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On a Lemma of Kontsevich

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To V. I. Arnold on the occasion of his sixtieth anniversary

Kontsevich has recently constructed a universal quantization theory. Among the facts discovered, the following assertion is contained. Let $f_1, \dots, f_{2n}$ be $2n$ rational functions on a complete complex algebraic variety M whose complex dimension is equal to n. Let M_0 be the set of nonsingular points of the algebraic variety M from which the union of the supports of the divisors of the functions $f_1, \dots, f_{2n}$ is deleted. Denote by $\arg f_i$ the argument of the function f_i.

Kontsevich's Lemma. $\displaystyle\int_{M_0} d\arg f_1 \wedge \dots \wedge d\arg f_{2n} = 0\,.$

In the last list of Arnold's problems, there is the following problem: to give a visual proof of Kontsevich's lemma. In this note we present such a proof.

1. Transformation of the Differential Form.

Lemma 1. *The following identity holds*:

$$d\arg f_1 \wedge \dots \wedge d\arg f_{2n} \equiv d\ln|f_1| \wedge \dots \wedge d\ln|f_{2n}|\,.$$

Proof. Let I_x be the operator of multiplication by the imaginary unit in the tangent space to the variety M at a nonsingular point x. We shall regard I_x as a real linear transformation. First, the determinant of this transformation is equal to one. Second, since the function $\ln f_i$ is analytic, we have the identity

$$I_x^* d\arg f_i = d\ln|f_i|\,.$$

This implies Lemma 1.

In contrast to $\arg f_i$, the functions $\ln|f_i|$ are single-valued. Therefore, Lemma 1 explains the Kontsevich lemma almost completely. The only remaining difficulty lies in the singularities of the functions $\ln|f_i|$. Let us show how it can be overcome.

2. Resolution of Singularities. Consider a rational mapping $F\colon M \to (\mathbb{CP}^1)^{2n}$ that takes a point x to $(f_1(x), \dots, f_{2n}(x))$. According to the Hironaka theorem, there exists a nonsingular compact complex variety N and a regular mapping $\pi\colon N \to M$ such that

1) a generic point of the variety M has a unique preimage under the mapping π;

2) the mapping $G = F \circ \pi$ is regular;

3) the union Γ of the supports of the divisors of the functions $g_i = f_i \circ \pi$ consists of transversally intersecting smooth hypersurfaces in the variety N.

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It suffices to prove the Kontsevich lemma for the variety N and the functions $g_1, \dots, g_{2n}$. Indeed, the varieties M and N differ only by a set of lower dimension, and this distinction does not affect the integrals.

3. Polar Coordinates. It is convenient to introduce polar coordinates near the points of the hypersurface Γ on the variety N. Let U be the coordinate neighborhood on the variety N and let $z_1, \dots, z_n$ be coordinates such that the hypersurface Γ in the chart U is described by the equation $z_1 \cdots z_k = 0$, where k is a nonnegative number. Let us identify the domain U with its image in $\mathbb{C}^n$ under the embedding specified by the coordinate functions z_j. Let $\mathbb{R}^{2n}$ be the space with coordinate functions $r_1, \varphi_1, \dots, r_k, \varphi_k, x_{k+1}, y_{k+1}, \dots, x_n, y_n$ and let $V \subseteq \mathbb{R}^{2n}$ be the domain defined by the inequalities $r_1 \geq 0, \dots, r_k \geq 0,\ 0 \leq \varphi_1 \leq 2\pi, \dots, 0 \leq \varphi_k \leq 2\pi$.

Consider the mapping ρ of the domain V into $\mathbb{C}^n$ determined by the formulas $z_j = r_j e^{i\varphi_j}$ for $1 \leq j \leq k$ and $z_j = x_j + iy_j$ for $k < j \leq n$.

Lemma 2. *The form $\rho^* d\ln|g_1| \wedge \cdots \wedge \rho^* d\ln|g_{2n}|$ is smooth in the preimage of the domain U.*

Proof. Let g be one of the functions $g_1, \dots, g_{2n}$. By assumption, the function g can be represented in the domain U as the product of a nowhere vanishing function by a monomial $z_1^{m_1} \cdots z_k^{m_k}$. Therefore, $d\ln|g| = \sum m_j \dfrac{dr_j}{r_j} + \alpha$, where α is a smooth 1-form in the domain U. Let us expand the smooth 1-form $\rho^*\alpha$ with respect to the coordinate basis in the domain V. The coefficient A_j in $d\varphi_j$ entering this expansion is divisible by r_j, i.e., $A_j = r_j B_j$, where B_j is a smooth function. Hence, the negative powers of all coordinate functions r_j in the product $\rho^* d\ln|g_1| \wedge \cdots \wedge \rho^* \ln|g_{2n}|$ are cancelled out and the resulting form is smooth.

Corollary. *The integral in the Kontsevich lemma is absolutely convergent.*

After the resolution of singularities, the corollary follows from Lemma 2. Indeed, the compact variety N is covered by finitely many charts U to which this lemma applies.

4. Zero-Degree Mapping. With a complex number we associate its modulus. This mapping can be extended by continuity to a mapping of the complex projective line into the real projective line. Denote by $\mu\colon (\mathbb{CP}^1)^{2n} \to (\mathbb{RP}^1)^{2n}$ the Cartesian power of this mapping. The image of the variety $(\mathbb{CP}^1)^{2n}$ under the mapping μ is the closure of the positive cone $(\mathbb{R}_+)^{2n}$ in $(\mathbb{RP}^1)^{2n}$.

Let $G\colon N \to (\mathbb{CP}^1)^{2n}$, $G = (g_1, \dots, g_{2n})$, be a regular mapping of the n-dimensional variety N into $(\mathbb{CP}^1)^{2n}$ and let Γ be the union of the supports of the divisors of the functions $g_1, \dots, g_{2n}$.

Lemma 3. *The restriction of the mapping $\mu \circ G$ to the domain $N \setminus \Gamma$ is a proper mapping of this domain into the interior of the positive cone. The degree of this proper mapping is equal to zero.*

Proof. The preimage of the boundary of the positive cone in the variety $(\mathbb{RP}^1)^{2n}$ under the mapping $\mu \circ G$ coincides with the hypersurface Γ. Therefore, the restriction of this mapping to the domain $N \setminus \Gamma$ is proper.

We consider the highest-degree form

$$\omega_{2n} = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{2n}}{x_{2n}}$$

on the cone $(\mathbb{R}_+)^{2n}$. The volume of the cone with respect to this form is infinite. However, the integral $\int_{N\setminus\Gamma}(\mu\circ G)^*\omega_{2n}$ is absolutely convergent (see the corollary in Sec. 2). Hence, almost every point of the positive cone lies outside of the image of the domain $N\setminus\Gamma$. Thus, the degree of the mapping $\mu\circ G$ is equal to zero.

5. End of the Proof. According to the corollary in Sec. 2, the integral in the Kontsevich lemma is absolutely convergent. By Lemma 3, it is equal to zero.

6. Remark. After this note was written I got occasionally acquainted with the original proof of Kontsevich (*Kontsevich M.* Deformation quantization of Poisson manifolds. Preliminary version). It is based on a similar idea. However, the details are different, and they are simpler in our version.

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