MAPPING DEGREE AND EULER CHARACTERISTIC

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Abstract

Let \( V_\delta \) denote a local level surface for function-germ \( f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0) \). A mapping degree formula for difference of the Euler characteristics of \( V_\delta \cap \{ g \leq 0 \} \) and \( V_\delta \cap \{ g \geq 0 \} \) is given, when level surfaces of a function \( g : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0) \) are parallelizable.

It is classically known that mapping degree is closely related to Euler characteristics. One of such relation is the following celebrated formula due to G. N. Khimshiashvili ([7]): Let \( (x_0, x_1, \ldots, x_n) \) be a coordinate system of \( \mathbb{R}^{n+1} \). Let \( B^{n+1}_e \) denote the open ball centered at 0 in \( \mathbb{R}^{n+1} \) with radius \( e \). Let \( f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0) \) be an analytic function-germ and \( V_\delta \) denote the local level surface of \( f \), i.e.,

\[
V_\delta = B^{n+1}_e \cap f^{-1}(\delta) \quad \text{for} \quad 0 < |\delta| \ll e \ll 1.
\]

We denote its Euler characteristic by \( \chi(V_\delta) \). Then the Khimshiashvili’s formula asserts that, when \( f \) defines an isolated singularity at 0,

\[
\deg(df) = \text{sign}(-\delta)^{n+1}(1 - \chi(V_\delta))
\]

where \( df \) is the map-germ defined by

\[
df : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0), \quad x \mapsto (f_{x_0}(x), f_{x_1}(x), \ldots, f_{x_n}(x)).
\]

Here \( f_{x_i} \) denote the partial derivative of \( f \) by \( x_i, \; i = 0, 1, \ldots, n \).

We consider a relative version of this formula. In [3], the first author considered the mapping degree of map-germs

\[
F : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0), \quad x \mapsto (f(x), f_{x_1}(x), \ldots, f_{x_n}(x))
\]

and showed that, if \( F \) is finite, then

\[
\deg(F) = \text{sign}(-\delta)^{n+1}(\chi(V_\delta(x_0 \leq 0)) - \chi(V_\delta(x_0 \geq 0)))
\]

where \( V_\delta(x_0 \leq 0) = \{ x \in V_\delta : x_0 \leq 0 \} \), and \( V_\delta(x_0 \geq 0) = \{ x \in V_\delta : x_0 \geq 0 \} \).

In this paper, we consider an analytic function \( g : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0) \) so that there are \( C^\infty \)-vector fields \( v_1, \ldots, v_n \) which span the tangent space of a level set

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of $g$ at each regular point of $g$. We assume that $V_g, v_1, \ldots, v_n$ agree with the orientation of $(\mathbb{R}^{n+1}, 0)$ at each regular point of $g$ where $V_g$ is the gradient vector of $g$. We define a map $F$ by

$$F: (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0), \quad x \mapsto (f(x), v_1 f(x), \ldots, v_n f(x)).$$

The purpose is to show (Theorem 4.1) that, if $F$ is finite, and $V_\delta \cap \Sigma(g) = \emptyset$, then

$$\deg(F) = \text{sign}(-\delta)^{n+1}(\chi(V_\delta(g \leq 0)) - \chi(V_\delta(g \geq 0)))$$

where $V_\delta(g \leq 0) = \{x \in V_\delta : g(x) \leq 0\}$, and $V_\delta(g \geq 0) = \{x \in V_\delta : g(x) \geq 0\}$.

This formula will be proved in §4 applying Morse theory to the restriction of $g$ to a level of $f$. In §1 we investigate the condition on the existence of such vector fields $v_1, \ldots, v_n$ and discuss explicit construction of them in some special case in §2. Applying Morse theory to the restriction of $f$ to a level of $g$, we also show another topological interpretation of $\deg F$ in §3. In §4 we investigate the condition that $g|_{V_\delta}$ is Morse and give a proof of (0.1) and its variant.

In the last section, we consider a kind of ‘product’ of $dg$ and $df$ and give a topological interpretation of its mapping degree. It is motivated by Remark 2.1 which is a consequence of the explicit form of $F$.

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1. Condition (P) and the definition of the map $F$

Let $L$ denote an oriented $(n + 1)$-dimensional $C^\infty$-manifold and $g: L \to \mathbb{R}$ be a $C^\infty$-function on $L$. We fix a Riemannian metric on $L$ and denote the gradient of $g$ by $V_g$. We always consider the orientation of the set of regular points of the level set of $g$ so that $V_g$ and the orientation of the level set of $g$ agree with the orientation of $L$.

We consider the following condition on $g$.

(P): There exist $C^\infty$-vector fields $v_1(x), \ldots, v_n(x)$ on $L$ which span the tangent space of the level set of $g$ at a regular point $x$ of $g$, and the orientation of a level of $g$ there coincides with the orientation defined by $v_1(x), \ldots, v_n(x)$.

**Definition 1.1.** Let $g: L \to \mathbb{R}$ be a $C^\infty$-function with Condition (P). We define the map

$$F: L \to \mathbb{R}^{n+1}, \quad x \mapsto (f(x), v_1 f(x), \ldots, v_n f(x)),$$

where $f: L \to \mathbb{R}$ is a $C^\infty$-function.

In later sections, we investigate several topological interpretations of the mapping degree of $F$. In the rest of this section, we investigate Condition (P) in general.
1.1. Existence of vector fields \( v_1, \ldots, v_n \) in Condition (P). We show the following

**Proposition 1.2.** Let \( g : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0) \) be a \( C^\infty \)-function which defines an isolated singularity at 0. Then, the following conditions are equivalent.

(i) There exist \( C^\infty \)-vector fields \( v_1(x), \ldots, v_n(x) \) near 0 which span the tangent space of the level set of \( g \) at a regular point \( x \) of \( g \).

(ii) One of the following conditions holds.

- \( n = 1, 3, 7 \).
- \( n \) is odd, \( n \neq 1, 3, 7 \), and \( \deg (dg) \) is even.
- \( n \) is even, and \( \deg (dg) \) is zero.

First we consider more general set-up. Let \( L \) be a manifold of dimension \( n + 1 \), and let \( g : L \to \mathbb{R} \) be a \( C^\infty \)-function. We denote \( L' = L - \Sigma(g) \), and assume that \( L' \) is parallelizable. Let \( E \) denote the vector bundle on \( L' \) whose fiber is the tangent space of each level of \( g \). We investigate the following question. When \( E \) is a trivial bundle?

If \( E \) is \( C^0 \)-trivial, then this bundle is \( C^\infty \)-trivial and there exist \( C^\infty \)-vector fields \( w_1(x), \ldots, w_n(x) \) on \( L' \) which span \( E \). Then \( v_i(x) = b(x)w_i(x), i = 1, \ldots, n \), satisfy Condition (P) where \( b \) is a \( C^\infty \)-function on \( L \) so that \( \Sigma(g) = b^{-1}(0) \) and that \( b \) is flat at \( \Sigma(g) \), that is, all partial derivatives of order \( k, k = 0, 1, 2, \ldots \), vanish at each point of \( \Sigma(g) \).

Since \( L' \) is parallelizable, there is an oriented orthonormal frame \( e_0, e_1, \ldots, e_n \) of the tangent bundle of \( L' \), and we can define the following Gauss map:

\[
\varsigma : L' \to S^n, \quad x \mapsto (a_0, a_1, \ldots, a_n) \quad \text{where} \quad \frac{\nabla g}{\|\nabla g\|} = a_0 e_0 + a_1 e_1 + \cdots + a_n e_n.
\]

Let \( \text{SO}(n) \) denote the group of orthogonal \( n \times n \) matrices with determinant 1. Let us consider the map defined by

\[
p : \text{SO}(n+1) \to S^n, \quad A \mapsto \text{the first column of } A.
\]

**Proposition 1.3.** Under the above assumption, the following conditions are equivalent.

(i) The vector bundle \( E \) is \( C^0 \)-trivial (and, thus \( C^\infty \)-trivial).

(ii) There is a continuous map \( \beta : L' \to \text{SO}(n+1) \) so that \( \varsigma = p \circ \beta \).

(iii) One of the following conditions holds.

- \( n = 1, 3, 7 \).
- \( n \) is odd, \( n \neq 1, 3, 7 \), and the induced map \( \pi_n(L') \to \pi_n(S^n) \) is even.
- \( n \) is even, and the induced map \( \pi_n(L') \to \pi_n(S^n) \) is zero.

Here we say that a map \( \varsigma : G_1 \to G_2 \) between two abelian groups \( G_1, G_2 \) is even if for any \( g_1 \in G_1 \) there is \( g_2 \in G_2 \) with \( f(g_1) = 2g_2 \).

We say that a map \( p : E \to B \) is a fibration in the sense of Serre if the
following condition holds: for a CW complex $X$ and a homotopy $\alpha_t : X \to B$, $0 \leq t \leq 1$, if there is a map $\beta_0 : X \to E$ with $p \circ \beta_0 = \alpha_0$, then there is a homotopy $\beta_t : X \to E$, $0 \leq t \leq 1$, with $p \circ \beta_t = \alpha_t$ for $0 \leq t \leq 1$.

We remark that the locally trivial fibration is a fibration in the sense of Serre.

**Proof of Proposition 1.3.** (i) $\Rightarrow$ (ii): If $E$ is trivial, then the associated $SO(n)$-bundle with $E$ is trivial, and thus have non-zero section. This means (ii).

(ii) $\Rightarrow$ (i): If there is a continuous map $\beta : L' \to SO(n + 1)$ so that $\alpha = p \circ \beta$, then there is an orthonormal frame which spans $E$, and we thus conclude that $E$ is trivial.

(ii) $\Rightarrow$ (iii): Since the map $p : SO(n + 1) \to S^n$ is a fibration with fiber $SO(n)$, we have the following homotopy exact sequence:

$$
\pi_n(SO(n + 1)) \xrightarrow{p_*} \pi_n(S^n) \xrightarrow{i_*} \pi_{n-1}(SO(n + 1)) \to 0
$$

where $i : SO(n) \to SO(n + 1)$ denote an inclusion. Remark that the map $\beta : L' \to SO(n + 1)$ induces $\beta_* : \pi_n(L') \to \pi_n(SO(n + 1))$ with $p \circ \beta_* = \alpha_*$. Then the following fact (see [6, Chapter 8, Ex. 8]) implies (iii).

$$(1.3) \quad \text{Kernel of } i_* = \begin{cases} 0, & \text{if } n = 1, 3, 7; \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } n \text{ is odd and } n \neq 1, 3, 7; \\ \mathbb{Z}, & \text{if } n \text{ is even.} \end{cases}$$

(iii) $\Rightarrow$ (ii): We show this implication as an application of the obstruction theory.

Let $S^k$, $k = 0, 1, \ldots, n - 1$, be a $k$-dimensional sphere in $L'$, and set $\alpha_k = \alpha|_{S^k}$. The map $\alpha_k : S^k \to S^n$ represents the zero element of $\pi_k(S^n)$, since $\pi_k(S^n) = 0$. Take a map $\beta^*_k : S^k \to SO(n + 1)$ which represents the zero element of $\pi_k(SO(n + 1))$. Since the map $p \circ \beta^*_k$ also represents zero of $\pi_k(S^n)$, there is a homotopy $\phi : S^k \to S^n$, $0 \leq t \leq 1$, with $\phi_0 = p \circ \beta^*_k$ and $\phi_1 = \alpha_k$. Since $p : SO(n + 1) \to S^n$ is a fibration in the sense of Serre, there is a map $\beta_k : S^k \to SO(n + 1)$ so that $p \circ \beta_k = \alpha_k$. If there is a $(k + 1)$-dimensional ball $B^{k+1}$ in $L'$ which bounds the sphere $S^k$ in $L'$, then $\beta_k$ can be extended to $B^{k+1}$, since $\beta_k$ represents the zero in $\pi_k(SO(n + 1))$.

Let $S^n$ be an $n$-dimensional sphere in $L'$ and set $\alpha_n = \alpha|_{S^n}$. By (iii), the homotopy class of $\alpha_n$ is in the kernel of $i_*$ in (1.2), because of (1.3). Since $p : SO(n + 1) \to S^n$ is a fibration in the sense of Serre, there is a map $\beta_n : S^n \to SO(n + 1)$ so that $p \circ \beta_n = \alpha_n$.

Since $L'$ is not compact, $L'$ has a homotopy type of a CW complex of dimension $\leq n$, and we complete the proof.

**Remark 1.4.** Let $g_1 : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a $C^\infty$-function. Let $U$ be a neighborhood of $0$ and assume that $g_1$ is defined on $U$. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a linear projection. Setting $g = g_1 \circ \pi$ and $L = \pi^{-1}(U)$, we have $L' = L - \Sigma(g) \simeq (U - \Sigma(g_1)) \times \mathbb{R}$, that has a homotopy type of a CW complex of dimension $\leq$
There is a continuous map the map \( c \) of homotopy the proof.

\[ x = [x_0 : x_1 : \cdots : x_n] \rightarrow \frac{1}{S}(2x_0^2 - S, 2x_1x_0, \ldots, 2x_nx_0) \quad \text{where} \quad S = \sum_{i=0}^{n} x_i^2. \]

Let \( q : S^n \rightarrow P^n(\mathbf{R}) \) denote the map defined by \( (x_0, x_1, \ldots, x_n) \rightarrow [x_0 : x_1 : \cdots : x_n] \). For a unit vector \( x = (x_0, x_1, \ldots, x_n) \) and \( y = \varphi \circ q(x) \), we see that \( 0, e_0, x, \text{and} y \) are in the same plane and \( 2\mathbf{e}_0x = \mathbf{e}_0y \). We remark that the map \( \varphi \) is generically one-to-one and sends the set defined by \( \{x_0 = 0\} \) to a point.

**Proof.** Let \( x = (x_0, x_1, \ldots, x_n) \) be a non-zero vector in \( \mathbf{R}^{n+1} \). Let \( \psi_x : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^{n+1} \) denote the reflection sending the vector \( x \) to \(-x\). We remark that the map \( \psi_x \) is represented by the matrix

\[
\begin{pmatrix}
\delta_{i,j} - \frac{2x_ix_j}{S} \\
\end{pmatrix}_{i,j=0,1,\ldots,n} \quad \text{where} \quad \delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\
0, & \text{otherwise},
\end{cases}
\]

and the first column of the matrix for \( \psi_{e_0} \circ \psi_x \) represents the map \( \varphi : P^n(\mathbf{R}) \rightarrow S^n \). Let \( h_1 \) denote a homotopy with \( h_0 = \varphi \circ \gamma \) and \( h_1 = \varphi \). We remark that there is a continuous map \( \gamma_1 : L' \rightarrow SO(n+1) \) with \( \varphi = p \circ \gamma_1 \). In fact, the map \( \gamma_1 = \psi \circ \gamma \) satisfies \( \varphi = p \circ \gamma_1 \) where \( \psi : P^n(\mathbf{R}) \rightarrow SO(n+1) \) is the embedding defined by \( [x] \mapsto \psi_{e_0} \circ \psi_x \). Since \( p : SO(n+1) \rightarrow S^n \) is a fibration in the sense of Serre, we obtain there is a continuous map \( \alpha_1 : L' \rightarrow SO(n+1) \) with \( \alpha = p \circ \alpha_1 \), and we complete the proof.

**Proposition 1.6.** Under the same assumption as Proposition 1.3, the vector bundle \( E \) is trivial if one of the following conditions holds.

- \( n \) is odd, and the induced map \( \alpha^* : H^n(S^n; \mathbf{Z}) \rightarrow H^n(L'; \mathbf{Z}) \) is even.
- \( n \) is even, and the induced map \( \alpha^* : H^n(S^n; \mathbf{Z}) \rightarrow H^n(L'; \mathbf{Z}) \) is zero.

**Proof.** Assume first that \( n \) is even and the induced map \( \alpha^* : H^n(S^n; \mathbf{Z}) \rightarrow H^n(L'; \mathbf{Z}) \) is zero. Then, by Hopf’s theorem (see [5, Chapter II, 8]) there is a homotopy \( A : L' \times [0, 1] \rightarrow S^n \), \( A(x, t) = \alpha(t) \), with the following properties:

- \( \alpha_0 = \alpha \).
- If \( n \) is odd, then there are continuous maps \( a : L' \rightarrow S^n \) and \( b : S^n \rightarrow S^n \) so that \( b \) is of degree two and \( \alpha = b \circ a \). We may assume that \( b \) factors through the map \( \varphi \).
· If \( n \) is even, then \( \text{Im} \, z_1 \) is a point.
Since \( p : \text{SO}(n+1) \to S^n \) is a fibration in the sense of Serre, we complete the proof as in the same way in the previous proposition.

Proof of Proposition 1.2. (i) \( \Rightarrow \) (ii): If (i) holds, then (iii) of Proposition 1.3 holds, and (ii) holds.
(ii) \( \Rightarrow \) (i): The implication (iii) \( \Rightarrow \) (i) of Proposition 1.3 implies (ii) \( \Rightarrow \) (i).
The explicit construction of \( v_1, \ldots, v_n \) in the next section gives another proof when \( n = 1, 3, 7 \). When \( n \neq 1, 3, 7 \), Proposition 1.6 also gives another proof by Hopf’s theorem (ibid.).

2. Explicit construction of vector fields \( v_1, \ldots, v_n \) in Condition (P)
Let \( g : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0) \) be a polynomial (resp. analytic) function. Assume that one of the conditions in Proposition 1.3 (iii) (or in Proposition 1.2 (ii) when \( g \) defines isolated singularity at \( 0 \)) holds. Then are there polynomial (resp. analytic) vector fields \( v_1, \ldots, v_n \) which span the tangent space of the level of \( g \) at a regular point of \( g \)? The answer is affirmative if one of the following conditions holds.
(a) \( n = 1, 3, 7 \).
(b) \( g_{x_0} \) is not negative.
We are going to prove this assertion to construct vector field \( v_1, \ldots, v_n \) explicitly.
Let \( L = \mathbb{R}^{n+1} \) and we denote by \( \partial_{x_i} \) the unit vector \( e_i = (0, \ldots, 1, \ldots, 0) \) for \( i = 0, 1, \ldots, n \).

2.1. Case (a). If \( n = 1, 3, 7 \), our explicit construction of the vector fields \( v_1, \ldots, v_n \) is based on the multiplicative structure of complex, quaternion, Cayley numbers, respectively.

Case \( n = 1 \): We consider the complex numbers \( \mathbb{C} = \mathbb{R} + \mathbb{R}i \) where \( i^2 = -1 \), and identify it with \( \mathbb{R}^2 \). Under this identification \( \nabla g = g_{x_0} + g_{x_1} i \). Then \( i \nabla g = -g_{x_1} + g_{x_0} i \) span the tangent space of the level set of \( g \) at a regular point of \( g \).
In other words, the vector field \( v_1 \) in Condition (P) is given by the following:

\[
v_1 = i \nabla g = -g_{x_1} \partial_{x_1} + g_{x_0} \partial_{x_1}.
\]

Case \( n = 3 \): We consider the quaternion numbers \( \mathbb{Q} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k \) with

\[
i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.
\]
We set \( x = a_0 - a_1 i - a_2 j - a_3 k \) when \( x = a_0 + a_1 i + a_2 j + a_3 k \). Since \( x \bar{x} = \sum_{i=0}^{3} a_i^2 \), \( x \) has the inverse \( x/(x \bar{x}) \) if \( x \neq 0 \). We identify \( \mathbb{Q} \) with \( \mathbb{R}^4 \). We remark that \( \langle x, y \rangle := \text{Re}(x \bar{y}) \) \((x, y \in \mathbb{Q})\) is the Euclidean inner product of \( \mathbb{R}^4 \). Under this identification we have that \( \nabla g = g_{x_0} + g_{x_1} i + g_{x_2} j + g_{x_3} k \). Since \((1, i, j, k)\) forms an orthonormal frame of the tangent space of \( \mathbb{R}^4 \), \( (\nabla g, i \nabla g, j \nabla g, k \nabla g) \)
forms also an orthogonal frame of the tangent space of $\mathbb{R}^4$, when $Vg \neq 0$. This implies that $i\nabla g$, $j\nabla g$, $k\nabla g$ span the tangent space of the level set of $g$ at a regular point of $g$. In other words, the vector fields $v_1$, $v_2$, $v_3$ in Condition (P) are given by the following:

$$\begin{align*}
v_1 &= i\nabla g = -gx_3\partial_{x_0} + gx_0\partial_{x_3} - gx_3\partial_{x_2} + gx_2\partial_{x_3}, \\
v_2 &= j\nabla g = -gx_2\partial_{x_0} + gx_0\partial_{x_2} - gx_2\partial_{x_1} + gx_1\partial_{x_2}, \\
v_3 &= k\nabla g = -gx_1\partial_{x_0} - gx_2\partial_{x_1} + gx_1\partial_{x_2} + gx_2\partial_{x_3}.
\end{align*}$$

**Case $n = 7$:** We consider Cayley numbers $\mathcal{C} = Q + Qe$ with

$$(q + re)(s + te) = (qs - \bar{r}t) + (tq + r\bar{s})e, \quad q, r, s, t \in Q.$$

We set $\bar{x} = \bar{q} - re$ when $x = q + re$. Since $xx = q\bar{q} + rr$, $x$ has the inverse $x/(x\bar{x})$ if $x \neq 0$. We identify $\mathcal{C}$ with $\mathbb{R}^3$ and remark that $\langle x, y \rangle := \text{Re}(xy)$ $(x, y \in \mathcal{C})$ is the Euclidean inner product of $\mathbb{R}^3$. Under this identification we have that $Vg = gx_0 + gx_1i + gx_2j + gx_3k + (g_{x_1} + gx_0i + gx_2j + gx_3k)e$. Then $i\nabla g$, $j\nabla g$, $k\nabla g$, $ie\nabla g$, $je\nabla g$, $ke\nabla g$ span the tangent space of the level set of $g$ at a regular point of $g$. In other words, the vector fields $v_1, \ldots, v_7$ in Condition (P) are given by the following:

$$\begin{align*}
v_1 &= i\nabla g = -gx_3\partial_{x_0} + gx_0\partial_{x_3} - gx_3\partial_{x_2} + gx_2\partial_{x_3}, \\
v_2 &= j\nabla g = -gx_2\partial_{x_0} + gx_0\partial_{x_2} - gx_2\partial_{x_1} + gx_1\partial_{x_2}, \\
v_3 &= k\nabla g = -gx_1\partial_{x_0} - gx_2\partial_{x_1} + gx_1\partial_{x_2} + gx_2\partial_{x_3}, \\
v_4 &= e\nabla g = -gx_4\partial_{x_0} + gx_0\partial_{x_4} + gx_4\partial_{x_2} + gx_2\partial_{x_4} + gx_4\partial_{x_3} - gx_3\partial_{x_4}, \\
v_5 &= ie\nabla g = -gx_4\partial_{x_0} + gx_0\partial_{x_4} + gx_4\partial_{x_2} + gx_2\partial_{x_4} - gx_3\partial_{x_4}, \\
v_6 &= je\nabla g = -gx_4\partial_{x_0} - gx_4\partial_{x_1} - gx_4\partial_{x_2} + gx_2\partial_{x_4} + gx_4\partial_{x_3} + gx_3\partial_{x_4}, \\
v_7 &= ke\nabla g = -gx_4\partial_{x_0} - gx_4\partial_{x_1} - gx_4\partial_{x_2} - gx_4\partial_{x_3} + gx_3\partial_{x_4} + gx_2\partial_{x_4} - gx_3\partial_{x_4}.
\end{align*}$$

**Remark 2.1.** In the above construction, the map $F$ (Definition 1.1) coincides with

$$p(\nabla g, \nabla f) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}, \quad x \mapsto p(\nabla g(x), \nabla f(x))$$

except the first component, where $p : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ is the product of the complex, quaternion, Cayley numbers, respectively. In fact, the $e_i$ component, $i = 1, \ldots, n$, of $F$ is $\langle e_i \nabla g, \nabla f \rangle = \text{Re}(p(-e_i, \nabla g, \nabla f)) = \text{Re}(p(-e_i, p(\nabla g, \nabla f)))$, which is the $e_i$ component of $p((\nabla g, \nabla f))$. Here we use the fact $\text{Re}(ab) = \text{Re}(a(bc))$ for any complex, quaternion, Cayley numbers $a, b, c$, respectively.

**2.2. Case (b).** Assume that $g_{x_0}$ is not negative. This means that the mapping degree of $dg$ is zero. We define vector fields $v_i, i = 1, \ldots, n$, by
Then $v_1, \ldots, v_n$ span the tangent space of each level of $g$ at each regular point of $g$.

It is clear that these $v_1, \ldots, v_n$ are polynomial (resp. analytic) vector fields when $g$ is a polynomial (resp. analytic).

**Proof.** It is easy to see that $\langle \nabla g, v_i \rangle = 0$ for $i = 1, \ldots, n$. So it is enough to show that $\nabla g, v_1, \ldots, v_n$ are linearly independent on $\mathbb{R}^n - \Sigma(g)$. The coefficient matrix of vector fields $\nabla g, v_1, \ldots, v_n$ is

$$M = \begin{pmatrix} g_{x_0} & g_{x_i} \\ g_{x_j} & g_{x_i} g_{x_j} - \delta_{i,j} T \end{pmatrix}_{i,j=1,\ldots,n}$$

and its determinant is $T^{n-1} \sum_{i=0}^n g_{x_i}^2$. This implies that $\nabla g, v_1, \ldots, v_n$ are linearly dependent only on $\{ T = 0 \} \cup \Sigma(g)$. By assumption $\{ T = 0 \} \cup \Sigma(g) = \Sigma(g)$, and we are done. \qed

Remark that $Me_0 = \nabla g$, $M \nabla g = \| \nabla g \|^2 e_0$, and $Mv = -Tv$ when $\langle v, e_0 \rangle = \langle v, \nabla g \rangle = 0$.

**Remark 2.2.** The matrix appeared in the proof of Proposition 1.5 suggests another explicit construction of the vector field $v_1, \ldots, v_n$ in some special case. Let us find an $x$ with $\varphi(x) = \nabla g / \| \nabla g \|$ where $\nabla g$ denotes the gradient of $g$. Looking the first component, we have $\frac{2x_0^2}{S} - 1 = \frac{g_{x_0}}{\| \nabla g \|}$ and

$$\left( 1 - \frac{g_{x_0}}{\| \nabla g \|} \right) x_0^2 = (x_1^2 + \cdots + x_n^2) \left( 1 + \frac{g_{x_0}}{\| \nabla g \|} \right).$$

We then obtain

$$\left( \frac{x_0}{1 + \frac{g_{x_0}}{\| \nabla g \|}} \right)^2 = \frac{x_1^2 + \cdots + x_n^2}{\left( \frac{g_{x_1}}{\| \nabla g \|} \right)^2 + \cdots + \left( \frac{g_{x_n}}{\| \nabla g \|} \right)^2} = \frac{x_1^2 + \cdots + x_n^2}{\left( \frac{2x_1 x_0}{S} \right)^2 + \cdots + \left( \frac{2x_n x_0}{S} \right)^2} = \left( \frac{S}{2x_0} \right)^2.$$

We thus conclude

$$(x_0, x_1, \ldots, x_n) = k (\nabla g \pm \| \nabla g \| e_0) \quad \text{where} \quad k = \frac{S}{2x_0 \| \nabla g \|}.$$
Choosing the sign $+$, and setting $k = 1$, we have

$$v_i := \varphi(e_i) = \frac{1}{\|\nabla g\|} \left( g_{\lambda_0} \partial_{\lambda_0} + \sum_{j=1}^{n} \left( \frac{g_{\lambda_j} g_{\lambda_0}}{\|\nabla g\| + g_{\lambda_0}} + \frac{1}{\|\nabla g\|} \right) \partial_{\lambda_j} \right), \quad i = 1, \ldots, n.$$ 

They are the desired vector fields which make sense whenever $\nabla g + \|\nabla g\| e_0 \neq 0$.

Remark that the last condition implies the mapping degree of $dg$ is zero. But, in this construction, it is not clear that $v_1, \ldots, v_n$ are polynomial (resp. analytic) vector fields when $g$ is a polynomial (resp. analytic).

3. Restricting $f$ to the level of $g$

**Theorem 3.1.** Let $L$ be a $C^\infty$-manifold of dimension $n + 1$ and $f, g : L \to \mathbb{R}$ $C^\infty$-functions. We assume that $0$ is a regular value of $g : L \to \mathbb{R}$ and set $N = g^{-1}(0)$. We assume that $g$ satisfies Condition (P) and the map

$$F : N \to S^n, \quad x \mapsto \frac{(f(x), v_1 f(x), \ldots, v_n f(x))}{\|(f(x), v_1 f(x), \ldots, v_n f(x))\|},$$

is well-defined and finite.

(i) If $L_+ = \{x \in L : f(x) \geq 0\}$ is compact, then

$$\deg F = \chi(N(f \geq 0), N(f = 0))$$

where $N(f \geq 0)$ denotes the set $\{x \in N : f(x) \geq 0\}$, and so on.

(ii) If $L_- = \{x \in L : f(x) \leq 0\}$ is compact, then we obtain

$$\deg F = (-1)^{n+1} \chi(N(f \leq 0), N(f = 0)).$$

**Proof.** Take the point $(1, 0, \ldots, 0)$ and consider its preimage by $F$. They are the critical points of $f : N \to \mathbb{R}$ in the region $\{f > 0\}$. If $f|_N$ is Morse (we can assume this after small perturbation of $f$ if necessary), we obtain

$$\text{Hess}(f|_N)(x) = \frac{\partial F}{\partial y}(x),$$

where $y$ denotes an oriented coordinate system of $N$. This implies the first equality.

Next take the point $(-1, 0, \ldots, 0)$ and apply the similar discussion for $-f$ on the region $\{f \leq 0\}$. We then obtain the second equality.

When $F$ induces a finite map germ $F_0 : (L, F^{-1}(0)) \to (\mathbb{R}^{n+1}, 0)$, $\deg F = \deg F_0$.

**Remark 3.2.** Assume that $L$ is compact. If $n$ is odd, we have that $\deg F = \frac{1}{2} \chi(N(f = 0))$ and $\chi(N(f \geq 0)) = \chi(N(f \leq 0))$. We consider the following Gauss map
We choose $\mathbf{V}$.

Using the fact stated in [8, §6], we obtain that the degree of this Gauss map is equal to the sum of indices of $\nabla f$ in $N(f \geq 0)$, which is equal to $\deg F$. So we conclude that $\deg G = \frac{1}{2} \chi(N(f = 0))$.

4. Restricting $g$ to the level of $f$

**Theorem 4.1.** Let $f, g : B^{n+1}_e \to \mathbb{R}$ be analytic functions with $f(0) = g(0) = 0$. We assume that the singular set of $(f, g)$, which is defined by

$$X = \left\{ x \in B^{n+1}_e : \text{rank} \begin{pmatrix} f_{x_0}(x) & f_{x_1}(x) & \cdots & f_{x_n}(x) \\ g_{x_0}(x) & g_{x_1}(x) & \cdots & g_{x_n}(x) \end{pmatrix} < 2 \right\},$$

is of dimension 1. We choose $\varepsilon > 0$ small enough so that

- the number of connected components of $X \setminus \{0\}$ in $B^{n+1}_e$ does not change if $0 < \varepsilon' \leq \varepsilon$, and
- the functions $f$ and $g$ do not change the sign on each connected component of $X \setminus \{0\}$.

We choose $\delta$, a regular value of $f$, which is close enough to 0, and set $V_\delta = \{ x \in B^{n+1}_e : f(x) = \delta \}$. We assume that $g$ satisfies Condition (P). If $V_\delta \cap \Sigma(g) = \emptyset$, $g|_{V_\delta}$ is a Morse function, and the map-germ

$$F : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0), \quad x \mapsto (f(x), v_1 f(x), \ldots, v_n f(x)),$$

is finite, then we have the following:

$$\deg(F) = \text{sign}(\delta)^{n+1} \chi(V_\delta(g \leq 0)) - \chi(V_\delta(g \geq 0))$$

Here we denote by $V_\delta(g \leq 0)$ the set $\{ x \in V_\delta : g(x) \leq 0 \}$, and so on. We also denote by $\overline{v}_{\text{sign}(\delta)}$ the set $\{ x \in S^n : \text{sign}(\delta) f(x) \geq 0, \pm g(x) \geq 0 \}$ for $0 < \varepsilon \ll 1$.

**Remark 4.2.** Consider the jet space $J = J^1(\mathbb{R}^{n+1}, \mathbb{R}^2)$ with coordinates

$$(x_0, x_1, \ldots, x_n, y, z, p_0, p_1, \ldots, p_n, q_0, q_1, \ldots, q_n),$$

so that the jet section of a map $(f, g) : \mathbb{R}^{n+1} \to \mathbb{R}^2$ is defined by

$$y = f(x), \quad p_i = f_{x_i}(x), \quad z = g(x), \quad q_i = g_{x_i}(x), \quad i = 0, 1, \ldots, n.$$ 

Let $\Sigma_i$, $i = 0, 1, 2$, be the submanifolds of the jet space $J$ defined by

$$\text{rank} \begin{pmatrix} p_0 & p_1 & \cdots & p_n \\ q_0 & q_1 & \cdots & q_n \end{pmatrix} = i.$$ 

If the map $(f, g) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^2, 0)$ is transverse to $\Sigma_0$, $\Sigma_1$ and $\Sigma_2$ on $(\mathbb{R}^{n+1} - 0, 0)$, then the singular set $X$ of $(f, g)$ is of dimension 1, and the num-
ber of connected components of \((X - \{0\}) \cap B^{n+1}_{e^t}\) does not change if \(0 < e' \ll 1\). This means that the condition on \(X\) is a generic condition, if \(e > 0\) is small enough.

**Lemma 4.3.** Let \(\gamma : (R, 0) \rightarrow (X, 0)\) be a \(C^\infty\)-map with \(f \circ \gamma(t) \neq 0\) when \(0 < |t| \ll 1\). We then obtain \(\nabla f(\gamma(t))\) is not identically zero. Since \(\gamma(t) \in X\), there are real numbers \(\lambda(t)\) so that \(\nabla g(\gamma(t)) = \lambda(t) \nabla f(\gamma(t))\), when \(0 < |t| \ll 1\).

(i) If \(g \circ \gamma(t)\) is identically zero, then \(\lambda(t)\) is also identically zero.

(ii) If \(g \circ \gamma(t)\) is not identically zero, then \(\operatorname{sign} \lambda = \operatorname{sign}(g/f)\) along \(\gamma(t)\), \(0 < |t| \ll 1\).

**Proof.** Assume that \(\nabla f(\gamma(t))\) is identically zero. We then have

\[
\frac{d}{dt} f \circ \gamma(t) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(\gamma(t)) \frac{d}{dt}(x_i \circ \gamma(t)) = 0,
\]

which implies \(f \circ \gamma(t)\) is constant. This shows the first assertion. If \(g \circ \gamma(t)\) is identically zero, then

\[
0 = \frac{d}{dt}(f \circ \gamma(t) \cdot g \circ \gamma(t)) = g \circ \gamma(t) \frac{d}{dt} f \circ \gamma(t) + f \circ \gamma(t) \frac{d}{dt} g \circ \gamma(t) = \lambda(t) f \circ \gamma(t) \frac{d}{dt} f \circ \gamma(t)
\]

and we conclude \(\lambda(t)\) is identically zero. This completes the proof of (i). The assertion (ii) is a consequence of Cauchy’s mean value theorem.

Take a point \(x \in X - \{0\}\).

- If \(\delta = f(x)\) is a regular value of \(f\), then \(x\) is a critical point of \(g|_{\{f=\delta\}}\).
- If \(\delta' = g(x)\) is a regular value of \(g\), then \(x\) is a critical point of \(f|_{\{g=\delta'\}}\).

The following lemma clarifies when \(g|_{V_0}\) is a Morse function.

**Lemma 4.4.** Let \(\delta\) be a regular value of \(f\). For \(x \in X \cap V_0\) there exists a real number \(\lambda\) so that \(\nabla g(x) = \lambda \nabla f(x)\). Then \(g|_{V_0}\) is Morse at \(x\), if and only if

\[
\begin{vmatrix}
0 & f_{x_j} \\
0 & g_{x_i x_j} - \lambda f_{x_i x_j}
\end{vmatrix}
\neq 0 \quad \text{at} \quad x.
\]

**Proof.** It is enough to prove the lemma assuming \(f_{x_0}(x) \neq 0\). Then there is a function \(\varphi(x_1, \ldots, x_n)\) with

\[
f(\varphi(x_1, \ldots, x_n), x_1, \ldots, x_n) \equiv \delta.
\]

Differentiating (4.4) by \(x_i\), \(i = 1, \ldots, n\), we obtain

\[
f_{x_0} \varphi_{x_i} + f_{x_i} \equiv 0,
\]
and $\varphi_{x_i} = -f^{-1}_x f_{x_i}$. Differentiating \eqref{eq:Hessian} by $x_j$, $j = 1, \ldots, n$, we obtain that
\begin{equation}
\label{eq:Hessian1}
f_{x_0} \varphi_{x_i} \varphi_{x_j} + f_{x_0 x_i} \varphi_{x_j} + f_{x_0 x_j} \varphi_{x_i} + f_{x_0} \varphi_{x_i x_j} \equiv 0.
\end{equation}

We consider the Hessian of the function $G(x_1, \ldots, x_n) := g(\varphi(x_1, \ldots, x_n), x_1, \ldots, x_n)$ at its critical point $x$. Similar computation shows that $G_{x_i} = g_{x_0} \varphi_{x_i} + g_{x_i}$, $i = 1, \ldots, n$, and $\lambda = g_{x_0} f^{-1}_x$ at $x \in X$. We also obtain that
\begin{align*}
G_{x_i x_j} &= g_{x_0} \varphi_{x_i} \varphi_{x_j} + g_{x_0 x_i} \varphi_{x_j} + g_{x_0 x_j} \varphi_{x_i} + g_{x_0} \varphi_{x_i x_j} \\
&= (g_{x_0 x_0} - \lambda f_{x_0 x_0}) \varphi_{x_i} \varphi_{x_j} + (g_{x_0 x_i} - \lambda f_{x_0 x_i}) \varphi_{x_j} + (g_{x_0 x_j} - \lambda f_{x_0 x_j}) \varphi_{x_i} + (g_{x_i x_j} - \lambda f_{x_i x_j})
\end{align*}
at $x$ by \eqref{eq:Hessian1}. Therefore we conclude that
\begin{align*}
\det(G_{x_i x_j})_{i,j=1,\ldots,n} &= \\
&= \begin{vmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & G_{x_i x_j}
\end{vmatrix}_{i,j=1,\ldots,n} \\
&= \begin{vmatrix}
-1 & 0 & 0 & (g_{x_0 x_0} - \lambda f_{x_0 x_0}) \varphi_{x_j} \\
0 & 0 & -1 & \varphi_{x_i} \\
0 & -1 & 0 & g_{x_0} \varphi_{x_i} - \lambda f_{x_0} \varphi_{x_j} \\
\varphi_{x_i} & \varphi_{x_j} & g_{x_0 x_i} - \lambda f_{x_0 x_i} & g_{x_0 x_j} - \lambda f_{x_0 x_j}
\end{vmatrix}_{i,j=1,\ldots,n} \\
&= \begin{vmatrix}
-1 & 0 & g_{x_0 x_0} - \lambda f_{x_0 x_0} & 0 \\
0 & 0 & -1 & \varphi_{x_i} \\
1 & -1 & 0 & g_{x_0} \varphi_{x_i} - \lambda f_{x_0} \varphi_{x_j} \\
\varphi_{x_i} & \varphi_{x_j} & g_{x_0 x_i} - \lambda f_{x_0 x_i} & g_{x_0 x_j} - \lambda f_{x_0 x_j}
\end{vmatrix}_{i,j=1,\ldots,n} \\
&= \begin{vmatrix}
-1 & 0 & g_{x_0 x_0} - \lambda f_{x_0 x_0} & 0 \\
0 & 0 & -1 & \varphi_{x_i} \\
0 & -1 & 0 & g_{x_0} \varphi_{x_i} - \lambda f_{x_0} \varphi_{x_j} \\
\varphi_{x_i} & \varphi_{x_j} & g_{x_0 x_i} - \lambda f_{x_0 x_i} & g_{x_0 x_j} - \lambda f_{x_0 x_j}
\end{vmatrix}_{i,j=1,\ldots,n} \\
&= -f_{x_0}^{-2} \frac{\partial}{\partial x_j} \left. \frac{f_{x_j}}{f_{x_0}} \right|_{x_0 = g(x), j=0,1,\ldots,n},
\end{align*}
at $x$, which completes the proof. \hfill \Box

If $x$ is a regular point of $f$ and $g$, then we have

\textbf{Lemma 4.5.} \quad \text{sign Hess}(f|_{\{g=0\}}) = \text{sign}((\lambda)^n \text{ Hess}(g|_{\{f=0\}})) \quad \text{at } x \in X \text{ near } 0.

\textit{Proof.} Since $x$ is a regular point of $g$, there exists a coordinate system $(x_0, x_1, \ldots, x_n)$ centered at $x$ so that $x_0 = g(x)$. We consider $f$ as a functions
of \((x_0, x_1, \ldots, x_n)\) and write \(f = f(x_0, x_1, \ldots, x_n)\). By implicit function theorem, there exists a \(C^\infty\)-function \(\psi(x_1, \ldots, x_n)\) so that \(f(\psi(x_1, \ldots, x_n), x_1, \ldots, x_n) = \delta\). Then we obtain that
\[
\begin{align*}
  f_{x_0} \psi_{x_i} + f_{x_i} &= 0, \quad \text{for } i = 1, \ldots, n, \quad \text{and} \\
  f_{x_0} \psi_{x_i x_j} + f_{x_i x_j} &\equiv 0 \mod \psi_{x_i}, \quad \text{for } i, j = 1, \ldots, n.
\end{align*}
\]
This means \(f_{x_0 \psi_{x_i}}(x) = -f_{x_0} \psi_{x_i}(x)\), which implies the lemma.

**Proof of Theorem 4.1.** We choose a non-zero number \(\delta\) close enough to 0 so that the numbers of connected components of \(\{x \in X : 0 < \text{sign}(\delta)f(x) < \varepsilon\}\) do not depend on \(\varepsilon\) with \(0 < \varepsilon < |\delta|\). Let \(a(\varepsilon)\), \(0 < \text{sign}(\delta)\varepsilon < |\delta|\), denote the half-branch of \(X\) which contains \(x\). We assume that \(f(a(\varepsilon)) = 0\). We extend the function \(\varepsilon\) to a neighborhood of \(X\) near \(x\) and denote it by the same letter \(\varepsilon\). We consider functions \(g_1, \ldots, g_n\) so that \(X = \{g_1 = \cdots = g_n = 0\}\) near \(x\) and so that \(\nabla g_i(x) = v_i(x)\) for \(i = 1, \ldots, n\). We see that \((\varepsilon, g_1, \ldots, g_n)\) and \((g_1, \ldots, g_n)\) are systems of coordinates near \(x\). Then we obtain that
\[
\frac{\partial F}{\partial (g, g_1, \ldots, g_n)} = \frac{\partial F}{\partial (g, g_1, \ldots, g_n)} = \lambda^{-1} \begin{vmatrix} 1 & * \\ 0 & v_i v_j f \end{vmatrix}
\]
at \(x\), since \(v_i f(a(\varepsilon)) = 0\) and \(\langle \dot{\varepsilon}, \nabla f \rangle = 1\). By Lemma 4.5, we conclude that
\[
(4.7) \quad \text{sign } \frac{\partial F}{\partial (g, g_1, \ldots, g_n)} = \text{sign}((-\lambda)^{n+1} \text{Hess}(g|_{f=\delta}))(x).
\]
Applying Morse theory to \(g\) on \(\{f = \delta, g \geq 0\}\), we obtain that
\[
\chi(V_\delta(g \geq 0), V_\delta(g = 0)) = \sum_{x \in X \cap \Omega \setminus \Sigma|g(x) > 0} \text{sign Hess}(g|_{f=\delta})(x).
\]
Applying Morse theory to \(-g\) on \(\{f = \delta, g \leq 0\}\), we also obtain that
\[
\chi(V_\delta(g \leq 0), V_\delta(g = 0)) = (-1)^n \sum_{x \in X \cap \Omega \setminus \Sigma|g(x) < 0} \text{sign Hess}(g|_{f=\delta})(x).
\]
Taking the difference, we thus conclude that
\[
(4.8) \quad \chi(V_\delta(g \geq 0)) - \chi(V_\delta(g \leq 0)) = \sum_{x \in X \cap \Omega \setminus \Sigma|g(x) \neq 0} \text{sign}(g(x))^{n+1} \text{Hess}(g|_{f=\delta})(x).
\]
By Lemma 4.3 (i), the condition \(V_\delta \cap \Sigma(g) = \emptyset\) implies that 0 is a regular value of \(g|_{V_\delta}\), and we have
\[
(4.9) \quad (4.8) = \sum_{x \in X \cap \Omega} \text{sign}(g(x))^{n+1} \text{Hess}(g|_{f=\delta})(x).
\]
By Lemma 4.3 (ii), \( \text{sign}(f \lambda) = \text{sign}(g) \) along each connected component of \( X - \{0\} \), and we obtain that

\[
(4.9) = \sum_{x \in X \cap V_0} \text{sign}(f(x)\lambda)^{n+1} \text{Hess}(g|_{\{f=\delta\}})(x)
\]

\[
= \text{sign}(-\delta)^{n+1} \sum_{x \in X \cap V_0} \text{sign}(-\lambda)^{n+1} \text{Hess}(g|_{\{f=\delta\}})(x)
\]

\[
= \text{sign}(-\delta)^{n+1} \sum_{x \in X \cap V_0} \text{sign} (\frac{\partial F}{\partial (g, g_1, \ldots, g_n)})(x) \quad \text{(by (4.7))}
\]

\[
= \text{sign}(-\delta)^{n+1} \deg F,
\]

which implies the formula (4.2). The equality (4.3) follows from the deformation argument due to [9, §11].

\[\square\]

**Corollary 4.6.** Let \( V \) be an analytic set of dimension \( n + 1 \) defined near 0 in \( \mathbb{R}^{m+n+1} \). Let \( L \) be the nonsingular locus of \( V \cap B_{\varepsilon}^{m+n+1} \) for small \( \varepsilon > 0 \) and assume that \( L \) is oriented. Let \( g : (\mathbb{R}^{m+n+1}, 0) \to (\mathbb{R}, 0) \) be an analytic function-germ. We assume that there are \( C^\infty \)-vector fields \( v_1(x), \ldots, v_n(x) \) on \( B_{\varepsilon}^{m+n+1} \) so that \( v_1(x), \ldots, v_n(x) \) span the tangent space of \( g|_L \) at each \( x \in L \) and the orientation of the level of \( g|_L \) there coincides with the orientation defined by \( v_1(x), \ldots, v_n(x) \). Let \( f : (\mathbb{R}^{m+n+1}, 0) \to (\mathbb{R}, 0) \) be an analytic function-germ. We assume that

\( V_\delta = \{ x \in V \cap B_\varepsilon : f(x) = \delta \} \)

is nonsingular for a non-zero number \( \delta \) which is sufficiently close to 0. If \( V_\delta \cap \Sigma(g) = \emptyset \), the map-germ

\( F : (L, 0) \to (\mathbb{R}^{n+1}, 0), \quad x \mapsto (f(x), v_1 f(x), \ldots, v_n f(x)) \)

is finite and \( g|_{V_\delta} \) is Morse, then

\[
(4.10) \quad \deg(F) = \text{sign}(-\delta)^{n+1} (\chi(V_\delta(g \leq 0)) - \chi(V_\delta(g > 0)))
\]

\[
= \text{sign}(-\delta)^{n+1} (\chi(F_{\text{sign}(\delta)^-}) - \chi(F_{\text{sign}(\delta)^+}))
\]

where \( F_{\text{sign}(\delta)\pm} = \{ x \in V \cap \Sigma^n : \text{sign}(\delta) f(x) \geq 0, \pm g(x) \geq 0 \} \) for \( 0 < \varepsilon \ll 1 \).

**Remark 4.7.** We sketch how to find the formula (Theorem 4.3) in [2]. Let \( (x_0, x_1, \ldots, x_{m+n+q}) \) denote a coordinate system of \( \mathbb{R}^{m+n+q+1} \) at the origin. Let \( n = 1, 3, 7 \), and let \( m, q \) be non-negative integers. Let \( f, g : (\mathbb{R}^{m+n+q+1}, 0) \to (\mathbb{R}, 0) \) denote two analytic functions, and \( h = (h_1, \ldots, h_m) : (\mathbb{R}^{m+n+q+1}, 0) \to (\mathbb{R}^{m}, 0) \) a \( C^\infty \)-map. We assume that \( g \) and \( h \) do not depend on the last \( q \) variables \( x_{m+n+1}, \ldots, x_{m+n+q} \). Set \( V = h^{-1}(0) \) and \( L \) is the set of regular points of \( V \) (i.e., \( L = V - \Sigma(h) \)). Since \( L \) is orientable, we fix an orientation of \( L \). Define vector fields \( v_1, \ldots, v_{n+q} \) by...
be vector fields on (4.3). This observation is sometimes useful if we know the vectors defined in subsection 2.1 when

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By (4.10), we obtain that $\hat{V}$ where

If the map $F$ is Morse, then the same proof works and we obtain the formulas (4.2), (4.3) in [2].

Remark 4.8. Let $g : (R^{n+1}, 0) \to (R, 0)$ be a $C^\infty$-function and let $v_1, \ldots, v_n$ be vector fields on $(R^{n+1}, 0)$ so that $\langle \nabla g, v_i \rangle = 0$, $i = 1, \ldots, n$. We denote by $\Sigma_n$ the set of points where $v_1, \ldots, v_n$ are linearly dependent. Let $f : (R^{n+1}, 0) \to (R, 0)$ be a $C^\infty$-function so that $V_\delta \cap \Sigma(f) = 0$, $V_\delta \cap \Sigma(g) = 0$ and $V_\delta \cap \Sigma_n = 0$, where $V_\delta = \{ x \in (R^{n+1}, 0) : f(x) = \delta \}$. If the map $F$ defined by (4.1) is finite and $g|_V\delta$ is Morse, then the same proof works and we obtain the formulas (4.2), (4.3). This observation is sometimes useful if we know $\Sigma_n$ explicitly.

Here is an example that $\Sigma_n$ can be expressed explicitly. Set $p = 1, 3, 8$. Define

$$v_i = \begin{cases} \text{the same as in subsection 2.1 replacing } n \text{ by } p & \text{there} \quad i = 1, \ldots, p \\ g_{x_i} \nabla g - \| \nabla g \| \hat{e}_{x_i} & i = p + 1, \ldots, n \end{cases}$$

Then we obtain $\Sigma_p = \{ g_{x_0} = \cdots = g_{x_p} = 0 \}$. Suppose that $g(x) = \sum_{i=0}^n x_i^2$. Then $\Sigma_n = \{ x_0 = \cdots = x_p = 0 \}$. If $f : (R^{n+1}, 0) \to (R, 0)$ defines an isolated singularity with $f(\Sigma_n) = 0$, and the map $F$ defined by (4.1) is finite, then we obtain that

$$\deg F = (-1)^p \delta \{ x \in S^p_\delta : f(x) \geq 0 \}.$$
DEFINITION 4.9. Let $M$ be a $C^\infty$-manifold and let $\varphi : M \rightarrow \mathbb{R}$ be a $C^\infty$-function. We say that Morse theory is applicable to $\varphi$ on the closed interval $[a,b]$ if the following two conditions hold.

1. $\varphi$ has at most finitely many critical points in $\varphi^{-1}[a,b]$, and all critical points are Morse singularities, that is, the Hessian determinant $\text{Hess}(\varphi)(x)$ of $\varphi$ is non zero at each critical point $x$.
2. There is “no surgery at infinity” on $[a,b]$, which means that $\{x \in M : \varphi(x) \leq c - \varepsilon\}$ and $\{x \in M : \varphi(x) \leq c + \varepsilon\}$ are diffeomorphic each other for sufficiently small $\varepsilon > 0$ when $c$ is not a critical value of $\varphi$ with $c \in [a,b]$.

THEOREM 4.10. Let $L$ be a real analytic manifold of dimension $n+1$ and let $f, g : L \rightarrow \mathbb{R}$ be analytic functions. We assume that $V_\delta = \{x \in L : f(x) = \delta\}$ is nonsingular for a non-zero number $\delta$ with $0 < |\delta| < 1$ and Morse theory is applicable for $g|_{V_\delta}$ on $[b_0, b_k]$. We assume that $g$ satisfies Condition (P), and that the map

$$F : L \rightarrow \mathbb{R}^{n+1}, \quad x \mapsto (f(x), v_1 f(x), \ldots, v_{n+1} f(x)),$$

is finite. We set $F^{-1}(0) = \{P_1, \ldots, P_k\}$ and $c_i = g(P_i)$ for $i = 1, \ldots, k$, and assume that $b_0 < c_1 < c_2 < \cdots < c_k < b_k$. Taking $b_i$ with $c_i < b_i < c_{i+1}$ for $i = 1, \ldots, k - 1$, we have

$$\deg(F) = \text{sign}(-\delta)^{n+1} \sum_{i=1}^k (\chi(V_\delta(b_{i-1} \leq g \leq c_i)) - \chi(V_\delta(c_i \leq g \leq b_i))).$$

Moreover, if $n$ is odd, we have

$$\deg(F) = \chi(V_\delta(b_0 \leq g \leq b_k), V_\delta(g = b_0)).$$

Proof. By Theorem 4.1, we obtain that

$$\deg(F) \text{ at } P_i = \text{sign}(-\delta)^{n+1} (\chi(V_\delta(b_{i-1} \leq g \leq c_i)) - \chi(V_\delta(c_i \leq g \leq b_i))).$$

This implies (4.11). When $n$ is odd, the proof of Theorem 4.1 implies

$$\deg(F) = \sum_{x \in X \cap V_\delta} \text{Hess}(g|_{V_\delta})(x)$$

and the right hand side is equals to

$$\chi(V_\delta(b_0 \geq g \geq b_k), V_\delta(g = b_0)),$$

which completes the proof of (4.12). \qed

5. Mapping degree of $\bar{p}([dg], [df])$

We denote by $\pi : \mathbb{R}^{n+1} - \{0\} \rightarrow S^n$ the projection defined by $x \mapsto x/\|x\|$. Let $f : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}, 0)$ be a $C^\infty$-function-germs. We define a map $[df]$:
$S^n \to S^n$ by $x \mapsto \pi \circ df(x)$ where $S^n$ denotes the $n$-sphere centered at 0 with radius $\varepsilon$ and $S^n$ denotes the unit sphere centered at 0. Suggested by Remark 2.1, we are interesting in the following: Let $f, g : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ be two $C^\infty$-function-germs. We consider a smooth map $p : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and set $Z = p^{-1}(0)$. We investigate the mapping degree of the map

$$p([dg], [df]) : S^n \to S^n, \quad x \mapsto \pi \circ p([dg](x), [df](x))$$

when $Z := Z \cap (S^n \times S^n)$ is empty.

**Lemma 5.1.** Let $M$ be an oriented manifold of dimension $\geq n$ and let $\omega$ be the volume form of the sphere $S^n$ so that $\int_{S^n} \omega = 1$. We consider a $C^\infty$-map $f : M \to S^n$. Then $\deg(f|_X) = \int_X f^* \omega$ for any oriented $n$-cycle $X$ of $M$ so that $f|_X$ is proper and finite.

The proof is similar to the proof of Theorem 12 in [10, Chapter 8].

**Proof.** Let $y$ be a regular value of $f|_X$ and let $U$ be an open neighborhood of $y$. Let $\omega'$ be an $n$-form of $S^n$ which is cohomologous to $\omega$ and $\text{supp}(\omega') \subset U$. Let $\{x_1, \ldots, x_k\}$ be the preimage of $y$. Choosing $U$ small we may assume that $(f|_X)^{-1}(U) = U_1 \cup \cdots \cup U_k$ where each $U_i$ is an open neighborhood of $x_i$ in $X$ and each $U_i$ is diffeomorphic to $U$. Then we have

$$\int_{U_i} (f|_X)^* \omega' = \pm \int_U \omega' = \pm 1$$

where the sign is $+$ (resp. $-$) when $f|_{U_i}$ is orientation preserving (resp. reversing). Thus we have

$$\deg(f|_X) = \sum_{i=1}^k \int_{U_i} (f|_X)^* \omega' = \int_X (f|_X)^* \omega' = \int_X f^* \omega,$$

and this completes the proof. $\square$

Let $e_i, i = 0, 1, \ldots, n$, denote the unit vector $(0, \ldots, 1, \ldots, 0)$ in $\mathbb{R}^{n+1}$. We investigate when $Z$ is empty. When $Z = \emptyset$, we can consider the following map:

$$\bar{p} : S^n \times S^n \to S^n, \quad (x, y) \mapsto \pi \circ p(x, y).$$

We define the class of $\bar{p}$, denoted by $h(\bar{p})$, the image of the fundamental class of $S^n$ by the map

$$H^n(S^n; \mathbb{Z}) \xrightarrow{\bar{p}^*} H^n(S^n \times S^n; \mathbb{Z}) = \mathbb{Z}^2,$$

where the last equality presents the natural identification between the cohomology group $H^n(S^n \times S^n; \mathbb{Z})$ and the free $\mathbb{Z}$-module generated by the cohomology classes corresponding to $S^n \times e_0$ and $e_0 \times S^n$.

**Proposition 5.2.** There is a $C^\infty$-map $\bar{p} : S^n \times S^n \to S^n$ so that $h(\bar{p}) = (k_1, k_2)$ if and only if one of the following conditions holds.

- $n = 1, 3, 7$.
- $n$ is odd, $n \neq 1, 3, 7$, and $k_1 k_2 \equiv 0 \pmod{2}$.
- $n$ is even, and $k_1 k_2 = 0$. 

Theorem 1.1.1. Let $\pi$ be a head product.

Let $\pi^* \omega$ be cohomologous to $k_1(p_1)^* \omega + k_2(p_2)^* \omega$. The assertion comes from the following:

$$0 = (\pi^* \omega) \wedge (\pi^* \omega) = (k_1(p_1)^* \omega + k_2(p_2)^* \omega) \wedge (k_1(p_1)^* \omega + k_2(p_2)^* \omega) = 2k_1k_2(p_1)^* \omega \wedge (p_2)^* \omega.$$

Proof. Assume first that $n$ is even. Let $\omega$ denote the volume form of $S^n$. Let $p_i : S^n \times S^n \to S^n$, $i = 1, 2$, denote the $i$-th projection. We remark that $\bar{p}^* \omega$ is cohomologous to $k_1(p_1)^* \omega + k_2(p_2)^* \omega$. The assertion comes from the following:

$$0 = (\bar{p}^* \omega) \wedge (\bar{p}^* \omega) = (k_1(p_1)^* \omega + k_2(p_2)^* \omega) \wedge (k_1(p_1)^* \omega + k_2(p_2)^* \omega) = 2k_1k_2(p_1)^* \omega \wedge (p_2)^* \omega.$$

We next consider the case that $n$ is odd. Let $f_i : S^n \to S^n$ be a $C^\infty$-map of degree $k_i$. We remark that their homotopy classes is $k_i t_n$ where $t_n$ is the identity map of $S^n$. It is enough to determine all $(k_1, k_2)$ so that the Whitehead product $[k_1 t_n, k_2 t_n] = k_1 k_2 [t_n, t_n]$ vanishes. By the theorem of J. Adams [1, Theorem 1.1.1], $[t_n, t_n] = 0$ if and only if $n = 1, 3, 7$. This implies the second assertion. Since $[t_n, t_n]$ is of order 2 when $n \neq 1, 3, 7$, we obtain the last assertion.

When $n = 1, 3, 7$, and a map $\bar{p}$ with $(k_1, k_2) = (1, 1)$, is induced by the product of complex, quaternion, Cayley numbers respectively.

When $n = 1$, we identify $\mathbb{R}^2$ with $\mathbb{C}$ by $(x, y) \mapsto z = x + yi$. The map $p_{k_1, k_2} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(z_1, z_2) \mapsto z_1^{k_1} z_2^{k_2}$ represents a map which class is $(k_1, k_2)$. Remarking $z^{-1} = \bar{z}$ on $S^1$, we see all the classes $(k_1, k_2)$ are represented by polynomial maps.

When $n$ is odd, a map $S^n \times S^n \to S^n$ with $(k_1, k_2) = (1 - k, k)$ is represented by the following way: Take $x, y \in S^n$, and consider the great circle containing $x, y$. The image of $(x, y)$ is $z$ in the great circle defined by $\angle x0z = k \angle x0y$ described in the following picture in the case $k = -2$.

An explicit formula for this map is described by the following: For $x, y \in S^n$, we set $z = p_k(u, v) x + q_k(u, v)(y - ux)$ where $u = \langle x, y \rangle$, $v = |y - ux|$. Here $p_k(u, v)$ and $q_k(u, v)$ denote real polynomials defined by $(u + iv)^k = p_k(u, v) + q_k(u, v)iv$.

Proposition 5.3. Let $\bar{p} : S^n \times S^n \to S^n$ be a $C^\infty$-map with $\bar{h}(\bar{p}) = (k_1, k_2)$, and let $f_i : S^n \to S^n$, $i = 1, 2$, be two $C^\infty$-maps. We define a map by
Then we have \( \deg(f) = k_1 \deg(f_1) + k_2 \deg(f_2) \).

Proof. Let \( \omega \) denote the volume form of \( S^n \) with \( \int_{S^n} \omega = 1 \). Let \( p_i : S^n \times S^n \to S^n, i = 1, 2 \), denote the \( i \)-th projection. We remark that \( p^* \omega \) is cohomologous to \( k_1(p_1)^* \omega + k_2(p_2)^* \omega \). Then we have \( \deg(f) = \int_{S^n} f^* \omega = \int_{S^n}(k_1(p_1)^* \omega + k_2(p_2)^* \omega) = k_1 \deg(f_1) + k_2 \deg(f_2) \). □

References