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ANALOGUES OF THE ALEKSANDROV-FENCHEL INEQUALITIES FOR HYPERBOLIC FORMS

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A number of relations that connect mixed volumes of various systems of convex bodies are known. They are all formal consequences of the Aleksandrov-Fenchel inequalities [1], [2]. A. D. Aleksandrov discovered that these relations connect mixed determinants of symmetric matrices [1d)]. In this note we prove that these relations are also valid for mixed values of any hyperbolic forms. In addition, we derive a number of inequalities that have no analogue in the theory of mixed volumes.

As this note was being prepared for the press I found out that Gårding's article [3] contains the most interesting of the inequalities given below [inequality 3 of Theorem 2 for $k = m$]. Since [3] does not contain parallels with the theory of mixed volumes or the remaining inequalities, I decided nevertheless to publish this note.

1. The mixed value of a form and differentiation. Let P be a form (a homogeneous polynomial in the coordinates of a vector) of degree m on a real linear space. A *polarization* of P is a symmetric multilinear form (depending on m vectors) that coincides with P on the diagonal. The value of the polarization on the vectors $x_1, \ldots, x_m$ is denoted by $P(x_1, \ldots, x_m)$ and called the *mixed value* of P on these vectors (in our notation $P(x, \ldots, x) = P(x)$).

The derivative of P along the constant vector field that is equal to x_1 everywhere is denoted by P'_{x_1} (the mixed value of this form of degree $m - 1$ on the vectors $x_2, \ldots, x_m$ is denoted by $P'_{x_1}(x_2, \ldots, x_m)$). It is easy to verify that the form $P^{(m)}_{x_1, \ldots, x_m}/m!$ of degree 0 is identically equal to the number $P(x_1, \ldots, x_m)$. This relation implies the series of equalities

$$P(x_1, \ldots, x_m) = \frac{P'_{x_1}(x_2, \ldots, x_m)}{m} = \frac{P^{(r)}_{x_1, x_2}(x_3, \ldots, x_m)}{m(m-1)} = \cdots = \frac{P^{(m)}_{x_1, \ldots, x_m}}{m!}.$$

Hyperbolic forms. A form P of degree m on a real space L is said to be *hyperbolic in the direction* a (or *a-hyperbolic*) if for any $x \in L$ the polynomial $\varphi(\lambda) = P(x + \lambda a)$ in one variable λ has exactly m roots (taking account of multiplicity). A vector b is said to be *positive* (*nonnegative*) if for $\lambda \geqslant 0$ ($\lambda > 0$) the polynomial $P(b + \lambda x)$ does not vanish. Positive vectors form a cone C_a. The cone C_a coincides with the component of connectedness of the point a of the complement L to the hypersurface Γ defined by $P = 0$.

We denote by $O_k(P)$ the subset of points of Γ at which all the partial derivatives of P of order less than k vanish.

ASSERTION 1 [3]. 1) *A form P that is hyperbolic in the direction a is b-hyperbolic for all $b \in C_a$.*

2) *The cone C_a is convex.*

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3) *At any point* $x \in \Gamma$ *and for any* $b \in C_a$ *the line* $x + \lambda b$ *does not lie in the tangent cone to* Γ *at* x.

4) $P(x+y) = P(y)$ *for any* $y \in L$ *if and only if* $x \in O_m(P)$, *where* m *is the degree of* P. *In particular, the set* $O_m(P)$ *is a linear space, called the* **kernel** *of* P.

It is sufficient to verify Assertion 1 for forms in three-dimensional space L. The proof of Gårding uses a complex domain. A much simpler proof can be obtained by considering Γ as a real projective curve. It is then easy to derive Assertion 1 from Bézout's theorem. Theorem 1 given below is a direct consequence of Assertion 1 and Rolle's lemma [with the exception of 2) it is contained in [3]].

THEOREM 1. *For any positive vector* $b \in C_a$ *the derivative* P'_b *of an a-hyperbolic form* P *is a b-hyperbolic form. In addition*:

1) *The cone* C_a *of* P *is contained in the cone* C_b *of* P'_b.

2) *For* $k > 1$ *the set* $O_k P$ *coincides with* $O_{k-1}P'_b$; *the set* $O_1 P$ *does not intersect the hypersurface* $P'_b = 0$.

3) *If the degree of* P *is greater than 2, then the kernels of* P *and* P'_b *coincide.*

4) *The signs of* P *and* P'_b *in* C_a *and* C_b *coincide.*

Inequalities.

THEOREM 2. *Let* P *be an a-hyperbolic form on* L *of degree* m *with kernel* K, *positive in* C_a, *let* $x_1, \dots, x_m, p \in C_a$ *be positive vectors,* $q \in \overline{C}_a$ *a nonnegative vector, and* $r \in L$ *any vector. Then*:

1) $P(q, x_2, \dots, x_m) \geq 0$; *equality is attained if and only if* $q \in K$.

2) *The "Aleksandrov-Fenchel" inequality is valid*:

$$P^2(p, r, x_3, \dots, x_m) \geq P(p, p, x_3, \dots, x_m) \cdot P(r, r, x_3, \dots, x_m);$$

equality is equivalent to the collinearity of p *and* r *modulo* K.

3) *For any* k *with* $1 \leq k \leq m$

$$P^k(x_1, \dots, x_k, x_{k+1}, \dots, x_m) \geq \prod_{1 \leq i \leq k} P(x_i, \dots, x_i, x_{k+1}, \dots, x_m);$$

equality is equivalent to the collinearity of all the vectors $x_1, \dots, x_k$ *modulo* K.

4) *The "Brunn-Minkowski theorem" is true: the function* $p^{1/m}$ *is convex in the cone* C_a. *At a point* $x \in C_a$ *it is strictly convex in all directions not collinear with* x *modulo* K.

REMARK. The formal differences of convex bodies in R^m form a linear space with respect to Minkowski addition. The volume extends to the space of differences of bodies as a homogeneous form of degree m. The mixed value of this form on m bodies is called their *mixed* volume. The inequailties of Theorem 2 are satisfied for mixed volumes of convex bodies (see [1b)]) [for nonsmooth bodies it is not known when the inequalities 2) and 3) become equalities]. We observe that *volume form is not hyperbolic* (for example, the cubic polynomial that is equal to the volume of the body $B_1 + \lambda B_2$ in R^3 for $\lambda > 0$, where B_1 is a unit disk and B_2 is a unit ball, has only one real root).

PROOF OF THEOREM 2. 1) By 4) of Theorem 1 the linear function $Q = P^{(m-1)}_{x_2,\dots,x_m}$ is positive on the cone C_a, and so $Q \geq 0$ on $\overline{C}_a$. By 2) of Theorem 1 the equality $0 = P^{(m-1)}_{x_2,\dots,x_m}(y)$ implies the chain of equalities $0 = P^{(m-2)}_{x_3,\dots,x_m}(y) = \cdots = P(y)$, which are satisfied only for points y of the kernel $O_m(P)$ of P.

2) The quadratic form $Q = P^{(m-2)}_{x_3,\dots,x_m}$ is a-hyperbolic [by 2) of Theorem 1], and is therefore p-hyperbolic. Hence the discriminant of the quadrant polynomial $\varphi(\lambda) = Q(r + \lambda p)$ is nonnegtive, which is what 2) says. Equality is attained only if the line $r + \lambda p$ intersects the set $O_2(Q)$, which by 3) of Theorem 1 coincides with the kernel of Q.

The inequality 2) is analogous to the Aleksandrov-Fenchel inequality in the theory of mixed volumes. Parts 3) and 4) of Theorem 2 are formal consequences of this inequality (see [1b)]).

EXAMPLES. 1) The determinant on the space of symmetric matrices of order m is a homogeneous form of degree m. *This form is hyperbolic in the direction of the matrix* E (a selfadjoint operator has a real spectrum). The cone C_E for this form consists of positive-definite matrices, the set O_k consists of matrices of rank $\leqslant m - k$, and the kernel consists of one point 0. Theorem 2 for this form was published by Aleksandrov [1d)] (see also [4]). It is used in the proof of the uniqueness of the convex body with given curvature function ([1d)], [5]), and in the derivation of inequalities in the theory of mixed volumes [1d)].

2) The product of the coordinate functions in R^m is a hyperbolic form of degree m. The mixed value of this form on a set of m vectors coincides with the permanent of the matrix whose columns are the vectors of the set. The discovery of an "Aleksandrov-Fenchel inequality" for this form has been reduced to the solution of an old problem of van der Waerden ([6], [7]). The convexity of the function $(x_1 \cdot \cdots \cdot x_m)^{1/m}$ is a key step in the proof of the Brunn-Minkowski theorem in the theory of convex bodies [8].

More general inequalities. The Hurwitz matrix of the polynomials $\varphi = \varphi_0 t^k + \cdots + \varphi_k$ and $\psi = \psi_0 t^{k-1} + \cdots + \psi_{k-1}$ is the matrix A_{ij} of order $2k$ in which $A_{2p,j} = \varphi_{j-p}$ for $0 \leqslant j - p \leqslant k$, $A_{2p-1,j} = \psi_{j-p-1}$ for $0 \leqslant j - p - 1 \leqslant k - 1$, and all the other elements are zero. The principal minors of this matrix of even order are called the *Hurwitz determinants* of ϕ and ψ. The next assertion is well known.

ASSERTION [9]. *A polynomial φ of degree k with positive leading coefficient has k distinct real roots if and only if all the Hurwitz determinants of φ and its derivative φ' are positive. The Hurwitz determinant of degree $2k$ vanishes for a polynomial φ with multiple roots (it differs from the discriminant only by a factor φ_0^2).*

THEOREM 3. *In the notation of Theorem 2 for any k with $2 \leqslant k \leqslant m$, all the Hurwitz determinants of the polynomial φ of degree k defined by the formula*

$$\varphi(\lambda) = P\Big(\underbrace{r + \lambda p, \dots, r + \lambda p}_{k \text{ times}}, x_{k+1}, \dots, x_m\Big)$$

are nonnegative. They are strictly positive if the straight line $r + \lambda p$ does not intersect the set $O_{m-k+2}(P)$.

Theorem 3 gives a series of inequalities connecting the mixed values of a hyperbolic form, since

$$\varphi(\lambda) = \sum C_k^l p\Big(\underbrace{p, \dots, p}_{l \text{ times}}, \underbrace{r, \dots, r}_{k - l \text{ times}}, x_{k+1}, \dots, x_m\Big)\lambda^l.$$

For $k = 2$ the inequality of Theorem 3 is equivalent to the "Aleksandrov-Fenchel inequality" of Theorem 2. For $k = m$ the inequalities of Theorem 3 are equivalent to the p-hyperbolicity of P if it is also known that the set $O_2(p)$ has codimension no less than 2 (for nonsingular forms the set $O_2(p)$ consists of the point 0). Generally speaking, the inequalities of Theorem 3 are not satisfied for mixed volumes of convex bodies.

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