. The map $M \mapsto (M^{-1})^t$ is an involution of $G = \mathrm{SL}(n,\mathbb{C})$ with the fixed-point subgroup $H = \mathrm{SO}(n,\mathbb{C})$. The symmetric homogeneous space G/H can be identified with the space of nondegenerate quadrics in $\mathbb{C}P^{n-1}$.

Any symmetric homogeneous space is an affine spherical G-variety (with the left G-action).

Under mild conditions on the L_i, analogues of Theorem 13.3 and Corollary 13.4 hold for symmetric varieties. Finally, the above theorems extend to subspaces of sections of G-line bundles.

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