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# Torsors on Smooth Algebras over Valuation Rings

*Torseurs sur les algèbres lisses sur les anneaux de valuation*

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‘এই তো জীবন, এইভাবেই তো জীবনকে ভোগ করতে চেয়েছিল সে। মানুষের আয়ু মানুষের জীবনের ভুল মাপকাঠি। দশ বৎসরের জীবন উপভোগ করেছে সে এই দেড় বছরে। আজ সে শুধু একজন ভবঘুরে পথিক নয়, একটা জীবন্ত আগ্নেয়গিরির সহ-আবিষ্কারক।’

চাঁদের পাহাড় - বিভূতিভূষণ বন্দ্যোপাধ্যায়

# Notation and conventions

Let $I$ be an ideal in a ring $A$.

- We shall denote the $I$-adic completion $\lim_n A/I^n$ of $A$ by $\widehat{A}^I$ or abusively by $\widehat{A}$, when $I$ is clear from the context. We say that $A$ is *$I$-adically complete* if it is $I$-adically complete and separated, i.e., if the canonical map is an isomorphism $A \xrightarrow{\sim} \lim_n A/I^n$.
- The vanishing locus of $I$ is denoted by $V(I) \subseteq \operatorname{Spec} A$.
- The maximal spectrum of $A$ is denoted by $\operatorname{MaxSpec}(A) \subseteq \operatorname{Spec} A$.
- If $I$ is prime, then the localisation of $A$ at $I$ shall be denoted by $A_I$.
- If $A$ is an integral domain, then the fraction field of $A$ is denoted by $\operatorname{Frac} A$.
- For an $A$-group scheme $G$, the unipotent radical of $G$ is denoted by $\mathscr{R}_u(G)$.
- An $A$-algebra is called *essentially smooth* if it can be obtained as the semilocalisation at finitely many primes of a smooth $A$-algebra.

Given two points $x$ and $y$ in a scheme $X$, the fact that $y$ is a specialisation of $x$ is denoted by $x \rightsquigarrow y$.

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# Chapitre 1

# Introduction en français

Le but de cette thèse est d'étudier les torseurs sous les schémas en groupes réductifs. Un des problèmes importants de ce sujet est la conjecture de Grothendieck–Serre énoncée ci-dessous. Elle est apparue dans les travaux du séminaire Chevalley de Serre dans [Ser58, page 31, Remarque] et de Grothendieck dans [Gro58, pages 26-27, Remarques 3]. Avant d'entrer dans le vif du sujet de cette thèse (voir section 1.1), nous entamons une discussion portant sur le contexte de la conjecture de Grothendieck–Serre. Nous y discutons des cas déjà connus et certaines de leurs conséquences, mais en premier lieu, nous y présentons l'énoncé de la conjecture.

**Conjecture 1.0.1** (Grothendieck–Serre)**.** *Pour un anneau local régulier $R$ avec le corps des fractions $K$ et un schéma en groupes réductifs $G$, un $G$-torseur $E$ qui est génériquement trivial sur* $\operatorname{Spec} R$ *est trivial, c'est-à-dire,*

$$\ker(\theta := H^1(R, G) \to H^1(K, G)) = \{*\}.$$

*De manière équivalente, étant donné un schéma régulier Noethérien $X$ et un $X$-schéma en groupes réductifs $G$, un $G$-torseur $E$ qui est génériquement trivial sur $X$ est Zariski localement trivial.*

Nous commençons notre discussion sur cette conjecture en soulignant une subtilité concernant son énoncé. Les schémas en groupes qui nous intéressent sont réductifs, en particulier, ils ne sont pas commutatifs, en général. Cela signifie que les ensembles de cohomologie $H^1(R, G)$ ont une structure d'ensembles pointés. Dans la catégorie des ensembles pointés, un morphisme ayant un noyau trivial n'est pas nécessairement injectif. Cette défaillance est due au manque d'une structure additive présente dans les groupes abéliens. Cependant, la remarque suivante met en évidence le fait que si la conjecture 1.0.1 est vraie, alors le morphisme $\theta$ est effectivement injectif.

*Remarque* 1.0.2. A fortiori, le morphisme canonique $\theta$ est injectif. Pour le démontrer, on prend $\eta, \xi \in H^1(R, G)$ tels qu'il existe un isomorphisme $\eta|_K \cong \xi|_K$. Il suffit de montrer que le faisceau $\tilde{\eta} := \underline{\mathrm{Isom}}_G(\eta, \xi)$ a une section sur $\operatorname{Spec} R$. En fait, le faisceau $\tilde{\eta}$ muni par composition d'une action à gauche de $\tilde{G} := \underline{\mathrm{Aut}}_G(\eta)$, est un torseur sous $\tilde{G}$. Grâce à la bijection de changement d'origine

$$H^1(R, G) \xrightarrow{\sim} H^1(R, \tilde{G})$$

$$\psi \longmapsto \underline{\mathrm{Isom}}_G(\eta, \psi),$$

il suffit d'appliquer la conjecture 1.0.1 au $\tilde{G}$. En effet, le groupe $\tilde{G}$ est une forme interne de $G$, et donc, il est réductive.

Pour avoir une idée de la conjecture 1.0.1, on peut tester son hypothèse en remplaçant $G$ par certains groupes réductifs dont les torseurs ont été bien étudiés. Les groupes réductifs dont les torseurs sont les mieux connus dans la littérature sont les groupes $\mathrm{GL}_n$ et $\mathrm{SL}_n$. En effet, les torseurs sous $\mathrm{GL}_n$ (resp., $\mathrm{SL}_n$) correspondent aux fibrés vectoriels de rang $n$ (resp., aux fibrés vectoriels de rang $n$ dont le déterminant est trivial). Nous vérifions l'hypothèse de la conjecture 1.0.1 ci-dessous.

*Remarque* 1.0.3. Lorsque $G = \mathrm{GL}_n$ ou lorsque $G = \mathrm{SL}_n$, le théorème de Hilbert 90 donne, sur les anneaux locaux, la propriété $H^1(R, G) = H^1(K, G) = \{*\}$. En effet, les $\mathrm{GL}_n$-torseurs (resp., $\mathrm{SL}_n$-torseurs) sur $\operatorname{Spec} R$ correspondent à des faisceaux localement libres de rang $n$ (resp., faisceaux localement libres de rang $n$ dont le déterminant est trivial) sur $\operatorname{Spec} R$. Puisque $R$ est local, les faisceaux localement libres sont libres, ainsi l'affirmation suit.

Le groupe $\mathrm{PGL}_n$ est un autre joli exemple d'un groupe réductif dont les torseurs ont été bien étudiés dans la littérature; Grothendieck en a fait l'étude dans [Gro68a, §I.1] (voir aussi [CS21, §3.1]), son intérêt pour ces torseurs venant de leur connexion avec le groupe de Brauer. Puisque $\mathrm{PGL}_n$ est le groupe d'automorphismes de l'algèbre matricielle $\mathrm{Mat}_n$, ses torseurs correspondent aux formes de $\mathrm{Mat}_n$, c'est-à-dire aux algèbres d'Azumaya de rang $n$. En prenant la suite exacte longue de cohomologie associée à la suite exacte courte $1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$ et en utilisant le fait que $H^1(R, \mathrm{GL}_n) = \{*\}$, Grothendieck a montré que

$$H^1(R, \mathrm{PGL}_n) \hookrightarrow H^2(R, \mathbb{G}_m). \tag{1.0.3.1}$$

Le cas où $G = \mathrm{PGL}_n$, qui était l'une des principales motivations de la conjecture 1.0.1 (voir [Gro68b, Remarques 1.11 a]), est étudié ci-dessous.

*Remarque* 1.0.4. Pour $G = \mathrm{PGL}_n$, la conjecture prédit qu'une algèbre d'Azumaya sur $R$ qui est isomorphe à une algèbre matricielle sur $K$ l'est déjà sur $R$. Ceci est confirmé par les travaux de Grothendieck [Gro68b, Corollaire 1.8] sur le groupe de Brauer (voir [Čes22$_{\text{Surv}}$, Example 3.1.3]). Plus précisément, nous avons le diagramme suivant :

$$\begin{tikzcd} H^1(R,G) \arrow[r] \arrow[d, hook] & H^1(K,G) \arrow[d, hook] \\ H^2(R,\mathbb{G}_m) \arrow[r] & H^2(K,\mathbb{G}_m), \end{tikzcd}$$

pour lequel nous devons montrer que le morphisme horizontal du haut est injectif. L'injectivité des flèches verticales, qui découle du diagramme (1.0.3.1), nous permet de n'avoir à montrer que l'injectivité de la flèche horizontale du bas. Ceci est une conséquence du travail effectué par Grothendieck, dont la méthode consistait à étudier la suite exacte longue de cohomologie associée à la suite exacte courte

$$1 \to \mathbb{G}_m \xrightarrow{\star} \eta_* \mathbb{G}_m \xrightarrow{\text{div}} \bigoplus_{\iota\colon \nu \to \operatorname{Spec} R} \iota_* \underline{\mathbb{Z}} \to 0, \tag{1.0.4.1}$$

où $\eta$ désigne l'inclusion du point générique de $\operatorname{Spec} R$, les inclusions $\iota$ sont indexées par les points $\nu$ de hauteur 1 de $\operatorname{Spec} R$ et div est défini en associant à chaque $f \in K^\times = \eta_* \mathbb{G}_m(\operatorname{Spec} R)$ son diviseur de Weil. Nous notons que la régularité de $R$ est utilisée dans la définition de div afin de traduire les diviseurs de Cartier en diviseurs de Weil. D'autre part, si l'anneau intègre $R$ n'est pas régulier, on peut toujours considérer $\mathscr{F} := \operatorname{coker} \star$ dans (1.0.4.1), bien que nous perdions l'agréable description de $\mathscr{F}$ en termes des diviseurs de Weil. En général, il existe une suite exacte

$$0 \to H^1(R, \mathscr{F}) \to H^2(R, \mathbb{G}_m) \to H^2(K, \mathbb{G}_m).$$

Dans le cas où les singularités de $R$ ne sont pas trop affreuses, par exemple, lorsque $\dim(R) \leq 1$, le problème d'annulation de $H^1(R, \mathscr{F})$ reste abordable (voir [CS21, Section 3.6.2]). Dans le remarque remark 2.0.11, nous discutons cette annulation dans le cas où $R$ est un anneau local d'une courbe sur un corps algébriquement fermé ou sur un corps fini, et dans le théorème theorem 4.7, nous la prouvons dans le cas où $R$ est un anneau local d'une algèbre lisse sur un anneau de valuation.

Il existe une riche histoire autour des preuves de différents cas de la conjecture 1.0.1, et ce, depuis les premiers travaux de Nisnevich dans les années 70. Parmi la vaste littérature, j'ai tenté d'énumérer chronologiquement les principaux acteurs du développement des techniques utilisées dans la preuve du théorème principal de cette thèse (Theorem A). Mon objectif ici n'est pas d'offrir au lecteur une liste exhaustive (voir [Čes22$_{\text{Surv}}$, §3.1] pour une liste complète des contributeurs), mais plutôt de signaler les articles qui joueront un rôle dans cette thèse.

## Cas déjà connus de la conjecture 1.0.1

(1) Le cas où $R$ est un anneau de valuation discrète a été démontré par Nisnevich dans sa thèse [Nis82]. L'idée est d'utiliser des résultats d'approximation de type de Harder pour réduire la preuve de la conjecture 1.0.1 sur $R$ à son homologue sur la complétion $\hat{R}$. La théorie de Bruhat–Tits est ensuite employée pour résoudre la conjecture lorsque $R$ est un anneau de valuation discrète complet. Inspiré par ces travaux, Guo a démontré dans sa thèse [Guo22] un cas de la conjecture 1.0.1 proche de celui de Nisnevich. En effet, il a étudié la conjecture lorsque $R$ est un anneau de valuation non nécessairement discrète. Nous parlerons de la relation entre les anneaux de valuation et les anneaux locaux réguliers dans Section 2.1. Dans le chapitre 3, nous utilisons les techniques de Guo pour étendre son résultat aux domaines de Prüfer semi-locaux.

(2) Le cas où $G$ est un tore a été démontré par Colliot-Thélène et Sansuc dans [CS87]. Introduisant le concept des tores flasques, c'est-à-dire des tores dont les données galoisiennes sont 'simples', ils ont montré que tout tore possède une résolution par des tores flasques. En prenant une résolution flasque, on peut, dans le cas d'un tore, réduire l'énoncé de la conjecture 1.0.1 à un énoncé sur l'annulation de la cohomologie locale en degré 2 avec coefficients dans un tore flasque. Nous utiliserons leurs méthodes dans le chapitre 4 pour montrer la pureté pour les torseurs sous les tores sur les algèbres lisses sur les anneaux de valuation.

(3) Le cas équi-caractéristique, c'est-à-dire lorsque $R$ contient un corps, a été résolu par Fedorov et Panin dans [FP15]. En combinant les idées sous-jacentes aux résultats d'Artin sur les bons voisinages provenant de [SGA $4_{\text{III}}$, Exposé XI] avec les "standard triples" de Voevodsky issus de [MVW06, Theorem 11.5], ils définissent la notion de "nice triples" (qui sont des courbes relatives lisses sur $R$ équipées d'une section et d'un sous-schéma fermé $R$-fini). Cela leur permet de passer par un recollement de type Nisnevich pour étudier les torseurs sur la droite affine relative $\mathbb{A}^1_R$. Ils concluent par des résultats de type Horrocks pour montrer que des tirés-en-arrière convenables d'un tel torseur correspond au torseur trivial.

(4) En caractéristique mixte, le cas où $R$ est non-ramifié[1] et $G$ a un $R$-sous-groupe de Borel a été résolu par Česnavičius dans [Čes22]. En optimisant la stratégie de Panin–Fedorov, il a remplacé les bons voisinages d'Artin par un lemme de présentation sur les anneaux de valuation discrète dans le style de Gabber. Nous suivrons la stratégie de Česnavičius afin de démontrer Theorem A. Entre autre,

[1] Un anneau local $R$ avec un idéal maximal $\mathfrak{m} \subset R$ est *non-ramifié* s'il contient un corps ou si $\mathrm{char}(R/\mathfrak{m}) \notin \mathfrak{m}^2$

nous utilisons sa méthode pour généraliser son lemme à un lemme de présentation sur les anneaux de valuation de rang 1 (voir Presentation Lemma 6.5).

A la suite de cette présentation des cas connus de la conjecture 1.0.1, nous donnons certaines applications que l'on peut trouver dans la littérature (voir [Česn22$_{\mathrm{Surv}}$] pour plus de détails). Puisque la conjecture est connue dans le cas équi-caractéristique (voir (3) ci-dessus), les applications mentionnées ci-dessous ne nécessitent pas de conditions lorsque l'on se place en équi-caractéristique. Grâce à la résolution du cas non-ramifié quasi-déployé de la conjecture 1.0.1 dans [Česn22], en caractéristique mixte, certaines applications continueront à être valables, sans nécessité de conditions supplémentaires, pour les groupes réductifs quasi-déployés et lorsque la base est une algèbre lisse sur un anneau de valuation discrète. Nous discuterons de quelques-unes de ces applications pour les groupes quasi-déployés dans le chapitre 8, où la base sera remplacée par une algèbre lisse sur un anneau de valuation de rang 1 de caractéristique mixte.

## Applications de la conjecture 1.0.1

Soit $G$ un schéma en groupes sur un schéma $X$, soit $H^1_{\mathrm{Nis}}(X,G)$ (resp., $H^1_{\mathrm{Zar}}(X,G)$) l'ensemble des torseurs qui se trivialisent Nisnevich localement (resp., Zariski localement). Nous commençons par énoncer la conséquence suivante de la Conjecture de Grothendieck–Serre.

**Corollaire 1.0.5.** *Si la conjecture 1.0.1 est vraie, étant donné un schéma régulier $X$ et un $X$-schéma en groupes réductifs $G$,*

$$H^1_{\mathrm{Nis}}(X,G) = H^1_{\mathrm{Zar}}(X,G).$$

En effet, un recouvrement ouvert pour la topologie de Nisnevich possède, par définition, une section générique. Par conséquent, un $G$-torseur Nisnevich localement trivial sur $X$ est génériquement trivial. En appliquant la la conjecture 1.0.1, on obtient qu'un tel torseur est Zariski localement trivial.

Comme deuxième application, nous présentons une application de la conjecture de Grothendieck–Serre dans l'étude des modèles entiers de groupes réductifs.

**Corollaire 1.0.6** (see [Česn22$_{\mathrm{Surv}}$, Proposition 3.1.5] et [Pan19]). *Si la conjecture 1.0.1 est vraie, alors deux $R$-schémas en groupes réductifs $G_1$ et $G_2$ sur un anneau local régulier $R$ sont isomorphes s'ils le sont lorsqu'on les restreint au corps des fractions $K$. En d'autres termes, un $K$-schéma en groupes réductifs a au plus un modèle réductif sur $R$.*

*Démonstration.* Puisque $G_1$ et $G_2$ ont les fibres génériques isomorphes et que la donnée radicielle est localement constante sur la base, les groupes $G_1$ et $G_2$ ont la même donnée radicielle sur le schéma connexe $\mathrm{Spec}\, R$. Par conséquent, le théorème d'unicité des groupes réductifs épinglés (voir [SGA 3$_{\mathrm{III}}$, Exposé XXIII, Théorème 4.1 et Corollaire 5.1]) donne que le groupe $G_2$ est une forme de $G := G_1$, c'est-à-dire il correspond à un $\mathrm{Aut}(G)$-torseur. Il nous faut montrer que le noyau de $H^1(R, \mathrm{Aut}(G)) \to H^1(K, \mathrm{Aut}(G))$ est trivial. Ceci serait vrai, grâce à la conjecture 1.0.1, si $\mathrm{Aut}(G)$ était réductif. Bien que $\mathrm{Aut}(G)$ ne soit pas réductif en général, il existe une suite exacte de groupe d'automorphismes

$$1 \to G/Z \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1,$$

où $Z \subset G$ est le centre et $\mathrm{Out}(G)$ désigne le groupe d'automorphismes extérieurs. En inspectant la suite exacte longue de cohomologie associée à la suite exacte courte présentée ci-dessus, il suffit d'appliquer la conjecture 1.0.1 pour le groupe réductif $G/Z$ et de montrer qu'il existe des isomorphismes $H^q(R, \mathrm{Out}(G)) \cong H^q(K, \mathrm{Out}(G))$ pour $q = 0, 1$. Ceci peut être fait en utilisant le fait que $\mathrm{Out}(G)$ est un $R$-schéma en groupes localement constant, comme dans la preuve de [Pan19] ou de [Guo20, Proposition 6.1].

Voici une autre application, elle met en avant le fait que certaines propriétés des groupes réductifs peuvent être vérifiées au niveau de la fibre générique.

**Corollaire 1.0.7.** *Étant donné un groupe réductif $G$ sur un anneau local régulier $R$ avec le corps des fractions $K$, la conjecture 1.0.1 implique que $G$ est déployé (resp., quasi-déployé) si $G_K$ est déployé (resp., quasi-déployé).*

*Démonstration.* Tout d'abord, supposons que $G_K$ est un groupe réductif déployé. Nous allons montrer que, dans ce cas, $G$ est également déployé. Dans la même veine que le début de la preuve du corollaire corollaire 1.0.6, la donnée radicielle du groupe $G$ est constante sur le schéma connexe $\operatorname{Spec} R$. Soit $\mathscr{G}$ un $R$-schéma en groupes réductifs déployé de ce type. Une fois encore, l'argument du début de la preuve du corollaire corollaire 1.0.6 nous donne que $G$ est une forme de $\mathscr{G}$. En particulier, $G$ correspond à une classe dans $H^1(R, \mathrm{Aut}(\mathscr{G}))$. Par le théorème d'unicité des groupes réductifs épinglés, le choix peut être fait de telle sorte que $G_K$ corresponde à la classe triviale dans $H^1(K, \mathrm{Aut}(\mathscr{G}))$. Il suffit de montrer que $G$ correspond à une classe triviale dans $H^1(R, \mathrm{Aut}(\mathscr{G}))$. Ceci se déduit de la suite de la preuve du corollaire corollaire 1.0.6, où l'on a montré que le noyau de $H^1(R, \mathrm{Aut}(\mathscr{G})) \to H^1(K, \mathrm{Aut}(\mathscr{G}))$ est trivial.

L'argument pour le cas quasi-déployé est similaire à l'argument ci-dessus. Pour plus de détails, nous renvoyons le lecteur à [Čes22, Theorem 9.5].

La conjecture 1.0.1 peut être utilisé pour démontrer l'énoncé suivant (cf. [Čes22, Corollary 1.3]), qui est difficile à démontrer directement, par un 'patching result' pour les torseurs.

**Corollaire 1.0.8.** *Étant donné un anneau local régulier $R$, un élément $r \in R$ et un $R$-schéma en groupes réductifs $G$, la conjecture 1.0.1 implique que*

$$G(\widehat{R}[\tfrac{1}{r}]) = G(\widehat{R}) \cdot G(R[\tfrac{1}{r}]).$$

*Démonstration.* Par contradiction, supposons qu'il existe un $g \in G(\widehat{R}[\frac{1}{r}]) \setminus (G(\widehat{R}) \cdot G(R[\frac{1}{r}]))$ nontrivial. Soit $\mathscr{E}$ le torseur sur $\operatorname{Spec} R$ obtenu en recollant formellement le torseur trivial sur $\operatorname{Spec} R[\frac{1}{r}]$ et le torseur trivial sur $\operatorname{Spec} \widehat{R}$ en utilisant la donnée de recollement fournie par $g$. Le torseur $\mathscr{E}$ est nontrivial, étant donné le fait que $g \neq 1$. Cependant, $\mathscr{E}$ est génériquement trivial par construction, ce qui est une contradiction avec la 1.0.1, et qui achève notre démonstration.

Étant donné un groupe réductif $G$ sur un anneau $R$, le foncteur de la *grassmannienne affine* $Gr_G$ associe à une $R$-algèbre $R'$ l'ensemble des $G$-torseurs $E$ sur $R'[\![t]\!]$ avec une trivialisation $\alpha$ sur $R'(\!(t)\!)$. Soit $LG(R')$ (resp., $L^+G(R')$) le foncteur qui associe à une $R$-algèbre $R'$ le groupe $G(R'(\!(t)\!))$ (resp., $G(R'[\![t]\!])$). Alors, le sous-ensemble

$$LG(R')/L^+G(R') \subseteq Gr_G \text{ paramètre les } G\text{-torseurs triviaux sur } R'[\![t]\!].$$

Pour plus de détails sur ces foncteurs, nous recommandons [Čes22$_{\text{Surv}}$, §5.3.1]. L'application de la conjecture 1.0.1 dans l'étude de la grassmannienne affine est présentée ci-dessous.

**Corollaire 1.0.9.** *Si la conjecture 1.0.1 est vraie, pour toute $R$-algèbre $R'$ qui est un anneau local régulier, nous avons une égalité $Gr_G(R') = LG(R')/L^+G(R')$.*

En effet, la conjecture 1.0.1 montre qu'aucun $G$-torseur non-trivial sur l'anneau local régulier $R'[\![t]\!]$ ne se trivialise sur $R'(\!(t)\!)$. On peut trouver cette application de la conjecture 1.0.1 dans [Bac19, Proposition 19] et dans [HR20, Section 3].

Ci-dessous, nous commençons notre discussion sur la conjecture analogue (conjecture 1.1.2) à la conjecture 1.0.1 lorsque $R$ est remplacé par un anneau local d'une algèbre lisse sur un anneau de valuation de rang 1. Nous énonçons le théorème principal de cette thèse (théorème A) après avoir discuté du fondement pour la conjecture 1.1.2 reposant sur la conjecture d'uniformisation de Zariski (conjecture 1.1.1).

## 1.1 La géométrie algébrique des anneaux de valuation

L'étude des anneaux de valuation a récemment gagné en élan grâce, entre autres, aux travaux de Bhatt, Gabber, Kelly, Mathew, Morrow et Scholze. Il n'est pas exagéré de dire qu'ils occupent la place centrale d'un bon nombre de leurs récents travaux. Par exemple, les anneaux de valuation apparaissent comme des anneaux locaux des diverses topologies de Grothendieck nouvellement découvertes, telles que la 'arc topology' (voir [BM21]). La géométrie des anneaux de valuation est assez simple, car les idéaux premiers de tels anneaux forment une chaîne linéaire. Malgré cela, leur propriété d'être non-noethérien rend leurs structures algébriques difficiles à étudier. Toutefois, de manières différentes les anneaux de valuation se comportent comme des anneaux locaux réguliers, en particulier, pour un anneau de valuation $V$ sur $\mathbb{F}_p$, le complexe cotangent $\mathbb{L}_{V/\mathbb{F}_p}$ est concentré en degré 0 (voir [GR03, Theorem 6.5.12(ii)]). À cet égard, on peut étendre de nombreux résultats établis pour les anneaux locaux réguliers aux anneaux de valuation. Cette heuristique est renforcée par la conjecture suivante, qui est une version faible de la conjecture d'uniformisation locale de Zariski, qui est elle-même une version faible de la résolution des singularités.

**Conjecture 1.1.1** (Zariski)**.** *Un anneau de valuation s'écrit comme une colimite filtrée d'anneaux locaux réguliers.*

En conséquence, de nombreuses propriétés cohomologiques des anneaux locaux réguliers continuent à s'appliquer aux anneaux de valuation. La conjecture 1.1.1 est largement ouverte, cependant, elle est connue, entre autres, pour les anneaux de valuation dont le corps des fractions est algébriquement clos (cf. les altérations de de Jong [Tem17, Theorem 1.2.5]). Le fondement de la conjecture 1.1.2 repose sur la conjecture 1.1.1.

**Conjecture 1.1.2.** *Soient $V$ un anneau de valuation, $R$ un anneau local, intègre, $V$-essentiellement lisse avec le corps des fractions $K$ et $G$ un $R$-schéma en groupes réductifs. Alors un $G$-torseur sur $R$ est trivial s'il est génériquement trivial.*

Par passage à la limite en utilisant la conjecture 1.1.1, heuristiquement la conjecture 1.1.2 est une conséquence de la conjecture 1.0.1. Dans cette thèse, nous démontrons un cas particulier de la conjecture 1.1.2 sans utiliser la conjecture 1.1.1. Récemment, Guo et Liu ont obtenu indépendamment le théorème A dans leur prépublication [GL23].

**Théorème A** (Théorème 8.1)**.** *Soit $V$ un anneau de valuation de rang 1, soit $A$ un anneau qui est obtenu comme la semi-localisation à un nombre fini de premiers d'un anneau intègre $V$-lisse et soit $G$ un $A$-schéma en groupes réductifs qui possède un $A$-sous-groupe de Borel. Alors un $G$-torseur sur $A$ est trivial s'il est génériquement trivial.*

Ci-dessous, nous présentons l'ingrédient géométrique principal dans la démonstration du théorème A. En suivant les arguments de [Čes22, Proposition 4.1], nous démontrons le théorème B, qui peut être considéré comme une version non-noethérienne du lemme de présentation géométrique. Découvert à l'origine par Quillen dans [Qui73, Lemma 5.12] pour démontrer la conjecture de Gersten dans le cas des algèbres lisses sur un corps; affiné par Gabber dans [Gab94, Lemma 3.1] et Gabber–Gros–Suwa dans [CHK97, Theorem 3.1.1], le lemme de présentation joue un rôle clé dans la démonstration de la conjecture de Grothendieck-Serre. On peut retracer les origines de ce lemme au concept de bons voisinages d'Artin dans [SGA $4_{\mathrm{III}}$, Exposé XI]. Nous présentons ci-dessous l'énoncé de notre version du lemme de présentation.

**Théorème B** (Lemme du présentation 6.5)**.** *Soient*

- *$V$ un anneau de valuation de rang 1,*
- *$X$ un $V$-schéma affine, lisse de dimension relative $d > 0$,*
- *$x_1, \ldots, x_m$ points dans $X$, et*
- *$Y$ un sous-schéma fermé de $X$ de codimension au moins 2.*

*Alors il existe des ouverts affines $x_1, \ldots, x_m \in U \subset X$ et $S \subset \mathbb{A}_V^{d-1}$ et un $V$-morphisme $\pi\colon U \to S$ de dimension relative pure 1 tel que $Y \cap U$ soit $\pi$-fini.*

*Remarque* 1.1.5.

(1) Le fait que $\pi$ soit fini lorsqu'on le restreint à $Y \cap U$ est un point clé du théorème B.

(2) Cette version du lemme de présentation est différente de versions démontrées dans diverses parties de la littérature, notamment [Qui73, Lemma 5.12], [Gab94, Lemma 3.1] et [CHK97, Theorem 3.1.1], en exigeant que $Y$ soit de codimension au moins 2. Dans le cadre de la caractéristique mixte, si nous supprimons cette hypothèse, il n'est pas clair comment obtenir le fait que $Y \cap U$ soit $\pi$-finie (même si nous pouvons l'arranger pour qu'elle soit $\pi$-quasi-finie).

Ceci termine notre introduction en français. La thèse sera écrite en anglais. Nous débuterons par réécrire l'introduction.

# Chapter 2

# Introduction in English

The goal of this thesis is the study of torsors under reductive group schemes. One of the central problems in this subject is the conjecture of Grothendieck–Serre stated below. It appeared in the Chevalley seminar papers of Serre in [Ser58, page 31, Remarque] and Grothendieck in [Gro58, pages 26–27, Remarques 3]. Before delving into the subject of this thesis (see Section 2.1), we initiate a discussion on the background of the Grothendieck–Serre conjecture. We discuss the known cases and their consequences, but at the outset, we present the statement of the conjecture.

**Conjecture 2.0.1** (Grothendieck–Serre)**.** *For a regular local ring $R$ with a fraction field $K$ and a reductive $R$-group scheme $G$, a generically trivial $G$-torsor $E$ on* $\operatorname{Spec} R$ *is trivial, i.e.,*

$$\ker(\theta := H^1(R, G) \to H^1(K, G)) = \{*\}.$$

*Equivalently, given a Noetherian, regular scheme $X$ and a reductive $X$-group scheme $G$, a generically trivial $G$-torsor $E$ on $X$ is Zariski locally trivial.*

We begin our discussion about this conjecture by pointing out a subtlety regarding its statement. The group schemes of our interest are reductive, in particular, they are not commutative, in general. This means that the cohomology sets $H^1(R, G)$ have a structure of pointed sets. In the category of pointed sets, a morphism that has trivial kernel is not necessarily injective. This failure is due to the lack of an additive structure that, otherwise, an abelian group enjoys. However, the following remark showcases the fact that, if Conjecture 2.0.1 is true, the morphism $\theta$ is, indeed, injective.

*Remark* 2.0.2. A fortiori, the canonical morphism $\theta$ is injective. To show this, we take $\eta, \xi \in H^1(R, G)$ with the property that there exists an isomorphism $\eta|_K \cong \xi|_K$. It suffices to show that the sheaf $\tilde{\eta} := \underline{\mathrm{Isom}}_G(\eta, \xi)$ has a section over $\operatorname{Spec} R$. In fact, the sheaf $\tilde{\eta}$, which is endowed with a left action of $\tilde{G} := \underline{\mathrm{Aut}}_G(\eta)$ by composition, is a torsor under $\tilde{G}$. Thanks to the change of origin bijection

$$\begin{aligned} H^1(R, G) &\xrightarrow{\ \sim\ } H^1(R, \tilde{G}) \\ \psi &\longmapsto \underline{\mathrm{Isom}}_G(\eta, \psi), \end{aligned}$$

it suffices to apply Conjecture 2.0.1 in the case of $\tilde{G}$. This is a valid application because $\tilde{G}$ is an inner form of $G$, and hence, reductive.

To get a feeling for Conjecture 2.0.1, we could test the hypothesis by replacing $G$ by certain reductive groups for which the torsors have been well studied. The best examples of reductive groups whose torsors are thoroughly studied in the literature are the groups $\mathrm{GL}_n$ and $\mathrm{SL}_n$. Torsors under $\mathrm{GL}_n$ (resp., $\mathrm{SL}_n$) correspond to vector bundles of rank $n$ (resp., vector bundles of rank $n$ whose determinant is trivial). We test out the hypothesis of Conjecture 2.0.1 below.

*Remark* 2.0.3. In the case when $G = \mathrm{GL}_n$ or when $G = \mathrm{SL}_n$, the Hilbert theorem 90 yields that on local rings $H^1(R, G) = H^1(K, G) = \{*\}$. Indeed, $\mathrm{GL}_n$-torsors (resp., $\mathrm{SL}_n$-torsors) over $\operatorname{Spec} R$ correspond to locally free sheaves of rank $n$ (resp., locally free sheaves of rank $n$ whose determinant is trivial) on $\operatorname{Spec} R$. Since $R$ is local, locally free sheaves are free, whence the claim follows.

Another example of a reductive group whose torsors are well studied in the literature is $\mathrm{PGL}_n$, whose torsors were studied by Grothendieck in [Gro68a, §I.1] (see also [CS21, §3.1])) for their connection with the Brauer group. Since $\mathrm{PGL}_n$ is the automorphism group of the matrix algebra $\mathrm{Mat}_n$, its torsors correspond to forms of $\mathrm{Mat}_n$, i.e., Azumaya algebras of rank $n$. By taking the long exact sequence of cohomology associated to the short exact sequence $1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$ and using the fact that $H^1(R, \mathrm{GL}_n) = \{*\}$, Grothendieck showed that

$$H^1(R, \mathrm{PGL}_n) \hookrightarrow H^2(R, \mathbb{G}_m). \tag{2.0.3.1}$$

The case when $G = \mathrm{PGL}_n$, which was one of the main motivations for Conjecture 2.0.1 (see [Gro68b, Remarques 1.11 a]), is studied below.

*Remark* 2.0.4. For $G = \mathrm{PGL}_n$, the conjecture predicts that an Azumaya algebra over $R$ that is isomorphic to a matrix algebra over $K$ is already isomorphic to a matrix algebra over $R$. This is confirmed by Grothendieck's work [Gro68b, Corollaire 1.8] on the Brauer group (see [Čes22$_{\mathrm{Surv}}$, Example 3.1.3]). In more details, we draw the following diagram

$$\begin{tikzcd} H^1(R, G) \arrow[r] \arrow[d, hook] & H^1(K, G) \arrow[d, hook] \\ H^2(R, \mathbb{G}_m) \arrow[r] & H^2(K, \mathbb{G}_m), \end{tikzcd}$$

for which we have to show that the top horizontal map is injective. In fact, it is enough to show the injectivity of the bottom horizontal map. Indeed, thanks to the diagram (2.0.3.1), the vertical maps are injective. The required injectivity is a consequence of the work done by Grothendieck, whose method comprised of studying the long exact sequence of cohomology associated to the short exact sequence

$$1 \to \mathbb{G}_m \xrightarrow{\star} \eta_* \mathbb{G}_m \xrightarrow{\mathrm{div}} \bigoplus_{\iota\colon \nu \to \operatorname{Spec} R} \iota_* \underline{\mathbb{Z}} \to 0, \tag{2.0.4.1}$$

where $\eta$ is the inclusion of the generic point of $\operatorname{Spec} R$, the inclusions $\iota$ run through the height 1 points $\nu$ of $\operatorname{Spec} R$ and div is defined by associating to an $f \in K^\times = \eta_* \mathbb{G}_m(\operatorname{Spec} R)$ its Weil divisor. We note that the regularity of $R$ is used in the definition of div in order to translate Cartier divisors into Weil divisors. On the other hand, if the integral domain $R$ is not regular, one can still consider $\mathscr{F} := \operatorname{coker} \star$ in (2.0.4.1), although we loose the nice description of $\mathscr{F}$ in terms of Weil divisors. In general, there is an exact sequence

$$0 \to H^1(R, \mathscr{F}) \to H^2(R, \mathbb{G}_m) \to H^2(K, \mathbb{G}_m).$$

If the singularities of $R$ are not too horrible, for example, when $\dim(R) \leq 1$, the problem of vanishing of $H^1(R, \mathscr{F})$ is still tractable (see [CS21, Section 3.6.2]). In Remark 2.0.11, we discuss the vanishing in the case when $R$ is a local ring of a curve over an algebraically closed field or a finite field, and in Theorem 4.7, we prove it in the case when $R$ is a local ring of a smooth algebra over a valuation ring.

There is a rich history surrounding the proofs of several instances of Conjecture 2.0.1, starting from the early work of Nisnevich in the 70's. Among the vast literature, I have made an attempt to chronologically enlist the main players in the development of techniques that is used in proof of the main theorem of this thesis (Theorem A). My task here is not to produce a comprehensive list (see [Čes22$_{\mathrm{Surv}}$, §3.1] for an all-rounded list of contributors), rather to point out the papers that will play a role in this thesis.

## Known Cases of Conjecture 2.0.1

(1) The case when $R$ is a discrete valuation ring was proved by Nisnevich in his thesis [Nis82]. The idea is to use Harder-type approximation results to reduce the proof of Conjecture 2.0.1 over $R$ to its counterpart over the completion $\hat{R}$. Subsequently, Bruhat–Tits theory is exploited to settle the conjecture when $R$ is a complete discrete valuation ring. Inspired by this work, Guo in his thesis [Guo22] proved the related case of Conjecture 2.0.1 when $R$ is, instead, replaced by a not-necessarily-discrete valuation ring. We speak about the relation of valuation rings with regular local rings in Section 2.1. In Chapter 3, we make use of Guo's techniques to extend his result to semilocal Prüfer domains.

(2) The case when $G$ is a torus was proved by Colliot-Thélène and Sansuc in [CS87]. Introducing the concept of a flasque torus, i.e., a torus whose Galois theoretic data is 'simple', they showed that any torus has a resolution by flasque tori. Taking a flasque resolution, one can reduce the statement of Conjecture 2.0.1 in the case of a torus to a statement about vanishing of the local cohomology in degree 2 with coefficients in a flasque torus. We shall employ their methods in Chapter 4 to show purity for torsors under tori over smooth algebras over valuation rings.

(3) The equicharacteristic case, i.e., when $R$ contains a field, was settled by Fedorov and Panin in [FP15]. Combining the ideas underlying Artin's results on good neighbourhoods from [SGA $4_{\mathrm{III}}$, Exposé XI] with Voevodsky's "standard triples" from [MVW06, Definition 11.5], they define the notion of "nice triples" (which are smooth relative curves over $R$ equipped with a section and an $R$-finite closed subscheme). This enables them to pass via Nisnevich-type gluing to the study of torsors over the relative affine line $\mathbb{A}^1_R$. They conclude via Horrocks-style results to show that such torsors pull-back to the trivial one.

(4) In mixed characteristic, the case when $R$ is unramified (a local ring $R$ with a maximal ideal $\mathfrak{m} \subset R$ is *unramified* if it contains a field or if $\mathrm{char}(R/\mathfrak{m}) \notin \mathfrak{m}^2$) and $G$ has a Borel $R$-subgroup was settled by Česnavičius in [Čes22]. Streamlining Panin–Fedorov's strategy, he replaced Artin's good neighbourhoods by a presentation lemma over discrete valuation rings in the style of Gabber. We shall follow Česnavičius' strategy to prove Theorem A, for instance, we use extra mileage from his method to generalise to a presentation lemma over valuation rings of rank 1 (see Presentation Lemma 6.5).

Following the discussion about the known cases, we present some of the applications of Conjecture 2.0.1 that may be found in the literature (see [Čes22$_{\mathrm{Surv}}$] for a detailed treatment). Since the conjecture is known in equicharacteristic case (see (3) above), the applications mentioned below hold unconditionally in the equicharacteristic. Thanks to the resolution of the quasi-split unramified case of Conjecture 2.0.1 in [Čes22], in the mixed-characteristic, some of the applications will continue to hold unconditionally for quasi-split reductive groups and when the base is a smooth algebra over a discrete valuation ring. We shall discuss a few of these applications for quasi-split groups in Chapter 8, where the base will be replaced by a smooth algebra over a mixed-characteristic valuation ring of rank 1.

## Applications of Conjecture 2.0.1

Given a group scheme $G$ on a scheme $X$, let $H^1_{\mathrm{Nis}}(X,G)$ (resp., $H^1_{\mathrm{Zar}}(X,G)$) denote the set of torsors that trivialise Nisnevich locally (resp., Zariski locally). We start with the following consequence of the Grothendieck–Serre Conjecture.

**Corollary 2.0.5.** *If Conjecture 2.0.1 is true, given a regular scheme $X$ and a reductive $X$-group scheme $G$,*

$$H^1_{\mathrm{Nis}}(X,G) = H^1_{\mathrm{Zar}}(X,G).$$

Indeed, a Nisnevich cover, by definition, has a generic section. Therefore, a Nisnevich locally trivial $G$-torsor on $X$ is generically trivial. By applying Conjecture 2.0.1, we get that such a torsor is Zariski locally trivial.

As a second application, we discuss an application of the Grothendieck–Serre conjecture in the study of integral models of reductive groups.

**Corollary 2.0.6** (see [Čes22$_{\mathrm{Surv}}$, Proposition 3.1.5] and [Pan19])**.** *If Conjecture 2.0.1 is true, then, two reductive $R$-group schemes $G_1$ and $G_2$ over a regular local ring $R$ are isomorphic if they are isomorphic when restricted to the fraction field $K$. In other words, a reductive $K$-group scheme has at most one reductive model on $R$.*

*Proof.* Since $G_1$ and $G_2$ have isomorphic generic fibres and since root datum is locally constant on the base, the groups $G_1$ and $G_2$ have the same root datum on the connected scheme $\operatorname{Spec} R$. Therefore, by the uniqueness theorem of pinned reductive groups (see [SGA 3$_{\mathrm{III}}$, Exposé XXIII, Théorème 4.1 and Corollaire 5.1]), the group $G_2$ is a form of $G := G_1$, i.e., it corresponds to an $\mathrm{Aut}(G)$-torsor. We need to show that the kernel of $H^1(R,\mathrm{Aut}(G)) \to H^1(K,\mathrm{Aut}(G))$ is trivial. This would be true, thanks to Conjecture 2.0.1, if $\mathrm{Aut}(G)$ were reductive. Although $\mathrm{Aut}(G)$ is not reductive in general, there is an exact sequence of automorphisms groups

$$1 \to G/Z \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1,$$

where $Z \subset G$ is the centre and $\mathrm{Out}(G)$ is the outer automorphism group. Inspecting the long exact sequence of cohomology associated to the above displayed short exact sequence, it suffices to apply Conjecture 2.0.1 for the reductive group $G/Z$ and to show that there are isomorphisms $H^q(R,\mathrm{Out}(G)) \cong H^q(K,\mathrm{Out}(G))$ for $q = 0, 1$. This can be done by using the fact that $\mathrm{Out}(G)$ is a locally constant $R$-group scheme, as in the proof of [Pan19] or [Guo20, Proposition 6.1]. □

The following is another application, which states that certain properties of reductive groups can be checked at the generic fibre.

**Corollary 2.0.7.** *Given a reductive group $G$ on a regular local ring $R$ with a fraction field $K$, Conjecture 2.0.1 implies that $G$ is split (resp., quasi-split) if $G_K$ is split (resp., quasi-split).*

*Proof.* At first, let $G_K$ be a split reductive group. We shall show that $G$ is split as well. In similar vein as the beginning of the proof of Corollary 2.0.6, the root datum of the group $G$ is constant on the connected scheme $\operatorname{Spec} R$. Let $\mathscr{G}$ be a split reductive $R$-group scheme of this type. Once again, the argument in the beginning of the proof of Corollary 2.0.6 implies that $G$ is a form of $\mathscr{G}$. In particular, $G$ corresponds to a class in $H^1(R, \operatorname{Aut}(\mathscr{G}))$. By the uniqueness theorem of pinned reductive groups, the choice can be made so that $G_K$ corresponds to the trivial class in $H^1(K, \operatorname{Aut}(\mathscr{G}))$. It suffices to show that $G$ corresponds to the trivial class in $H^1(R, \operatorname{Aut}(\mathscr{G}))$. This follows from the argument of Corollary 2.0.6, which shows that the kernel of $H^1(R, \operatorname{Aut}(\mathscr{G})) \to H^1(K, \operatorname{Aut}(\mathscr{G}))$ is trivial.

The argument for the quasi-split case is similar to the above argument. For more details, we refer the reader to [Česnavičius22, Theorem 9.5]. □

Conjecture 2.0.1 can be used to prove the following (cf. [Česnavičius22, Corollary 1.3]), which is otherwise difficult to prove directly, by a patching result of torsors.

**Corollary 2.0.8.** *Given a regular local ring $R$, an element $r \in R$ and a reductive $R$-group scheme, Conjecture 2.0.1 implies that there is an equality*

$$G(\widehat{R}[\tfrac{1}{r}]) = G(\widehat{R}) \cdot G(R[\tfrac{1}{r}]).$$

*Proof.* On the contrary, let us assume that there is a nontrivial $g \in G(\widehat{R}[\frac{1}{r}]) \setminus (G(\widehat{R}) \cdot G(R[\frac{1}{r}]))$. Let $\mathscr{E}$ be the torsor on $\operatorname{Spec} R$ obtained by formally gluing the trivial torsor on $\operatorname{Spec} R[\frac{1}{r}]$ and the trivial torsor on $\operatorname{Spec} \widehat{R}$ using the gluing datum provided by $g$. The torsor $\mathscr{E}$ is nontrivial, given by the nontriviality of $g$. However, $\mathscr{E}$ is generically trivial by construction, which is a contradiction, and we are done. □

Given a reductive group $G$ on ring $R$, the *affine Grassmannian* functor $Gr_G$ associates to an $R$-algebra $R'$ the set of $G$-torsors $E$ over $R'[\![t]\!]$ with a chosen trivialisation $\alpha$ over $R'(\!(t)\!)$. Let $LG(R')$ (resp., $L^+G(R')$) denote the functor that associates to an $R$-algebra $R'$ the group $G(R'(\!(t)\!))$ (resp., $G(R'[\![t]\!])$). Then, the subset

$$LG(R')/L^+G(R') \subseteq Gr_G \text{ parametrises the trivial } G\text{-torsors over } R'[\![t]\!].$$

For more details on these functors, we refer the reader to [Česnavičius22$_{\mathrm{Surv}}$, §5.3.1]. The application of Conjecture 2.0.1 in the study of the affine Grassmannian is presented below.

**Corollary 2.0.9.** *If Conjecture 2.0.1 is true, for any $R$-algebra $R'$ that is a regular local ring, we have an equality $Gr_G(R') = LG(R')/L^+G(R')$.*

Indeed, Conjecture 2.0.1 shows that no nontrivial $G$-torsor on the regular local ring $R'[\![t]\!]$ trivialises over $R'(\!(t)\!)$. This particular application of Conjecture 2.0.1 can be found in [Bac19, Proposition 19] and [HR20, Section 3].

We conclude our discussion on the background of Conjecture 2.0.1 by a brief subsection dedicated to further questions that might arise while studying the conjecture. Wondering about the role of the regularity of $R$ in Conjecture 2.0.1, we can try to loosen the hypothesis by allowing certain singularities in $\operatorname{Spec} R$. It shall be evident in Remark 2.0.10 that nodal singularities pose problems to injectivity of the Brauer group (for a more detailed study on the Brauer group, see [CS21, §3.5-3.6]). Therefore, one might wonder about the optimal hypothesis on $R$ such that Conjecture 2.0.1 is satisfied. On the other hand, the role of the reductive property of $G$ is not clear either. Loosening the hypothesis on $G$, we show in Remark 2.0.12 that the corresponding conjecture for smooth, affine group schemes over $R$ is not true.

## Further Questions/Non-examples

In the following remark, recalling an example from [CS21, Chapter 8.6], we construct a local ring with nodal singularities for which the injectivity of the Brauer group fails to hold, showing that Conjecture 2.0.1 is not satisfied when we drop the hypothesis of normality from $R$.

*Remark* 2.0.10. Consider the nodal singular curve $C \subset \mathbb{A}^2_{\mathbb{R}}$ defined by the equation

$$y^2 = x^2(x-1).$$

This curve will be our candidate for a counterexample to Conjecture 2.0.1 in the case when $R$ is not regular. More precisely, we furnish a local ring $R = \mathcal{O}_{C,p}$, where $p \in C$ is the nodal point, which shows that the morphism $H^1(R, \mathrm{PGL}_n) \to H^1(\mathrm{Frac}(R), \mathrm{PGL}_n)$ is not necessarily injective for non-regular rings. The discussion in Remark 2.0.4 asserts that, for any local ring $A$, there in an injection $H^1(A, \mathrm{PGL}_n) \hookrightarrow H^2(A, \mathbb{G}_m)$. Therefore, it suffices to show that the kernel of the morphism

$$H^2(R, \mathbb{G}_m) \to H^2(\mathrm{Frac}(R), \mathbb{G}_m) \text{ contains a class } 0 \neq [\xi] \in H^1(R, \mathrm{PGL}_n).$$

Let $U := C \setminus \{(1,0)\}$. By op. cit. Theorem 3.5.7, there is an injection $H^2(U, \mathbb{G}_m) \hookrightarrow H^2(\mathcal{O}_{C,p}, \mathbb{G}_m)$, hence it suffices to study the kernel of the morphism $H^2(U, \mathbb{G}_m) \to H^2(\mathrm{Frac}(C), \mathbb{G}_m)$. The Hamiltonian quaternions $\mathbb{H}$, which are generated over $\mathbb{R}$ by $i, j$ and $k$ with relations

$$i^2 = -1, j^2 = -1 \text{ and } ij = k = -ji,$$

represent a nontrivial class in the image of $H^1(\mathbb{R}, \mathrm{PGL}_2) \to H^2(\mathbb{R}, \mathbb{G}_m)$. There is a quaternion algebra $\xi$ defined by the relation $i^2 = -1$ and $j^2 = x - 1$ that lies in the preimage of $\mathbb{H}$ under the canonical morphism $H^2(U, \mathbb{G}_m) \to H^2(\mathbb{R}, \mathbb{G}_m)$ defined by restriction to $p := (0,0)$. However, the image of $\xi$ under $H^2(U, \mathbb{G}_m) \to H^2(U \setminus \{(1,0)\}, \mathbb{G}_m)$ is trivial, since it is represented by a quaternion algebra that satisfies $i^2 = -1$ and $j^2 = (\frac{y}{x})^2$. Consequently, we have produced a nontrivial class $[\xi] \in H^1(R, \mathrm{PGL}_2)$ that is generically trivial, as promised.

We prove below that the curve $C$ in Remark 2.0.10 cannot be replaced by any curve that is defined over an algebraically closed field or over a finite field. More precisely, we show the injectivity for the Brauer group for such curves. In particular, thanks to Tsen's theorem [Sta22, Tag 03RF], we show that the Brauer group of a curve over an algebraically closed field vanishes.

*Remark* 2.0.11. Let $\mathbb{F}$ be either an algebraically closed field or a finite field. Suppose that $C$ is a finite type, integral $\mathbb{F}$-scheme of dimension 1, and that $p \in C$ is a point. We claim that for the ring $R := \mathcal{O}_{C,p}$, there is an injection

$$H^2(R, \mathbb{G}_m) \hookrightarrow H^2(\mathrm{Frac}(R), \mathbb{G}_m).$$

By a limit argument, it suffices to show that the morphism $H^2(U, \mathbb{G}_m) \to H^2(\mathrm{Frac}(C), \mathbb{G}_m)$ is injective, for any small enough open neighbourhood $U \subset C$ of $p$. Let $\tilde{C}$ be the normalisation of $C$. In the algebraically closed case, the Brauer group $H^2(\kappa(q), \mathbb{G}_m)$ vanishes for the residue field at any non-generic point $q \in C$ (same for $q \in \tilde{C}$). The corresponding vanishing in the finite field is a consequence of Lang's theorem. This means that, in either case, thanks to [CS21, Proposition 8.5.2], there is an injection

$$H^2(U, \mathbb{G}_m) \hookrightarrow H^2(U \times_C \tilde{C}, \mathbb{G}_m).$$

However, letting $K$ be the fraction field of $C$ (equivalently, of $\tilde{C}$), the smoothness of $\tilde{C}$ ensures that there is an injection $H^2(U \times_C \tilde{C}, \mathbb{G}_m) \hookrightarrow H^2(K, \mathbb{G}_m)$.

Following an idea due to Česnavičius, we produce a smooth, affine group scheme $G$ over a regular local ring $R$ for which the morphism $H^1(R, G) \to H^1(\mathrm{Frac}(R), G)$ has a nontrivial kernel.

*Remark* 2.0.12. Taking the curve $C$ defined in Remark 2.0.10, we write $f$ for the morphism $C \to L := \mathbb{A}^1_{\mathbb{R}}$ defined by $(x, y) \mapsto y$. We check by hand that $f$ induces a finite morphism of algebras $\varphi\colon \mathcal{O}_L(L) \to \mathcal{O}_C(C)$ that is surjective on the level of spectra. Therefore, $f$ is a finite, faithfully flat morphism. Indeed, the flatness of $f$ is a consequence of the fact that $\varphi$ is a finite morphism whose source is a principal ideal domain and whose target is an integral domain. The algebraic group $G := \mathrm{PGL}_2$ is linear, in particular, it is a quasi-projective scheme. Thanks to [CGP15, Proposition A.5.2], the Weil restriction $\mathrm{Res}_{C/L}(G_C) := f_*(G_C)$ exists and is a smooth, affine $L$-group scheme (we use the fact that $G$ is a smooth, affine $\mathbb{Z}$-scheme). Letting $p \in C$ be the origin and $R$ be the semi-localisation of $\mathcal{O}_C(C)$ at the points in $f^{-1}(f(p))$, the previous discussion implies that the induced morphism $f_p\colon \operatorname{Spec} R \to \operatorname{Spec} \mathcal{O}_{L,f(p)}$ is finite and faithfully flat. In addition, we have that $\mathscr{G} := \mathrm{Res}_{R/\mathcal{O}_{L,f(p)}}(G_R)$ is the localisation of $\mathrm{Res}_{C/L}(G)$ at $f(p)$. We shall show that the canonical morphism

$$H^1(\mathcal{O}_{L,f(p)}, \mathscr{G}) \to H^1(\mathrm{Frac}(\mathcal{O}_{L,f(p)}), \mathscr{G}) \text{ has a nontrivial kernel.}$$

Let $\mathrm{Frac}(R)$ be the total ring of fractions of the semilocal ring $R$. Since $f_p$ is finite, its higher direct images vanish. This implies that there is an identification of the above displayed morphism with the morphism

$$h\colon H^1(R, \mathrm{PGL}_2) \to H^1(\mathrm{Frac}(R), \mathrm{PGL}_2).$$

Thanks to Remark 2.0.10, the morphism $h_0\colon H^1(\mathcal{O}_{C,p}, \mathrm{PGL}_2) \to H^1(\mathrm{Frac}(\mathcal{O}_{C,p}), \mathrm{PGL}_2)$ has nontrivial kernel. Since the set $f^{-1}(f(p))$ is a disjoint union of two closed points, the ring $R$ is a product of two integral domains. As a consequence, $h_0$ appears as a factor of $h$. Thus, we are done.

Below, we begin our discussion on the analogous conjecture (Conjecture 2.1.2) to Conjecture 2.0.1 when $R$ is replaced by a local ring of a smooth algebra over a valuation ring of rank 1. We state the main theorem of this thesis (Theorem A) after discussing a basis for Conjecture 2.1.2 via Zariski's uniformisation conjecture (Conjecture 2.1.1).

## 2.1 Algebraic Geometry of Valuation Rings

The study of valuation rings has lately gained momentum through the work of Bhatt, Gabber, Kelly, Mathew, Morrow and Scholze, among many others. The valuation rings occupy the centrepiece in many of their latest works. For example, they appear as the local rings of various newly discovered Grothendieck topologies like the arc topology [BM21]. The geometry of valuation rings is fairly simple, since the prime ideals of such rings form a linear chain. Despite this, their non-Noetherian nature makes their algebraic structure hard to study. However, in many ways valuation rings resemble regular local rings, in particular, for a valuation ring $V$ over $\mathbb{F}_p$, the cotangent complex $\mathbb{L}_{V/\mathbb{F}_p}$ is concentrated in degree 0 (see [GR03, Theorem 6.5.12(ii)]). In this regard, one expects that many results that have been established for regular local rings hold for valuation rings. This heuristic is buttressed by the following conjecture, which is a weak form of Zariski's local uniformisation conjecture, which is itself a weak form of the resolution of singularities.

**Conjecture 2.1.1** (Zariski)**.** *Any valuation ring is a filtered colimit of regular local rings.*

As a consequence, many cohomological properties of regular local rings pass onto valuation rings. Conjecture 2.1.1 is widely open, however, among other cases, it is known for valuation rings whose fraction field is algebraically closed (cf. de Jong's alterations [Tem17, Theorem 1.2.5]). Conjecture 2.1.1 is the basis of Conjecture 2.1.2.

**Conjecture 2.1.2.** *For a valuation ring $V$, a local, $V$-essentially smooth, integral domain $R$ with a fraction field $K$ and a reductive $R$-group scheme $G$, a generically trivial $G$-torsor $E$ over $R$ is trivial, i.e.,*

$$\ker(H^1(R, G) \to H^1(K, G)) = \{*\}.$$

By limit arguments using Conjecture 2.1.1, heuristically Conjecture 2.1.2 is a consequence of Conjecture 2.0.1. In this thesis, we give a proof of a special case of Conjecture 2.1.2 without using Conjecture 2.1.1. Recently, Guo and Liu independently obtained Theorem A in their preprint [GL23].

**Theorem A** (Theorem 8.1)**.** *Let $V$ be a valuation ring of rank one, let $A$ be a ring that is obtained as the semilocalisation of a smooth $V$-domain at finitely many primes and let $G$ be a reductive $A$-group scheme that has a Borel $A$-subgroup. Then any generically trivial $G$-torsor over* $\operatorname{Spec} A$ *is trivial, i.e.,*

$$\ker(H^1(A, G) \to H^1(\mathrm{Frac}(A), G)) = \{*\}.$$

We discuss the steps involved in the proof of Theorem A in Section 2.1.1. However, before the start of this discussion, we present the principal geometric input in the proof. Bootstrapping from the arguments of [Česnavičius22, Proposition 4.1], we prove a technical theorem that can be considered as a non-Noetherian version of the geometric presentation lemma. Originally discovered by Quillen in [Qui73, Lemma 5.12] to prove the Gersten's conjecture in the case of smooth algebras over fields; further refined by Gabber in [Gab94, Lemma 3.1] and Gabber–Gros–Suwa in [CHK97, Theorem 3.1.1], the presentation lemma plays a key role in the proof of the Grothendieck–Serre conjecture. One can trace the origins of this lemma to Artin's concept of good neighbourhoods in [SGA $4_{\mathrm{III}}$, Exposé XI]. We present the statement of our version of the presentation lemma below.

**Theorem B** (Presentation Lemma 6.5)**.** *Given*

- *a valuation ring $V$ of rank one,*
- *a smooth, affine $V$-scheme $X$ of relative dimension $d > 0$,*
- *points $x_1, \ldots, x_m \in X$, and*

- *a closed subscheme $Y \subset X$ that is of codimension at least 2,*

*there are affine opens $x_1, \dots, x_m \in U \subset X$ and $S \subset \mathbb{A}_V^{d-1}$ and a smooth $V$-morphism $\pi\colon U \to S$ of pure relative dimension 1 such that $Y \cap U$ is $\pi$-finite.*

The idea (due to Česnavičius) is to slice a compactification $X \hookrightarrow \overline{X} \subset \mathbb{P}_V^n$ by $d-1$ hyperplanes $H_1, \dots, H_{d-1}$ in generic positions. The hyperplanes are chosen so that they intersect the boundary $\overline{X} \setminus X$ in a nice fashion, for example, we ensure that

$$Y \cap H_1 \cap \dots \cap H_{d-1} \text{ is a finite set and that } (\overline{Y} \setminus Y) \cap H_1 \cap \dots \cap H_{d-1} \text{ is empty.}$$

The morphism $\pi\colon X \to \mathbb{A}_V^{d-1}$ is defined by projections onto the coordinates of $H_1, \dots, H_{d-1}$. By making sure that $\pi$ is smooth at the points of $x_1, \dots, x_m$, we shrink to a neighbourhood $x_1 \dots, x_m \in U \subset X$ where $\pi$ is smooth. The finiteness at $Y \cap U$ is confirmed by the properness of $\overline{Y}$ and the quasi-finiteness of $\pi$ at $Y \cap U$.

*Remark* 2.1.5.

(1) The fact that $\pi$ is finite when restricted to $Y \cap U$ is a key point of Theorem B (see Section 2.1.1).

(2) This presentation lemma differs from the versions proved in various parts of the literature, notably [Qui73, Lemma 5.12], [Gab94, Lemma 3.1] and [CHK97, Theorem 3.1.1], by demanding $Y$ to be of codimension at least 2. In the mixed characteristic setting, if we drop this hypothesis, it is not clear how to achieve that $Y \cap U$ is $\pi$-finite (even though we can arrange that it is $\pi$-quasi-finite).

We discuss below the steps involved in the proof of Theorem A. For the proof of Theorem A, see Chapter 7.1.

### 2.1.1 The steps involved in the proof of Theorem A

Let $E$ be a generically trivial $G$-torsor on $\operatorname{Spec} A$. We shall show that $E$ is trivial.

- The point of departure is the Presentation Lemma 6.5, which is used to produce a relative $A$-curve $C$ with a section $s \in C(A)$ such that there is a quasi-split reductive $C$-group scheme $\mathscr{G}$ whose $s$-pullback is $G$ (see Proposition 7.1.1).
- Moreover, Presentation Lemma 6.5 is used to obtain an $A$-finite closed subscheme $Z \subset C$ such that there is a $\mathscr{G}$-torsor $\mathscr{E}$ on $C$ whose $s$-pullback is $E$ and that $\mathscr{E}_{C \setminus Z}$ is 'trivial'. By an application of the valuative criterion of properness, the quasi-split property is exploited to satisfy the codimension at least 2 hypothesis on $Y$ in the Presentation Lemma 6.5. The set-up is depicted in the following diagram.

$$\begin{tikzcd}
 & & \mathscr{G} \circlearrowleft \mathscr{E} \\
Z \arrow[r, hook] \arrow[dr] & C \arrow[ur, dash] \arrow[d] & \\
 & \operatorname{Spec} A \arrow[u, bend left, dotted, "s"] \arrow[r, dash] & G \circlearrowleft E
\end{tikzcd}$$

- The next step is to modify $C$ and to replace it by the source of an étale morphism $\tilde{C} \to C$ that lets us assume that $\mathscr{G} = G_C$, without altering the other properties. Keeping in mind that $s^*\mathscr{G} = G$, this can be done by emulating a Bertini type slicing argument to a projective compactification of $\mathscr{G}$ (see Section 7.2).
- Once we have that, we build a Nisnevich-type gluing square

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "\wr"] & C \arrow[d, "f"] \\ Z' \arrow[r, hook] & \mathbb{A}^1_A, \end{tikzcd}$$

  where $f$ is étale. The argument is a little delicate when the residue field of $A$ is finite because $Z$ might have 'too many' rational points (see Section 7.3).
- Next, we glue $\mathscr{E}$ along the displayed square to obtain a $G_{\mathbb{A}^1_A}$-torsor $\mathscr{E}'$. A Beauville–Laszlo gluing result of Rydh is used in our non-Noetherian setting in order to perform the above mentioned gluing (see Section 7.4).
- Finally, it remains to study the $G_{\mathbb{A}^1_A}$-torsor $\mathscr{E}'$. Keeping in mind that $\mathscr{E}'$ is trivial away from an $A$-finite closed subscheme $Z' \subset \mathbb{A}^1_A$, a Horrocks' principle type statement shows that the pullback of $\mathscr{E}'$ along the zero section is trivial (see Proposition 8.2). This finishes the proof of Theorem A (see Chapter 8).

The strategy sketched above closely follows the strategy of the proof of Česnavičius in [Čes22, Theorem 1.2]. The proof of loc. cit. is inspired by earlier work on the Grothedieck–Serre conjecture, for example, the work of Fedorov and Panin in [FP15].

## Outline of the Thesis

- In Chapter 3, we prove statements concerning Prüfer domains that will be useful to us in subsequent chapters. Following arguments from [GR18], we show that valuation rings are universally catenary (see Proposition 3.8). We show that the local ring at a generic point in the special fibre of a smooth algebra over a valuation ring is a valuation ring (Lemma 3.10).
- In Chapter 4, employing similar techniques as [CS87], we prove the Grothendieck–Serre conjecture for torsors under tori over smooth algebras over valuation rings (Theorem 4.7). The method of the proof also shows the following result.

  **Theorem Brauer group** (Theorem 4.7)**.** *Let $V$ be a valuation ring and let $A$ be an essentially smooth $V$-domain. Then, the morphism*

$$H^2(A, \mathbb{G}_m) \hookrightarrow H^2(\mathrm{Frac}(A), \mathbb{G}_m) \ \textit{ is injective.}$$

  As an ingredient of the proof Theorem 4.7, we prove a weak version of the Auslander–Buchsbaum formula for smooth algebras over valuation rings, and show purity of torsors under tori over such rings (Proposition 4.5). The results of this chapter have been obtained independently in the recent work [GL23].

- In Chapter 5, we prove the Grothendieck–Serre conjecture for semilocal Prüfer domains. The case of valuation rings is proved in [Guo22], whose arguments we closely follow. A patching formula (Proposition 5.10), which lets us reduce to the case of complete valuation rings of rank 1, is proven by using techniques that parallel Harder-type approximations. Unlike the others, this chapter does not impose a quasi-split hypothesis on the reductive group $G$.

- In Chapter 6, we bootstrap from the arguments of [Čes22, Theorem 4.2] to obtain Theorem B. The non-Noetherian geometry poses certain difficulties that we overcome by using the results of Chapter 3. Although, a relative version of the presentation lemma (Proposition 6.4) is established for valuation rings of arbitrary finite rank, it is insufficient to remove the rank one hypothesis from Theorem B. Any improvement in this direction will enable progress in the resolution of Conjecture 2.1.2.

- In Chapters 7-8, the goal is to prove Theorem A. We proceed as in Section 2.1.1.

# Chapter 3

# Generalities on Prüfer Domains

The purpose of this chapter is to prove certain useful facts about rings that are algebras over valuation rings. A notable result proved here is the fact that valuation rings are universally catenary (Proposition 3.8).

**Definition 3.1** ([Gil92, §22])**.** A *Prüfer domain* is an integral domain whose localisation at every prime ideal is a valuation ring.

For equivalent definitions of Prüfer domains, see [Gil92, Theorem 22.1] and [Sta22, Tag 092S]. The notion of Prüfer domains generalises the one of Dedekind domains (an integral domain is called Dedekind if its localisation at every prime ideal is a discrete valuation ring). Since the increasing union of Prüfer domains is a Prüfer domain (see [Gil92, Proposition 22.6]), the class of Prüfer domains contains $\overline{\mathbb{Z}}$ and $\overline{k[X]}$, for any field $k$, both of which are non-Noetherian, Prüfer domains of Krull dimension 1. The ring of integer valued polynomials in one variable with rational coefficients is an example of a non-Noetherian, Prüfer domain of Krull dimension 2 (see [CC16, Theorem 17]). The ring of entire holomorphic functions is a non-Noetherian, Prüfer domain of infinite Krull dimension (see [Lop98]). A Prüfer domain of Krull dimension 0 is a field.

Below, we state certain permanence properties of Prüfer domains.

**Lemma 3.2.** *Given a Prüfer domain $R$, an ideal $I \subset R$ and a multiplicative subset $S \subset R$, the following are Prüfer domains:*

(*a*) *the localisation $S^{-1}R$, and*

(*b*) *the quotient $R/I$, given that $I \subset R$ is prime.*

*Proof.* Since both (a) and (b) commute with localisation at $\mathfrak{m}$, we reduce to proving the same for $R_{\mathfrak{m}}$. Therefore, without loss of generality, we may assume that $R$ is a valuation ring, whence part (a) and (b) follows from [Sta22, Tag 088Y]. □

A different (but similar) proof of (a) and (b) can be found in [Gil92, Proposition 22.5].

**Lemma 3.3.** *A Prüfer domain $R$ is the intersection in* $\mathrm{Frac}(R)$ *of the valuation rings obtained by the localisations at the maximal ideals. Additionally, if $R$ is semilocal,*

(*a*) *for a maximal ideal $\mathfrak{m} \subset R$ and a prime ideal $\mathfrak{m} \neq \mathfrak{p} \subset R$, there exists an element $a \in R$ so that $V(a) \cap (\mathrm{MaxSpec}(R) \cup \{\mathfrak{p}\}) = \{\mathfrak{m}\}$,* (*b*) *for a subfield* $K \subseteq \operatorname{Frac} R$*, the intersection* $R \cap K$ *is a semilocal Prüfer domain with* $\operatorname{Frac}(R \cap K) = K$*, and*

(*c*) *it is the increasing union of its subrings that are semilocal Prüfer domain of finite Krull dimension.*

*Proof.* The first claim is a consequence of the fact that any integral domain is the intersection of its localisations at the maximal ideals[2]. Indeed, the containment in one side being obvious, we need to verify that

$$R \supseteq \bigcap_{\mathfrak{m} \in \operatorname{MaxSpec}(R)} R_{\mathfrak{m}}.$$

For an $x \in \operatorname{Frac} R \setminus R$, consider the ideal $I = \{a \in R \mid ax \in R\}$, which is a proper ideal since $1 \notin I$. Letting $I \subseteq \mathfrak{m} \subset R$ be a maximal ideal, in the case that $x \in R_{\mathfrak{m}}$, there exists a $t \in R \setminus \mathfrak{m}$ such that $xt \in R$. This is a contradiction, since by the definition of $I$, we have $t \in I \subseteq \mathfrak{m}$. Therefore, $x \notin R_{\mathfrak{m}}$, and we are done.

(a): This is a consequence of the prime avoidance lemma [Sta22, Tag 00DS]. Indeed, putting $J = \mathfrak{m}$ and $\{I_i \mid i = 1, \dots, r\} = \{\mathfrak{p}\} \cup (\operatorname{MaxSpec}(R) \setminus \{\mathfrak{m}\})$, we get the required $a \in R$.

(b): This is a consequence of [Mat86, Theorem 12.2] (or [BouCA, Chapter 6, Section 7, Number 1, Proposition 2]). Indeed, by [Sta22, Tag 0AAV], the intersection $R_{\mathfrak{m}} \cap K$ is a valuation ring with fraction field $K$, for each maximal ideal $\mathfrak{m} \subset R$. Therefore,

$$R_K := R \cap K = \bigcap_{\mathfrak{m} \in \operatorname{MaxSpec}(R)} (R_{\mathfrak{m}} \cap K)$$

is a 'Krull ring' with fraction field $K$, namely, there are a surjection $\operatorname{MaxSpec}(R) \twoheadrightarrow \operatorname{MaxSpec}(R_K)$ given by $\mathfrak{m} \mapsto \mathfrak{m}_K := \mathfrak{m} \cap R_K$ and an isomorphism $(R_K)_{\mathfrak{m}_K} \cong R_{\mathfrak{m}} \cap K$ for each maximal ideal.

(c): The fraction field $K$ of $R$ can be written as $K = \bigcup K'$, the increasing union being taken over subfields $K' \subset K$ that are finitely generated extensions over the prime subfield $\mathbb{F} \subset K$; as a consequence, $R = \bigcup (R \cap K')$. Thanks to (b), the semilocal Prüfer domain $R' := R \cap K'$ has fraction field $K'$; therefore, the fact that $\operatorname{tr.deg}_{\mathbb{F}}(K') < \infty$ ensures that $R'$ has finite Krull dimension ([BouCA, Chapter VI, Section 10, Number 3, Corollary 1]). Therefore, we have exhibited $R$ as an increasing union of subrings $R'$ that are semilocal Prüfer domain of finite Krull dimension. Hence, we are done. $\square$

The following result will be useful in Chapter 5, where we will need to study formal neighbourhoods of maximal ideals in Prüfer domains.

**Lemma 3.4.** *Given a semilocal ring* $R$ *and an ideal (resp., a finitely generated ideal)* $I \subset R$ *of co-height* 0 *(i.e.,* $V(I) \subseteq \operatorname{MaxSpec}(A)$*), the* $I$*-adically continuous morphism of rings (resp.,* $I$*-adically complete rings)*

$$\widehat{R}^I \xrightarrow{\sim} \prod_{\mathfrak{m} \in V(I)} \widehat{R_{\mathfrak{m}}}^I \quad \text{is an isomorphism.} \tag{3.4.1}$$

*Proof.* By hypothesis, for any integer $n \geq 1$, the prime ideals of $R/I^n$ are maximal, consequently, the Jacobson radical of $R/I^n$ (which is the product of maximal ideals of $R/I^n$ thanks to [Mat86, Theorem 1.3] or [Sta22, Tag 00DT]) is the nilradical of $R/I^n$. By [Sta22, Tag 00JA][3], there is an isomorphism

$$R/I^n \xrightarrow{\sim} \prod_{\mathfrak{m} \in V(I)} (R/I^n)_{\mathfrak{m}/I^n} \cong \prod_{\mathfrak{m} \in V(I)} R_{\mathfrak{m}}/I^m R_{\mathfrak{m}}.$$

[2]The proof is taken from the MathStackExchange post #630752.

[3]By definition, the nilradical $\operatorname{Nil}(R/I^n)$ is a 'locally nilpotent ideal', i.e., for any element $x \in \operatorname{Nil}(R/I^n)$ there is an integer $m \geq 1$ such that $x^m = 0$ ([Sta22, Tag 00IL]).

By varying $n$ and taking the limit, we deduce the isomorphism (3.4.1), which is an isomorphism of $I$-adically complete rings when $I$ is finitely generated (see [Sta22, Tag 05GG]). $\square$

*Remark* 3.5. Let $R$ be a semilocal Prüfer domain of finite, positive Krull dimension. For each maximal ideal $\mathfrak{n} \subset R$, we can choose a prime ideal $\mathfrak{p} \subsetneq \mathfrak{n}$ of co-height 1, and an element $a_{\mathfrak{n}} \in R$ such that $V(a_{\mathfrak{n}}) \cap (\operatorname{MaxSpec}(R) \cup \{\mathfrak{p}\}) = \{\mathfrak{n}\} \implies V(a_{\mathfrak{n}}) = \{\mathfrak{n}\}$ (see Lemma 3.3(a)). Therefore, letting $\mathfrak{m} \subset R$ be a maximal ideal, we get that $R_{\mathfrak{m}} = R[\frac{1}{b_{\mathfrak{m}}}]$, where $b_{\mathfrak{m}} = \prod_{\mathfrak{n} \neq \mathfrak{m}} a_{\mathfrak{n}}$. Consequently, the spectra of the localisations of $R$ at its maximal ideals form a Zariski open covering of $\operatorname{Spec} R$. Furthermore, by the definition [Sta22, Tag 0A02], the Henselisation $R^h$ of $R$ along $\mathfrak{m}$ coincides with the Henselisation of $R_{\mathfrak{m}}$ ([Sta22, Tag 0A03]), which is a valuation ring ([Sta22, Tag 0ASK]). Moreover, since $V(a) = \{\mathfrak{m}\}$, [Sta22, Tag 0F0L] implies that $R^h$ can also be identified with the Henselisation of $R$ along $(a)$. On another note, Lemma 3.4 implies that $\widehat{R}^a \cong \widehat{R_{\mathfrak{m}}}^a$ is an $a$-adically complete valuation ring of rank 1 (see [Guo22, Proposition 8.9(iv)]).

Following the proof of [GR18, Lemma 11.5.8], the following lemma is applied to bound the fibres of finite type schemes over Prüfer domains in the proof of Proposition 3.8. The hypothesis of Lemma 3.6 is sufficient to ensure that $X$ is flat over $R$.

**Lemma 3.6** ([EGA $\mathrm{IV}_3$, Lemme 14.3.10])**.** *Let $R$ be a Prüfer domain, and let $\gamma, \eta \in \operatorname{Spec} R$ be points such that $\eta$ is the generic point. Given an irreducible, finite type, dominant $R$-scheme $X$, if $X_{\kappa(\gamma)} \neq \emptyset$ then $\dim X_{\kappa(\eta)} = \dim X_{\kappa(\gamma)}$.*

*Proof.* Since the statement is local we can localise at $\gamma$ and assume that $R$ is a valuation ring with closed point $\gamma$. We then apply loc. cit. $\square$

**3.7. Catenary.** A topological space $X$ is called *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exists a maximal chain of irreducible closed subsets $T = T_0 \subset T_1 \subset \ldots \subset T_n = T'$ and every such chain has the same length ([Sta22, Tag 02I1]). A scheme is called *catenary* if its underlying topological space is catenary ([Sta22, Tag 02IW]). A scheme $S$ is called *universally catenary* if any locally of finite type $S$-scheme is catenary. A ring is called *catenary* (resp., *universally catenary*) if its spectrum is catenary (resp., universally catenary).

The following result shows that valuation rings are universally catenary.

**Proposition 3.8.** *Let $R$ be a Prüfer domain of finite Krull dimension and let $f\colon X \to \operatorname{Spec} R$ be finite type morphism of schemes. The function*

$$\delta\colon |X| \to \mathbb{Z}, \text{ given by } \delta(x) = \operatorname{tr.deg}_{\kappa(f(x))}(\kappa(x)) - \operatorname{codim}(\overline{\{f(x)\}}),$$

*is a 'dimension' function (cf. [Sta22, Tag 02I8]), i.e., $x$ specialises to $y \neq x$ only if $\delta(x) > \delta(y)$, and a specialisation $x \rightsquigarrow y$ is immediate if and only if $\delta(x) = \delta(y) + 1$. Furthermore, if $Y$ is the spectrum of a semilocalisation of $X$, then $|Y|$ is a catenary topological space of finite Krull dimension.*

*Proof.* We show that it suffices to assume that $Y = X$ (i.e., it is a semilocalisation of $X$ at the empty set) to prove the final statement. First, we claim that it is enough to show that $X$ is catenary. Indeed, this is true since the semilocalisation of any catenary scheme is catenary (being catenary is a Zariski local property ([Sta22, Tag 02I2]) and any localisation of a catenary ring is catenary ([Sta22, Tag 00NJ])). Second, by definition of the Krull dimension, it is enough to check that $|X|$ has finite Krull dimension.

Therefore, without loss of generality, we may assume that $Y = X$. Henceforth, we show that $|X|$ is a catenary topological space of finite Krull dimension.

A sober topological space ([Sta22, Tag 004X]) with a dimension function is catenary (see [Sta22, Tag 02IA]). In fact, a sober topological space with a bounded dimension function is of finite Krull dimension. Indeed, consider a descending chain $|X| \supseteq X_0 \supsetneq X_1 \supsetneq \ldots \supsetneq X_m$ of irreducible closed subsets. For each $n$, let $x_n \in X_n$ be the generic point. The containment $X_n \supsetneq X_{n+1}$ implies that $x_n \rightsquigarrow x_{n+1}$ and $x_n \neq x_{n+1}$, and hence, $\delta(x_n) > \delta(x_{n+1})$. As a consequence, applying the dimension function to the sequence $\{x_n\}_{n=0,\ldots,m}$, we obtain a strictly descending sequence of integers $\{\delta(x_n)\}_{n=0,\ldots,m}$. However, since $\delta$ is bounded, we get a limit on the length $m$ of the descending chain $\{X_n\}$, implying that $|X|$ is of finite Krull dimension.

Hence, it suffices to show that $\delta$ is a bounded dimension function. Consider a specialisation $x \rightsquigarrow y$ in $X$. If $f(x) = f(y)$, then replacing $X$ by its fibre over $f(x)$, we may assume that $X$ is a finite type $\kappa(f(x))$-scheme; in which case, thanks to [Sta22, Tag 02JW], $\delta(x) \geq \delta(y)$ and the specialisation is immediate if and only if $\delta(x) = \delta(y) + 1$. Henceforth, we assume that $f(x) \neq f(y)$. Localising at $f(y)$, the function

$$\delta' \colon |X_{R_{f(y)}}| \to \mathbb{Z}, \text{ given by, } \delta'(x) = \operatorname{tr.deg}_{\kappa(f(x))}(\kappa(x)) - \operatorname{codim}_{\operatorname{Spec} R_{f(y)}}(\overline{\{f(x)\}})$$

equals $\delta|_{X_{R_{f(y)}}} \colon |X_{R_{f(y)}}| \to \mathbb{Z}$, up to a constant. Thus, localising at the prime ideal corresponding to $f(y)$, without loss of generality, we might assume that $R$ is a valuation ring with closed point $f(y)$. Further, dividing by the prime ideal corresponding to $f(x)$, we may also assume that $f(x)$ is the generic point. Therefore, the closed subscheme $Z := \overline{\{x\}} \subseteq X$ (with reduced structure) is dominant, producing the equality $\dim Z_{f(y)} = \dim Z_{f(x)}$ thanks to Lemma 3.6. Moreover, since $Z$ is a dominant, integral $R$-scheme, it is automatically $R$-flat (it follows from the fact that flatness can be checked locally and from [BouCA, Chapter I, §2.4, Proposition 3(ii)], which implies that an injection $R \hookrightarrow A$ into an integral domain is flat); additionally, since $Z$ is of $R$-finite type, by [RG71, Première partie, Corollaire 3.4.7], it is of $R$-finite presentation. On the other hand, since $x$ is the generic point of $Z$, it is also the generic point of $Z_{\kappa(x)}$; consequently, applying, for example Noether normalisation [Sta22, Tag 00P0], we deduce that

$$\operatorname{tr.deg}_{\kappa(f(x))}(\kappa(x)) = \dim Z_{f(x)} = \dim Z_{f(y)} \geq \operatorname{tr.deg}_{\kappa(f(y))}(\kappa(y)).$$

Finally, since $\overline{\{f(x)\}} \supsetneq \overline{\{f(y)\}}$, the inequality $\delta(x) > \delta(y)$ follows.

Lastly, we show that the specialisation $x \rightsquigarrow y$ is immediate if and only if $\delta(x) = \delta(y) + 1$. In similar vein as before, $\operatorname{tr.deg}_{\kappa(f(x))}(\kappa(x)) \geq \operatorname{tr.deg}_{\kappa(f(y))}(\kappa(y))$, with equality in the case that $y$ is the generic point of $Z_{f(y)}$. As a consequence, $\delta(x) = \delta(y) + 1$ is equivalent to the case when $\dim(\overline{\{f(x)\}}) = \dim(\overline{\{f(y)\}}) + 1$ and $\operatorname{tr.deg}_{\kappa(f(x))}(\kappa(x)) = \operatorname{tr.deg}_{\kappa(f(y))}(\kappa(y))$, which in turn is equivalent to the case when $f(y)$ is an immediate specialisation of $f(x)$ and $y$ is the generic point of $Z_{f(y)}$, in other words, $y$ is an immediate specialisation of $x$. Indeed, if $y$ is an immediate specialisation of $x$, then, $f(y)$ is an immediate specialisation of $f(x)$ (see [Sta22, Tag 0D4H]).

To verify that $\delta$ is bounded, we choose the generic point $x \in X$ of an irreducible component. Let $y \in \overline{\{x\}}$ be a closed point. Then, $0 \leq \delta(x) - \delta(y) \leq \operatorname{tr.deg}_{\kappa(f(x))}(\kappa(x)) + \dim(\overline{\{f(x)\}}) = \dim Z_{f(x)} + \dim(\overline{\{f(x)\}}) \leq \dim X_{f(x)} + \dim R$, and hence we are done. $\square$

*Remark* 3.9. In Proposition 3.8, additionally, if $R$ is semilocal, then $|Y|$ is even Noetherian. To prove this claim, it is enough to show that $|X|$ is Noetherian. Indeed, this follows since the subspace of any

Noetherian topological space is Noetherian [Sta22, Tag 0052] and since $|Y| \subseteq |X|$. To show that $|X|$ is Noetherian, we use the fact that the hypothesis implies that $\operatorname{Spec} R$ is a finite set (see, for example, Remark 3.5), which implies that the union $|X| = \cup_{\eta \in \operatorname{Spec} R} |X_\eta|$ is finite. Since for each $\eta$, the scheme $X_\eta$, being of finite type over a field, is Noetherian, the claim follows by using [Sta22, Tag 0053].

While working with smooth algebras $A$ over a valuation ring $V$, to use local arguments, it is often useful to localise at a generic point $x$ in the $V$-special fibre of $\operatorname{Spec} A$ (see Theorem 4.7 and Proposition 7.1.1). The following lemma shows that $A_x$ is, in fact, a valuation ring.

**Lemma 3.10** (cf. [Mor22, Théorème A])**.** *Given a valuation ring $V$, an integral domain $A$ that is an essentially smooth, faithfully flat $V$-algebra, a prime $\mathfrak{p} \subset A$ corresponding to a generic point of the $V$-special fibre of* $\operatorname{Spec} A$*, the morphism $V \to A_\mathfrak{p}$ is an extension of valuation rings which induces an isomorphism at the level of value groups.*

*Proof.* A limit argument shows that, without loss of generality, we can assume that $A$ is smooth ([Sta22, Tag 0AS4]). We shall reduce, by the local structure of smooth morphisms, to showing the claim when $A$ is étale, in which case it suffices to use [Sta22, Tag 0ASJ], and when $A$ is a polynomial algebra, in which we do an explicit computation.

By the local structure of smooth morphisms [SGA 1, Exposé II, Théorème 4.10(ii)] (cf. [Sta22, Tag 052E]), there exist an $n \geq 0$, an affine open neighbourhood $\operatorname{Spec} B \subseteq \operatorname{Spec} A$ of $\mathfrak{p}$ and an étale morphism $j\colon \operatorname{Spec} B \to \operatorname{Spec} V[x_1, \dots, x_n]$. Since the claim is local and $B_\mathfrak{p} = A_\mathfrak{p}$, without loss of generality, we may assume that $A = B$. If $V \to A$ is étale, then $n = 0$ and the statement is true by [Sta22, Tag 0ASJ]. Otherwise, $n \geq 1$ and we observe that since étale morphisms are quasi-finite, $j(\mathfrak{p})$ is the generic point of the $V$-special fibre of $\operatorname{Spec} V[x_1, \dots, x_n]$. As a result, we have a composite morphism

$$V \xrightarrow{g} V[x_1, \dots, x_n]_{j(\mathfrak{p})} \xrightarrow{h} A_\mathfrak{p}.$$

We reduce to showing that the claim for the morphism $g$. Indeed, supposing that the result is true when $A$ is replaced by $V[x_1, \dots, x_n]$, i.e., supposing that the claim is true for $g$, the $n = 0$ case applied to $j$ implies that $h$ is an extension of valuation rings which induces an isomorphism at the level of value groups. Therefore, we are done. Thus, we reduce to showing the statement for $g$.

Consequently, without loss of generality, we may assume that $A = V[x_1, \dots, x_n]$. We have that $\mathfrak{p} = \mathfrak{m}[x_1, \dots, x_n]$, where $\mathfrak{m} \subset V$ is the maximal ideal. To check that $A_\mathfrak{p}$ is a valuation ring, we verify that for any

$$t \in \operatorname{Frac}(A_\mathfrak{p}) = \operatorname{Frac}(V[x_1, \dots, x_n]), \text{ either } t \in A_\mathfrak{p} \text{ or } 1/t \in A_\mathfrak{p}.$$

Let $t = f/g \in \operatorname{Frac}(A_\mathfrak{p})$, where $f, g \in A$. Using the valuation on $V$, we define $\operatorname{val}(f) \in V$ (resp., $\operatorname{val}(g) \in V$) to be the element, which is well defined up to a unit in $V$, such that $f/\operatorname{val}(f) \in A \setminus \mathfrak{p}$ (resp., $g/\operatorname{val}(g) \in A \setminus \mathfrak{p}$). If $\operatorname{val}(g) \mid \operatorname{val}(f)$, then $t \in A_\mathfrak{p}$, otherwise, $1/t \in A_\mathfrak{p}$, and we are done. It remains to show that the morphism

$$\varphi\colon \operatorname{Frac}(V)^\times / V^\times \to \operatorname{Frac}(A)^\times / A_\mathfrak{p}^\times \text{ of value groups is an isomorphism.}$$

Since $\varphi$ is injective, it suffices to show that $\varphi$ is surjective. In a similar vein as the previous arguments, given a $t \in \operatorname{Frac}(A)^\times$, there exists a $u \in \operatorname{Frac}(V)^\times$ such that $t/u \in A_\mathfrak{p}^\times$. Thus, we are done. □

# Chapter 4

# The Toral Case of the Grothendieck–Serre Conjecture

In this chapter, extending the work of Colliot-Thélène and Sansuc from [CS87], we prove the conjecture of Grothendieck–Serre for torsors under tori over smooth algebras over valuation rings (Theorem 4.7). The techniques of our proof are heavily inspired by op. cit. and [GR18]. We prove purity of torsors under tori over smooth algebras over valuation rings (4.5.3) as an intermediary result (purity is used in the proof of Propositions 7.1.1 and 7.2.5). Before turning our discussion towards torsors, we motivate algebraic purity below.

Let $X$ be a Noetherian, normal, integral scheme and let $U$ be an open subscheme $U$ such that $X \setminus U \subset X$ is of codimension at least 2. The algebraic Hartogs' principle (cf. [Sta22, Tag 0BCS]) states that $H^0(X, \mathcal{O}_X) \cong H^0(U, \mathcal{O}_U)$; in other words, the inclusion $j\colon U \hookrightarrow X$ induces an isomorphism $\mathcal{O}_X \xrightarrow{\sim} j_*\mathcal{O}_U$. More generally, the 'algebraic purity' type statments, which incorporate the algebraic Hartogs' principle, concerns reflexive sheaves $\mathscr{F}$ (defined below) on $X$ and their pullback to open subschemes $U$ whose complement satisfies the Serre's $S_2$ condition [Sta22, Tag 033P].

Before introducing the notion of reflexive sheaves, we need to define the concept of coherent schemes, which we do below.

**4.1. Coherence.** Given a scheme $X$, an $\mathcal{O}_X$-module $\mathscr{F}$ is called *coherent* if it is of finite type and for every open $U \subseteq X$ and every finite collection $s_i \in \mathscr{F}(U)$, $i = 1, \dots, n$, the kernel of the associated morphism $\bigoplus_{i=1,\dots,n} \mathcal{O}_U \to \mathscr{F}$ is of finite type ([Sta22, Tag 01BV]). A coherent $\mathcal{O}_X$-module is finitely presented, and therefore, quasi-coherent ([Sta22, Tag 01BW]). A scheme $X$ is called *locally coherent* if $\mathcal{O}_X$ is a coherent module over itself ([GR18, Definition 8.1.54]). A ring $A$ is called *coherent* if any finitely generated ideal of $A$ is finitely presented ([Sta22, Tag 05CV]).

A scheme $X$ that is locally of finite presentation over a Prüfer domain $R$ is locally coherent. Indeed, since the property of being locally coherent is Zariski local, it suffices to check that any ring $A$ that is a finitely presented $R$-algebra is coherent. Let $f\colon A' := R[x_1, \dots, x_n] \twoheadrightarrow A$ be a presentation of $A$ such that $\ker(f) \subset A'$ is a finitely generated ideal. Since $\ker(f) \subset A'$ is finitely generated, it is enough to show that the ring $A'$ is coherent. Letting $I \subset A'$ be a finitely generated ideal, we shall show that $I$ is a finitely presented $A'$-module. Putting $X = \operatorname{Spec} A', S = \operatorname{Spec} R$ and $\mathscr{M} = \widetilde{I}$ in [RG71, Première partie, Théorème 3.4.6] (by [BouCA, Chapter I, §2.4, Proposition 3(ii)], [Sta22, Tag 090Q] and the fact that flatness is a local property [Sta22, Tag 0250], the $R$-torsion-free module $I$ is flat), we obtain that $I$ is a finitely presented $A'$-module, showing that $A'$ is coherent.

The schemes considered in this chapter are flat and finite type over Prüfer domains. Let $X$ be such a scheme; more precisely, let $X$ be a flat, finite type scheme over a Prüfer domain $R$. Thanks to [RG71, Première partie, Corollaire 3.4.7], $X$ is $R$-finitely presented, and hence, by the discussion above, it is a coherent scheme. As a result, the schemes considered in this chapter are coherent.

We introduce the notion of reflexive sheaves below, as promised.

**4.2. Reflexive sheaves.** Let $X$ be a scheme. The *dual* of an $\mathcal{O}_X$-module $\mathscr{F}$ is defined to be the $\mathcal{O}_X$-module $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$. A coherent $\mathcal{O}_X$-module $\mathscr{F}$ is called *reflexive* if for every $x \in X$, there is a neighbourhood $x \in U \subset X$ such that the canonical morphism

$$\beta_{\mathscr{F}|_U} \colon \mathscr{F}|_U \xrightarrow{\sim} \mathscr{F}|_U^{\vee\vee} \quad \text{is an isomorphism.}$$

Given a coherent $\mathcal{O}_X$-module $\mathscr{F}$ and a presentation

$$\mathcal{O}_X^{\oplus m} \longrightarrow \mathcal{O}_X^{\oplus n} \longrightarrow \mathscr{F} \longrightarrow 0, \tag{4.2.1}$$

we can dualise to obtain a short exact sequence

$$0 \longrightarrow \mathscr{F}^{\vee} \longrightarrow \mathcal{O}_X^{\oplus n} \longrightarrow \mathcal{O}_X^{\oplus m}. \tag{4.2.2}$$

Therefore, by [Sta22, Tag 01BY], if $X$ is a locally coherent scheme, then for a coherent sheaf (in particular, reflexive) $\mathscr{F}$, the dual $\mathscr{F}^{\vee}$ is coherent.

In [CS87], Colliot-Thélène and Sansuc introduced the concept of a flasque torus (discussed below) and showed that any torus has a resolution by flasque tori. Consequently, they were able to reduce problems involving torsors under tori to problems involving flasque tori. This simplification made it possible to employ Galois theoretic techniques in the study of torsors.

**4.3. Flasque Torus and Flasque Resolution.** We recall some definitions from [CS87, §0.5]. Let $G$ be a finite group. A finitely generated, free $\mathbb{Z}$-module $\mathcal{P}$ with a linear action of $G$ is called a *permutation module* if $\mathcal{P}$ admits a $G$-stable $\mathbb{Z}$-basis. A finitely generated, free $\mathbb{Z}$-module $\mathcal{F}$ with a linear action of $G$ is called *flasque* if

$$\mathrm{Ext}^1_{\mathbb{Z}[G]}(\mathcal{F}, \mathcal{P}) = 0 \quad \text{or, equivalently,} \quad H^1(G, \mathrm{Hom}_{\mathbb{Z}}(\mathcal{F}, \mathcal{P})) = 0,$$

for any permutation $\mathbb{Z}[G]$-module $\mathcal{P}$. For example, the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ is a permutation module, and as a consequence, for a flasque $\mathbb{Z}[G]$-module $\mathcal{F}$, we have $\mathrm{Ext}^1_{\mathbb{Z}[G]}(\mathcal{F}, \mathbb{Z}) = 0$.

Let $X$ be a scheme. An $X$-torus $T$ is called *isotrivial* if it is split by a finite étale surjection $\tilde{X} \to X$. The *character group* of an $X$-torus $T$ is the sheaf of abelian groups $T^{\vee} := \underline{\mathrm{Hom}}_{X\text{-gps}}(T, \mathbb{G}_{m,X})$. An isotrivial $X$-torus $T$ is called *quasi-trivial* (resp., *flasque*) if for any connected component $Z \subset X$, there exists a connected, Galois, finite étale cover $\tilde{Z} \to Z$ that splits $T$ such that the induced $\mathbb{Z}[\mathrm{Gal}(\tilde{Z}/Z)]$-module $T^{\vee}(\tilde{Z})$ is a permutation module (resp., is flasque) (see [CS87, Definition 1.2]). In fact, by op. cit. Lemma 1.1, any connected, Galois, finite étale cover $\tilde{Z} \to Z$ that splits $T$ can be chosen in the previous definition, and the choice of the closed subscheme structure on $Z \subset X$ is irrelevant. As we might expect, the notions of flasque and quasi-trivial tori are preserved under base change (cf., op. cit. Proposition 1.3). For a connected scheme $X$, any quasi-trivial torus $Q$ can be written as a finite product of Weil restrictions $\mathrm{Res}_{X_i/X}(\mathbb{G}_{m,X_i})$, for finite étale covers $X_i \to X$ (see [Čes22$_{\mathrm{Surv}}$, Lemma A.2.6]).

Thanks to [CS87, Proposition 1.3], given an isotrivial torus $T$ on a scheme $X$ whose connected components are open (for example, a scheme which has finitely many connected components, like the spectrum of a semilocal ring), there exists a *flasque resolution* of $T$, namely an exact sequence

$$1 \to F \to Q \to T \to 1, \text{ where } F \text{ is a flasque and } Q \text{ is a quasi-trivial } X\text{-torus.} \tag{4.3.1}$$

The following result, which takes inspiration from [GR18, Proposition 11.3.8] and [CS79, Lemma 2.1], is an important step in proving a weak version of the Auslander–Buchsbaum formula (4.5.1).

**Lemma 4.4.** *For a locally coherent scheme $X$, a quasi-compact open $j\colon U \hookrightarrow X$ such that at each point $z \in Z := X \setminus U$, we have*[4] $\operatorname{depth}(\mathscr{O}_{X,z}) \geq 2$*, and a reflexive $\mathscr{O}_X$-module $\mathscr{F}$, the restriction induces an isomorphism*

$$\mathscr{F} \xrightarrow{\sim} j_*j^*\mathscr{F}. \tag{4.4.1}$$

*Moreover, if $X$ is reduced, for a reflexive $\mathscr{O}_U$-module $\mathscr{G}$,*

$$\textit{the pushforward } j_*\mathscr{G} \textit{ is a reflexive } \mathscr{O}_X\textit{-module.} \tag{4.4.2}$$

*Proof.* (4.4.1): Thanks to [ČS21, Lemma 7.2.7(b)], the restriction induces an isomorphism

$$\mathscr{O}_X \xrightarrow{\sim} j_*\mathscr{O}_U. \tag{4.4.3}$$

We shall reduce to the special case $\mathscr{F} = \mathscr{O}_X$. Since it is enough to show (4.4.1) locally, given a reflexive $\mathscr{O}_X$-module $\mathscr{F}$, we may assume that there is a presentation (4.2.1) of $\mathscr{F}^\vee$, which can be dualised to obtain a short exact sequence like (4.2.2). Since $j_*$ is left exact, this gives us a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{O}_X^{\oplus n} \arrow[r] \arrow[d, "(4.4.3)"', "\sim"] & \mathscr{O}_X^{\oplus m} \arrow[d, "\sim"', "(4.4.3)"] \\
0 \arrow[r] & j_*j^*\mathscr{F} \arrow[r] & \mathscr{O}_X^{\oplus n} \arrow[r] & \mathscr{O}_X^{\oplus m},
\end{tikzcd}$$

from which we are reduced to the case when $\mathscr{F} = \mathscr{O}_X$, and we are done.

(4.4.2): In view of (4.4.1), it is enough to show that there exists a reflexive $\mathscr{O}_X$-module $\mathscr{F}$ such that $\mathscr{F}|_U = \mathscr{G}$. By [Sta22, Tag 0G41], there is a finitely presented $\mathscr{O}_X$-module $\mathscr{F}'$ such that $\mathscr{F}'|_U = \mathscr{G}$. Thanks to [Sta22, Tag 01BZ], keeping in mind that $X$ is locally coherent, the $\mathscr{O}_X$-module $\mathscr{F}'$ is automatically coherent. Taking $\mathscr{F} = \mathscr{F}'^{\vee\vee}$, we note that $\mathscr{F}|_U = \mathscr{F}'^{\vee\vee}|_U = \mathscr{G}^{\vee\vee} = \mathscr{G}$ (using the fact that $\mathscr{G}$ is reflexive). It remains to check that $\mathscr{F}$ is reflexive, for which we follow the proof of [Sta22, Tag 0AY4]. Since the result is local, it can be assumed that $X = \operatorname{Spec} A$ is affine. Choosing a presentation

$$A^{\oplus m} \to A^{\oplus n} \to \Gamma(\operatorname{Spec} A, \mathscr{F}') \to 0,$$

and dualising it, in order to conclude, it is sufficient to show the following claim.

[4] A module $M$ over a local ring $(A, \mathfrak{m})$ has $\operatorname{depth}_A(M) \geq d$, if there is an $M$-regular sequence $x_1, \ldots, x_d \in \mathfrak{m}$; the depth of $A$ is $\operatorname{depth}_A(A)$ (see [EGA IV$_1$, Chapitre 0, Définition 15.1.7 and §15.2.2)]). There is no condition on the quotients being nonzero.

Claim: Given an exact sequence

$$0 \to M \to M' \to M''$$

of finitely presented $A$-modules, the module $M$ is reflexive if $M'$ and $M''$ are reflexive.

Proof: We suppose that $M'$ and $M''$ are reflexive. Proceeding as in the proof of [Sta22, Tag 0EB8], we shall show that $M$ is reflexive. Double dualising the displayed short exact sequence in the claim and writing down canonical morphisms, we get the following morphism of complexes

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & M'' \arrow[d] \\ \mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A) \arrow[r, "\alpha"] & \mathrm{Hom}_A(\mathrm{Hom}_A(M', A), A) \arrow[r] & \mathrm{Hom}_A(\mathrm{Hom}_A(M'', A), A). \end{tikzcd}$$

By the assumption, the middle and the right vertical arrows are isomorphisms. We need to show that the left vertical arrow is an isomorphism. It suffices to show that $\alpha$ is injective. We consider module $Q$ defined by the exact sequence $\mathrm{Hom}_A(M', A) \to \mathrm{Hom}_A(M, A) \to Q \to 0$. Letting $K$ be the total ring of fractions of $A$ (see [Sta22, Tag 00EW]), by the finite presentation property [Sta22, Tag 0583], tensoring the exact sequence with $K$, we obtain the exact sequence

$$\mathrm{Hom}_K(M' \otimes_A K, K) \to \mathrm{Hom}_K(M \otimes_A K, K) \to Q \otimes_A K \to 0.$$

However, since $K$ is a product of fields, the injection $M \otimes_A K \hookrightarrow M' \otimes_A K$ is split, consequently, $Q \otimes_A K = 0$, implying that $Q$ is a torsion $A$-module. In that case, $\mathrm{Hom}_A(Q, A) = 0$, ensuring that $\alpha$ is injective. □

The following, which is inspired from [CS79, Lemma 2.1 and Corollary 6.9], proves purity for torsors under tori over smooth algebras over valuation rings (4.5.3). We can consider the following statement (4.5.1) as a weak version of the Auslander-Buchsbaum formula for smooth algebras over Prüfer domains. Given a flat (resp., smooth) group scheme $G$ over a scheme $S$ and an $S$-scheme $S'$, we let $\mathbf{B}G(S')$ denote the category of fppf locally (resp., étale locally) trivial $G$-torsors on $S'$. Likewise, let $\mathbf{B}G$ be the presheaf on the category of $S$-schemes defined by $S' \mapsto \mathbf{B}G(S')$ (see [Sta22, Tag 0048]).

**Proposition 4.5.** *Let $R$ be a Prüfer domain, let $X$ be a smooth, integral $R$-scheme and let $j\colon U \hookrightarrow X$ be a quasi-compact open such that at each point $x \in Z := X \setminus U$ with $f(x) = y$, we have $\dim(\mathcal{O}_{X_y,x}) + \min(1, \dim(R_y)) \geq 2$. Then, for a locally free $\mathcal{O}_U$-module $\mathscr{L}$ of rank $1$,*

$$\text{the pushforward } j_*\mathscr{L} \text{ is a locally free } \mathcal{O}_X\text{-module of rank } 1, \tag{4.5.1}$$

*in particular, for any étale $X$-scheme $X'$, the restriction induces an equivalence of categories*

$$\mathbf{B}\mathbb{G}_m(X') \xrightarrow{\sim} \mathbf{B}\mathbb{G}_m(X' \times_X U). \tag{4.5.2}$$

*More generally, for an $X$-torus $T$ and for any étale $X$-scheme $X'$, the restriction induces an equivalence of categories*

$$\mathbf{B}T(X') \xrightarrow{\sim} \mathbf{B}T(X' \times_X U), \tag{4.5.3}$$

*in particular,*

$$H^q(X', T) \cong H^q(X' \times_X U, T), \quad \text{for } q \leq 1. \tag{4.5.4}$$

*Proof.* We show that $\operatorname{depth}(\mathcal{O}_{X,z}) \geq 2$, at any point $z \in Z$. Let $f\colon X \to \operatorname{Spec} R$. Thanks to [EGA IV$_3$, Théorème 11.3.8] (specifically, (c) $\Longrightarrow$ (a)), it suffices to argue that $\operatorname{depth}(\mathcal{O}_{X_{f(z)},z}) + \min(1, \dim(R_{f(z)})) \geq 2$. It follows from the hypothesis and the equality $\operatorname{depth}(\mathcal{O}_{X_{f(z)},z}) = \dim(\mathcal{O}_{X_{f(z)},z})$, which is true because $f$ is smooth.

We show the key claim, i.e., (4.5.1), below. The claim (4.5.2) is a consequence of (4.4.1) and (4.5.1). Since the torus $T$ trivialises étale locally on $X$, the claims (4.5.3) and (4.5.4) reduce to (4.5.2) by an étale descent argument.

(4.5.1): We follow the proof of [GR18, Proposition 11.4.1(iv)]. A complex of sheaves on $X$ of abelian groups that is concentrated in cohomological degree 0 with the 0-th term $\mathscr{A}$ shall be denoted by $\mathscr{A}[0]$. Thanks to (4.4.2), the pushforward $\mathscr{M} := j_*\mathscr{L}$ is a reflexive $\mathcal{O}_X$-module, and also, torsion-free and hence, flat over $R$. Assuming that $\mathscr{M}[0]$ is a perfect $\mathcal{O}_X$-complex, the result follows. Indeed, letting $\det(\mathscr{M}[0])$ be the determinant line bundle of $\mathscr{M}[0]$ (see [KM76, Theorem 1]), there is a sequence of isomorphisms

$$\mathscr{M} \xrightarrow{\sim} j_*\mathscr{L} \xrightarrow{\sim} j_*\det(\mathscr{L}[0]) \xrightarrow{\sim} j_*j^*(\det(\mathscr{M}[0])) \xleftarrow[\sim]{(4.4.1)} \det(\mathscr{M}[0]),$$

from which the result follows. We verify that $\mathscr{M}[0]$ is a perfect $\mathcal{O}_X$-complex. It suffices to assume that $X = \operatorname{Spec} A$ is affine and to show that $\mathscr{M}[0]$ is quasi-isomorphic to a bounded complex of finite free $A$-modules ([Sta22, Tag 0BCJ]). Let $M := \Gamma(A, \mathscr{M})$. By [Sta22, Tag 0G9A], it suffices to show that

$$\operatorname{Ext}^q_A(M, N) = 0 \text{ for any finitely presented } A\text{-module } N \text{ and any } q \gg 0 \tag{4.5.5}$$

(the complex $M[0]$ is pseudo-coherent because $M$ is a coherent $A$-module). Using the finitely presented property of the variable $N$ in (4.5.5), it is enough to show that there exists an integer $n$ such that $\operatorname{proj.dim}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq n$ for any prime $\mathfrak{p} \subset A$ (see [Wei94, Lemma 3.3.8]). This follows from [GR18, Proposition 11.4.1(ii)] (or [Guo22, Lemma 7.2(i)]), which shows that taking $n = \dim_R(A)$ suffices. Hence, we are done.

(4.5.2): Let $j'\colon U' := X' \times_X U \hookrightarrow X$ be the inclusion. Since $\dim_X(X') = 0$, we obtain that $(X', U')$ satisfies the hypothesis of Proposition 4.5. We shall show that the pushforward $j'_*\colon \mathbf{B}\mathbb{G}_m(U') \to \mathbf{B}\mathbb{G}_m(X')$, which is well defined as a consequence of (4.5.1), is an inverse to the pullback $j'^*\colon \mathbf{B}\mathbb{G}_m(X') \to \mathbf{B}\mathbb{G}_m(U')$. Since the equality $j'^*j'_* = \mathrm{id}$ follows from the definition, it suffices to show that for a line bundle $\mathscr{L} \in \mathbf{B}\mathbb{G}_m(X')$, the restriction induces an isomorphism $\mathscr{L} \xrightarrow{\sim} j'_*j'^*\mathscr{L}$. However, this results from (4.4.1).

(4.5.3): Let $U' := X' \times_X U$. Thanks to [Sta22, Tag 04UK], $\mathbf{B}T_{X'}$ satisfies étale descent, consequently, the same holds for the presheaf $j_*\mathbf{B}T_{U'}$. By [SGA 3$_{\mathrm{II}}$, Exposé X, Corollaire 4.5], there exists an étale surjection $\tilde{X} \to X'$ that splits $T$. In view of the étale descent property of $\mathbf{B}T_{X'}$ and $j_*\mathbf{B}T_{U'}$, it suffices to show that for any étale $\tilde{X}$-scheme $X''$, the restriction induces an equivalence of categories $\mathbf{B}T_{\tilde{X}}(X'') \xrightarrow{\sim} \mathbf{B}T_{\tilde{X}}(X'' \times_{X'} U')$, which is the content of (4.5.2).

(4.5.4) This follows from (4.5.3). Indeed, $H^0(X', T)$ (resp., $H^0(U', T)$) is the automorphism group of the trivial torsor in $\mathbf{B}T(X')$ (resp., in $\mathbf{B}T(U')$) and $H^1(X', T)$ (resp., $H^1(U', T)$) is the group of isomorphism classes of objects in $\mathbf{B}T(X')$ (resp., in $\mathbf{B}T(U')$). $\square$

We follow the arguments in [Guo22, §2] to prove the Grothendieck–Serre conjecture for torsors under tori over semilocal Prüfer domains. As a consequence, we obtain that the Brauer group of a semilocal Prüfer domain injects into the Brauer group of its fraction field (putting $F = \mathbb{G}_m$ in Lemma 4.6).

**Lemma 4.6.** *Given a semilocal Prüfer domain $R$ with a fraction field $K$ and a flasque $R$-torus $F$,*

- $\theta_1\colon H^1(R,F) \to H^1(K,F)$ *is surjective, and*
- $\theta_2\colon H^2(R,F) \to H^2(K,F)$ *is injective.*

*Proof.* We reduce to proving the statements for semilocal Prüfer domains of finite Krull dimension. Thanks to Lemma 3.3(c), the ring $R$ is an increasing union of its subrings that are semilocal Prüfer domain of finite Krull dimension. Therefore, since étale cohomology commutes with filtered colimits of rings (see [Sta22, Tag 09YQ]), without loss of generality, we may assume that $R$ is of finite Krull dimension.

Following the last paragraph of the proof of [Guo22, Proposition 2.4(i)], to prove the assertion for both $\theta_1$ and $\theta_2$, we induct on

$$d(R) := \sum_{\mathfrak{m}\in \mathrm{MaxSpec}(R)} \dim(R_{\mathfrak{m}}).$$

If $d(R) = 0$, then $R$ is a field and the result is trivial. Assuming that the result is true for all semilocal Prüfer domains $R'$ such that $d(R') = d \geq 0$, we shall prove that it is true for a Prüfer domain $R$ such that $d(R) = d+1$. Letting $\mathfrak{m} \subset R$ be a maximal ideal, by [SGA $4_{\mathrm{II}}$, Exposé V, Proposition 6.5], there is a long exact sequence

$$\cdots \longrightarrow H^1(R,F) \longrightarrow H^1(\operatorname{Spec} R\setminus\{\mathfrak{m}\},F) \longrightarrow H^2_{\mathfrak{m}}(R,F) \longrightarrow H^2(R,F) \longrightarrow H^2(\operatorname{Spec} R\setminus\{\mathfrak{m}\},F) \longrightarrow \cdots.$$

The open subscheme $\operatorname{Spec} R\setminus\{\mathfrak{m}\} \subset \operatorname{Spec} R$ is the spectrum of a semilocal Prüfer domain (see Lemma 3.3(a) and Lemma 3.2(a)), say $R'$, such that $d(R') = d$. Consequently, by induction hypothesis, the corresponding morphism $\theta_1$ (resp., $\theta_2$) for $R'$ is surjective (resp., injective). Hence, it is enough to show that $H^2_{\mathfrak{m}}(R,F) = 0$. By excision [Mil80, Chapter III, Proposition 1.27], we have $H^2_{\mathfrak{m}}(R,F) \cong H^2_{\mathfrak{m}}(R_{\mathfrak{m}},F) \cong 0$. The second isomorphism follows from the fact that $R_{\mathfrak{m}}$ is a valuation ring by using [Guo22, Lemma 2.3]. ☐

Below we show that the Brauer group of an integral domain that is smooth over a valuation ring injects into the Brauer group of its fraction field (putting $F = \mathbb{G}_m$ in the isomorphism (4.7.2)). In similar vein as in the proof of Lemma 4.6, it suffices to show the vanishing of local cohomology $H^2_{\operatorname{Spec} A\setminus U}(A,F)$, which follows from the purity of torsors under tori (Proposition 4.5). The techniques that are used in the following proof to reduce to the local case, i.e., the coniveau spectral sequence and the local to global spectral sequence, are standard.

**Theorem 4.7.** *Given a semilocal Prüfer domain $R$, an integral domain $A$ that is $R$-essentially smooth, a quasi-compact open $U \hookrightarrow \operatorname{Spec} A$, and a flasque $A$-torus $F$,*

- $\theta_1^F\colon H^1(A,F) \to H^1(U,F)$ *is surjective, and* (4.7.1)
- $\theta_2^F\colon H^2(A,F) \to H^2(U,F)$ *is injective.* (4.7.2)

*Proof.* We reduce to showing (4.7.1) and (4.7.2) for integral domains that are $R$-smooth. Let $\mathcal{A}$ be an $R$-smooth, integral domain such that $A$ is a semilocalisation of $\mathcal{A}$.Assuming that Theorem 4.7 holds for integral domains of the form $\mathcal{A}[\frac{1}{f}]$, for some $f \in \mathcal{A}$, a limit argument and the facts that étale cohomology commutes with filtered colimits of rings (see [Sta22, Tag 09YQ]) and that colimits commute with cokernels will then show that (4.7.1) is true for $A$. In similar vein, we reduce to showing (4.7.2) for rings of the form $\mathcal{A}[\frac{1}{f}]$, for some $f \in \mathcal{A}$. Thus, without loss of generality, we assume that $A$ is $R$-smooth.

Similar to the beginning of the proof of Lemma 4.6, thanks to Lemma 3.3(c), by a limit argument (see [Sta22, Tag 09YQ]), we may assume that $R$ has finite Krull dimension. Moreover, writing a long exact sequence of cohomology with supports, in a similar vein to the proof of Lemma 4.6, it is enough to show that $H^2_Z(A, F) = 0$, where $Z := \operatorname{Spec} A \setminus U$. Thanks to the coniveau spectral sequence [ILO14, Exposé XVIII-A, §2.2.1] (see also [Gro68c, Section 10.1], which is applicable because, by Remark 3.9, the topological space $\operatorname{Spec} A$ is Noetherian)

$$E_1^{p,q}\colon \bigoplus_{z \in Z \text{ with } \dim A_z = p} H^{p+q}_{\{z\}}(A_z, F) \Rightarrow H^{p+q}_Z(A, F), \tag{4.7.3}$$

we are reduced to showing the vanishing

$$H^{q \le 2}_{\{z\}}(A_z, F) = 0, \text{ for each } z \in Z. \tag{4.7.4}$$

If $z$ is a generic point of an $R$-fibre, then Lemma 3.10 shows that $A_z$ is a valuation ring. Indeed, letting $x$ be the image of $z$ along $\operatorname{Spec} A \to \operatorname{Spec} R$, the image of $z$ in $\operatorname{Spec} A_x$ is a generic point of the special fibre over the valuation ring $R_x$. In this case, thanks to the proof of Lemma 4.6, we obtain the vanishing (4.7.4). If $z$ lies in the generic fibre, then $A_z$ is a $\operatorname{Frac}(R)$-algebra, for which we have to show the vanishing (4.7.4). Again writing a long exact sequence of cohomology with supports, we need to show that $\theta_1^F$ is surjective and $\theta_2^F$ is injective for $A_z$, which is the content of [CS87, Theorem 2.2].

Consequently, it reduces to showing the vanishing (4.7.4) for a $z \in Z$ that is not a generic point of any $R$-fibre of $A$ and that does not lie in the $R$-generic fibre. Thanks to the local to global spectral sequence ([SGA $4_{\mathrm{II}}$, Exposé V, Proposition 6.5])

$$E_2^{p,q}\colon H^p(A_z, \mathscr{H}^q_{\{z\}}(A_z, F)) \Rightarrow H^{p+q}_{\{z\}}(A_z, F),$$

it suffices to show that $\mathscr{H}^{q\le 2}_{\{z\}}(A_z, F) = 0$. By taking stalks, it is equivalent to show that $H^{q \le 2}_{\{\overline{z}\}}(A_{\overline{z}}, F) = 0$. The long exact sequence of cohomology with supports

$$\begin{aligned}
0 \longrightarrow H^0_{\{\overline{z}\}}(A_{\overline{z}}, F) &\longrightarrow H^0(A_{\overline{z}}, F) \longrightarrow H^0(\operatorname{Spec} A_{\overline{z}} \setminus \{\overline{z}\}, F) \\
\longrightarrow H^1_{\{\overline{z}\}}(A_{\overline{z}}, F) &\longrightarrow H^1(A_{\overline{z}}, F) \longrightarrow H^1(\operatorname{Spec} A_{\overline{z}} \setminus \{\overline{z}\}, F) \\
\longrightarrow H^2_{\{\overline{z}\}}(A_{\overline{z}}, F) &\longrightarrow H^2(A_{\overline{z}}, F) \longrightarrow \cdots
\end{aligned}$$

and the vanishing of the étale cohomology of strictly Henselian rings [Sta22, Tag 03QO], which has the consequence that $H^{n\ge 1}(A_{\overline{z}}, F) = 0$, implies that

$$H^2_{\{\overline{z}\}}(A_{\overline{z}}, F) \cong H^1(\operatorname{Spec} A_{\overline{z}} \setminus \{\overline{z}\}, F). \tag{4.7.5}$$

To conclude, it remains to show that $H^q(\operatorname{Spec} A_{\overline{z}}, F) \cong H^q(\operatorname{Spec} A_{\overline{z}} \setminus \{\overline{z}\}, F)$, for $q = 0, 1$. Since the étale cohomology commutes with limits of schemes, by the definition of the strict Henselisation, it suffices to apply Proposition 4.5 to $X = \operatorname{Spec} A$, $Z = \overline{\{z\}}$ and $T = F$ (Remark 3.9 implies that $\operatorname{Spec} A$ is a Noetherian topological space, and hence, any open subset of $\operatorname{Spec} A$ is quasi-compact). □

Below, we prove the Grothendieck–Serre conjecture for torsors under tori over smooth algebras over valuation rings (4.8.1). By using a flasque resolution (see (4.3.1)), we reduce to showing a liftability condition for torsors under flasque tori (4.7.1).

**Corollary 4.8.** *For a semilocal Prüfer domain $R$, a semilocal, integral domain $A$ that is $R$-essentially smooth and an $A$-torus $T$, the morphism*

$$\theta_1^T \colon H^1(A, T) \to H^1(K, T) \qquad \textit{is injective.} \tag{4.8.1}$$

*Proof.* Since $T$ is commutative, showing that $\ker \theta_1^T = 0$ is enough to prove (4.8.1). Letting $1 \to F \to Q \to T \to 1$ be a flasque resolution (4.3.1), we get the following morphism of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & H^1(A,F) \arrow[r] \arrow[d, "\theta_1^F"] & H^1(A,Q) \arrow[r] \arrow[d] & H^1(A,T) \arrow[r] \arrow[d, "\theta_1^T"] & H^2(A,F) \arrow[r] \arrow[d, "\theta_2^F"] & \cdots \\ \cdots \arrow[r] & H^1(K,F) \arrow[r] & H^1(K,Q) \arrow[r] & H^1(K,T) \arrow[r] & H^2(K,F) \arrow[r] & \cdots . \end{tikzcd} \tag{4.8.2}$$

Since $Q$ is quasi-trivial, there are connected finite étale covers $A \to A_i$ such that $Q \cong \prod \operatorname{Res}_{A_i/A}(\mathbb{G}_m)$. By the fact that higher direct images vanish along finite morphisms [Sta22, Tag 03QP], the cohomology rewrites itself as $H^1(A, Q) \cong \prod H^1(A_i, \mathbb{G}_m)$, and since all the rings $A_i$ are semilocal (since an étale morphism is quasi-finite), the cohomology vanishes thanks to the Hilbert theorem 90 [Mil80, Chapter III, Section 4, Proposition 4.9]. By a similar argument, the cohomology $H^1(K, Q)$ vanishes, and in view of (4.8.2), to prove (4.8.1), it is enough to show that $\theta_2^F$ is injective. By using the fact that the étale cohomology commutes with filtered colimits of rings, this is a consequence of Theorem 4.7. □

**Corollary 4.9.** *For a semilocal Prüfer domain $R$, a semilocal, integral domain $A$ that is $R$-essentially smooth, a flasque $A$-torus and a nonempty, quasi-compact open $U \subset \mathbb{A}^n_A$, the composite morphism*

$$H^1(A, F) \to H^1(\mathbb{A}^n_A, F) \to H^1(U, F) \quad \textit{is surjective.}$$

*Proof.* Thanks to (4.7.1), the second morphism is surjective. Consequently, it reduces us to proving the surjectivity of the first morphism. In fact, we shall show that

$$H^1(A, F) \xrightarrow{\sim} H^1(\mathbb{A}^n_A, F).$$

Assuming that $A$ is normal, by a limit argument, it is enough to show the displayed isomorphism for Noetherian, normal domains, which is the content of [CS87, Lemma 2.4]. Therefore, it remains to show that $A$ is normal. Let $X$ be an integral, smooth $R$-scheme such that $A$ is the semilocalisation of $X$. By the local structure of smooth morphisms [SGA 1, Exposé II, Théorème 4.10(ii)] (cf. [Sta22, Tag 052E]), given any $x \in X$, there are an open neighbourhood $V \subseteq X$ of $x$ and an étale morphism $V \to \mathbb{A}^d_R$, for some $d \geq 0$. Since normality can be checked locally (see [Sta22, Tag 00GY]), the permanence properties of normality [Sta22, Tag 030A] and [SGA 1, Exposé I, Théoréme 9.5(i)] (cf. [Mil80, Chapter I, Proposition 3.17(b)]) imply that $X$ is a normal scheme, which is enough to conclude that $A$ is normal. □

# Chapter 5

# The Grothendieck–Serre Conjecture for Semilocal Prüfer Domains

In this chapter, the main result is Theorem 5.11, where we prove that a generically trivial torsor under a reductive group $G$ over a semilocal Prüfer domain $R$ is trivial. Unlike the rest of this thesis, the theorem applies to non-quasi-split reductive groups $G$. Similarly, there is no assumption on the Krull dimension of $R$; it can be arbitrary. The theorem is an extension of [Guo20, Theorem 1.2] where the Grothendieck–Serre conjecture was settled in the case of semilocal Dedekind domains $R$. Theorem 5.11 can be regarded as the semilocal version of [Guo22, Theorem 1.3], where Conjecture 2.1.2 was proved in the case of valuation rings. Closely following the proof of loc. cit., the key result to show is the 'patching formula' or the 'weak approximation result' (Proposition 5.10). Using the patching formula, the proof of Theorem 5.11 reduces to the case of complete valuation rings of rank 1.

We start with recalling some necessary definitions from [SGA $3_{\mathrm{II}}$; SGA $3_{\mathrm{III}}$; Con14].

**5.1. Reminders on Reductive Groups.** Let $S$ be a scheme. A smooth, affine $S$-group scheme $G$ is *reductive* if it has reductive (assumed to be connected) geometric fibres ([SGA $3_{\mathrm{III}}$, Exposé XIX, Définition 2.7]). A smooth, affine $S$-subgroup $T$ of a reductive $S$-group scheme is called a *maximal torus* if its geometric fibres are maximal tori (see [Con14, Definition 3.2.1]). A smooth $S$-subgroup $P$ of a reductive $S$-group scheme $G$ is called *parabolic* if for each geometric point $\operatorname{Spec} \overline{k} \to S$, the quotient $G_{\overline{k}}/P_{\overline{k}}$ is represented by a proper $\operatorname{Spec} \overline{k}$-scheme (see [SGA $3_{\mathrm{III}}$, Exposé XXVI, Définition 1.1]). A parabolic subgroup $B \subset G$ is called a *Borel subgroup* if its geometric fibres $B_{\overline{s}} \subset G_{\overline{s}}$ are maximal connected, solvable, linear algebraic subgroups (see [Con14, Definition 5.2.10]). A smooth $S$-subgroup $L$ of a parabolic $S$-subgroup $P$ of a reductive $S$-group scheme is called *Levi* if $L \rtimes \mathscr{R}_u(P) \to P$ is an isomorphism (see [Con14, Definition 5.4.2]). A reductive $S$-group is called *anisotropic* if it contains no subgroup scheme isomorphic to $\mathbb{G}_{m,S}$ (cf. [SGA $3_{\mathrm{III}}$, Exposé XXVI, Définition 6.13 and Corollaire 6.14]).

By [SGA $3_{\mathrm{II}}$, Exposé XIV, Théorème 6.1], the functor of maximal subtori $\underline{\mathrm{Tor}}(G)$ is represented by a finitely presented, smooth, affine scheme. In fact, given $T \in \underline{\mathrm{Tor}}(G)(S)$ there is an isomorphism $G_S/\mathrm{N}_{G_S}(T) \xrightarrow{\sim} \underline{\mathrm{Tor}}(G_S)$ given by $g \mapsto gTg^{-1}$ ( [SGA $3_{\mathrm{III}}$, Exposé XXII, Corollaire 5.8.3]), where the normaliser $N_{G_S}(T)$ is represented by a smooth, affine scheme (see [SGA $3_{\mathrm{II}}$, Exposé XI, Corollaire 2.4bis]).

We commence with lemmas (Lemmas 5.3-5.9) that parallel Harder-type approximations in preparation for the proof of Proposition 5.10 after introducing their setting below.

**5.2. Setting of the Lemmas 5.3-5.9.** Let $R$ a semilocal Prüfer domain of finite, positive Krull dimension with a fraction field $K$, let $\mathfrak{m} \subset R$ a maximal ideal and let $G$ be a reductive $R$-group scheme. Let $a \in R$ be such that $V(a) = \{\mathfrak{m}\}$ (such an element exists thanks to Remark 3.5). We shall endow $R$ with the $a$-adic topology. Let $\widehat{R}^a$ be the $a$-adic completion (which is an $a$-adically complete valuation ring of rank $\leq 1$ thanks to Remark 3.5) of $R$. We endow $R[\frac{1}{a}]$ (resp., $K$) with the unique ring topology, called the *$a$-adic topology*, for which the morphism $R \hookrightarrow R[\frac{1}{a}]$ (resp., $R \hookrightarrow K$) is continuous and open (see [BČ20, §2.1.9]). We note that the $a$-adic topology on $R[\frac{1}{a}]$ (resp., $K$) is not linear. Following loc. cit., we define the *$a$-adic completion*, denoted $\widehat{R[\frac{1}{a}]}^a$ (resp., $\widehat{K}^a$), of $R[\frac{1}{a}]$ (resp., $K$) to be

$$\lim_{m\geq 1}(R[\tfrac{1}{a}]/\mathrm{Im}(a^m R \to R[\tfrac{1}{a}])) \quad (\text{resp., } \lim_{m\geq 1}(K/\mathrm{Im}(a^m R \to K))).$$

By, for example op. cit. Example 2.1.10(2), since $R$ is an integral domain (in particular, $a$-torsion free),

$$\widehat{R[\tfrac{1}{a}]}^a = \widehat{K}^a = \mathrm{Frac}(\widehat{R}^a).$$

Let $B$ be a commutative, unital, topological ring. We recall that, by [Con12, Proposition 2.1], there is a unique way, functorial in $X$ and $B$, to topologise (compatibly with the formation of fibre products) the set $X(B)$, for any finite type, affine $B$-scheme $X$; so that for $X = \mathbb{A}^n_B$, the topological space $\mathbb{A}^n_B(B)$ identifies itself with $B^n$. In the following lemmas, we shall endow the sections of $\underline{\mathrm{Tor}}(G)$ with the $a$-adic topology.

**Lemma 5.3.** *Assuming the notations of 5.2,*

$$\text{the image of } \underline{\mathrm{Tor}}(G)(R[\tfrac{1}{a}]) \to \underline{\mathrm{Tor}}(G)(\widehat{K}^a) \text{ is dense.} \tag{5.3.1}$$

*Proof.* We shall follow the proof of [Guo22, Lemma 3.7]. To prove the required density (5.3.1), we use the idea of Cauchy nets [BČ20, Section 2.1.12]. Given a Cauchy sequence $(x_n)_{n\geq 0}$ in $R[\frac{1}{a}]$ in the $a$-adic topology, we define the $m$-truncation as the Cauchy sequence $(x_n)_{n\geq m}$. The $m$-truncated Cauchy sequences form a set which we denote by $\mathrm{Cauchy}^{\geq m}(R[\frac{1}{a}])$, and in fact, it can be endowed with a ring structure under termwise addition and multiplication. The first sentence of this proof implies that there is a surjection

$$\mathrm{colim}_m \left(\mathrm{Cauchy}^{\geq m}(R[\tfrac{1}{a}])\right) \twoheadrightarrow \widehat{K}^a, \tag{5.3.2}$$

whose kernel is a maximal ideal, given that the target is a field. The fact that $\widehat{K}^a$ is $a$-adically separated ensures that any Cauchy sequence $(x_n)_{n\geq m} \in \mathrm{Cauchy}^{\geq m}(R[\frac{1}{a}])$ that converges to a nonzero element in $\widehat{K}^a$ is eventually nonzero, implying that the germ of the sequence has a multiplicative inverse obtained by inverting each of the terms of a germ of the sequence; in other words the source of the surjection (5.3.2) is a local ring. Finally, applying [Guo22, Lemma 3.3] (applicable because $\widehat{K}^a$ is an infinite field) to the finitely presented, affine $R[\frac{1}{a}]$-scheme $\underline{\mathrm{Tor}}(G)$, we obtain a surjection

$$\underline{\mathrm{Tor}}(G)(\mathrm{colim}_m \left(\mathrm{Cauchy}^{\geq m}(R[\tfrac{1}{a}])\right)) \cong \mathrm{colim}_m \left(\underline{\mathrm{Tor}}(G)(\mathrm{Cauchy}^{\geq m}(R[\tfrac{1}{a}]))\right) \twoheadrightarrow \underline{\mathrm{Tor}}(G)(\widehat{K}^a). \qquad \square$$

Before stating Lemma 5.6, we need to recall some facts about the Weil restrictions of quasi-projective schemes along finite, flat, finitely presented morphisms of schemes. The main references are [BLR90, Section 7.6], [CGP15, Appendix A.5] and [Sta22, Tag 05Y8]. The Weil restriction will again be needed in the proof of Proposition 7.2.1(b).

**5.4. Weil restriction.** Given a finite, flat, finitely presented morphism $f\colon X \to Y$ of schemes and an $X$-scheme $T$, the sheaf of sets $\mathrm{Res}_{Y/X}(T_Y) := f_*(T_Y)$ is represented by an $X$-algebraic space (see [Sta22, Tag 05YF]). Moreover, if $T$ is quasi-projective, thanks to [BLR90, Section 7.6, Theorem 4] (for a brief discussion on why quasi-projective schemes satisfy the hypothesis of loc. cit., see, for example, [CGP15, Appendix A.5]), $\mathrm{Res}_{Y/X}(T_Y)$ is represented by an $X$-scheme. In the latter case, by op. cit. Proposition A.5.2(1), the formation of $\mathrm{Res}_{Y/X}(T_Y)$ is naturally compatible with extension of scalars on $X$ (although, loc. cit. is for Noetherian schemes, its proof works in our setting).

We also need to define the notion of a norm morphism of a group scheme for a finite étale covering of schemes (cf. the trace morphism in the theory of the étale cohomology [Sta22, Tag 03SH]).

**5.5. Norm morphism.** For a Galois, finite étale cover $f\colon Y \to X$ of order $n$ of a scheme $X$ with Galois group $\Gamma$ and an $X$-group scheme $T$, we define the *norm morphism*

$$\mathrm{Norm}_{Y/X}\colon T(Y) \to T(X) \text{ by } t \mapsto \textstyle\prod_{g\in\Gamma} g\cdot t$$

(the action of $\Gamma$ on $T(Y)$ is induced by the action of the Galois group on $f$). Since $\mathrm{Norm}_{Y/X}$ is functorial in $X$, it upgrades to a morphism $u\colon \mathrm{Res}_{Y/X}(T_Y) \to T$. By the discussion above, if $T$ is quasi-projective, then

$$\mathrm{Res}_{(Y\times_X Y)/Y}(T_Y) \xrightarrow{\sim} \big(\mathrm{Res}_{Y/X}(T_Y)\big) \times_X Y.$$

As a consequence, since $Y\times_X Y \to Y$ is a split, Galois, finite étale cover with Galois group $\Gamma$,

$$T_Y^\Gamma \xrightarrow{\sim} \big(\mathrm{Res}_{Y/X}(T_Y)\big) \times_X Y.$$

Therefore, $u_Y$ identifies with the morphism $T_Y^\Gamma \to T_Y$ defined by $(t_g)_{g\in\Gamma} \mapsto \prod_{g\in\Gamma} t_g$. Consequently, $u$ is smooth if $T$ is, in addition, smooth. Indeed, it is enough to argue that the base change $u_Y$ of $u$ along $f$ is smooth (see [Sta22, Tag 02VL]). To show that $u_Y$ is smooth, fixing a set theoretic bijection $\Gamma \cong \{1,2,\dots,n\}$, we note that $u_Y$ can be written as composition of the sheaf-theoretic automorphism mult of $T_Y^\Gamma$, defined by $(t_1,\dots,t_n) \mapsto (t_1, t_1t_2, \dots, t_1t_2\dots t_n)$, and the projection onto the last factor $\mathrm{pr}_n\colon T_Y^\Gamma \to T_Y$. Since both mult and $\mathrm{pr}_n$ are smooth, $u_Y$ is smooth.

We are ready to state the lemma.

**Lemma 5.6.** *Assuming the notations of 5.2, for a $\widehat{K}^a$-torus $T$, a finite Galois extension $\widehat{K}^a \to L$, the norm map*

$$\mathrm{Norm}_{L/\widehat{K}^a}\colon T(L) \to T(\widehat{K}^a) \tag{5.6.1}$$

*is open in the $a$-adic topology.*

*Proof.* We follow the proof of [Čes15, Proposition 2.9(a)]. Since $T_{\widehat{K}^a}$ is a smooth, affine $\widehat{K}^a$-scheme, in particular, it is quasi-projective, thanks to §5.4-5.5, the sheaf $T' := \mathrm{Res}_{L/\widehat{K}^a}(T_L)$ is represented by a $\widehat{K}^a$-scheme and the norm morphism $T' \to T$ is smooth. To show that $\mathrm{Norm}_{L/\widehat{K}^a}$ is open, it is enough to argue locally in the source. By definition, $\widehat{K}^a$ is a Hausdorff topological field. Therefore, thanks to [Con12, Proposition 3.1], an open immersion $Y \hookrightarrow X$ of finite type, affine $\widehat{K}^a$-schemes induces a continuous, open morphism $Y(\widehat{K}^a) \hookrightarrow X(\widehat{K}^a)$. Hence, it is suffices to show that for each $x \in T'$, there exists an open neighbourhood $U \subseteq T'$ of $x$ such that the induced morphism $U(\widehat{K}^a) \to T(\widehat{K}^a)$ is open. Let $x \in T'$. By the local structure of smooth morphisms [SGA 1, Exposé II, Théorème 4.10(ii)] (cf. [Sta22, Tag 052E]), there are an open $x \in U \subseteq T'$ and an étale morphism $U \to \mathbb{A}^d_T$, for some $d \geq 0$. The canonical projection $\mathbb{A}^d_T(\widehat{K}^a) \to T(\widehat{K}^a)$ is open. Indeed, it is a composition of

$$\mathbb{A}^d_T(\widehat{K}^a) \xrightarrow{\sim} (\widehat{K}^a)^d \times T(\widehat{K}^a)$$

(to argue that this is an isomorphism we can embed $T \hookrightarrow \mathbb{A}^d_{\widehat{K}^a}$ and then use use functoriality properties of the topology as in [Čes15, Claim 2.2.1]) and the projection

$$(\widehat{K}^a)^d \times T(\widehat{K}^a) \to T(\widehat{K}^a)$$

(projections are open). Consequently, it suffices to argue that $\varphi\colon U(\widehat{K}^a) \to \mathbb{A}^d_T(\widehat{K}^a)$ is open. Since $\widehat{R}^a$ is a complete valuation ring of rank 1, in particular, it is Henselian, thanks to [GGM14, Proposition 3.1.4], $\varphi$ is open, and we are done. □

**Lemma 5.7.** *Assuming the notations of 5.2, for an $R[\frac{1}{a}]$-torus $T$,*

(*a*) *there is a minimal Galois, finite étale extension $R[\frac{1}{a}] \to S$ that splits $T$, and for such an extension we have an isomorphism*

$$S \otimes_{R[\frac{1}{a}]} \widehat{K}^a \cong \textstyle\prod_{i=0}^r L_i,$$

*where each $L_i$ is a minimal splitting field of $T_{\widehat{K}^a}$, and*

(*b*) *given a minimal finite Galois extension $\widehat{K}^a \to L_0$ that splits $T_{\widehat{K}^a}$, the image $U$ of the norm map (5.6.1) is contained in the a-adic closure $\overline{T(R[\frac{1}{a}])}$ of the image of $T(R[\frac{1}{a}])$ in $T(\widehat{K}^a)$.*

*Proof.* (a): The torus $T$ over the normal domain $R[\frac{1}{a}]$ is isotrivial, i.e., splits after a finite étale extension (use, for example, a limit argument to reduce to the Noetherian case [SGA $3_{\text{II}}$, Exposé X, Théorème 5.16] as in [Guo22, Lemma 2.2]), which can be assumed to be a minimal Galois, finite étale extension after taking the minimal subextension of the Galois closure [SGA 1, Exposé V, §4, page 100, part (g)] that splits $T$. To show the minimality of $L_i$, we follow the proof of [Guo22, Lemma 3.9(i)]. The equivalence [SGA $3_{\text{II}}$, Exposé X, Corollaire 1.2] associates the isotrivial torus $T$ with its character lattice with Galois action, more precisely, the $\pi_1^{\text{ét}}(R[\frac{1}{a}])$-module $\mathrm{Hom}_{R[\frac{1}{a}]\text{-gps}}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$, where $n$ is the rank of $T$ as a multiplicative group. Letting $\rho\colon \pi_1^{\text{ét}}(R[\frac{1}{a}]) \to \mathrm{GL}_n(\mathbb{Z})$ be the associated Galois representation, the minimality of $S$ is equivalent to the fact that $\pi_1^{\text{ét}}(S) = \ker(\rho)$. In the displayed isomorphism, the left hand side is a product of fields which are finite separable extensions of $\widehat{K}^a$. Composing $\rho$ with the canonical morphism $\pi_1^{\text{ét}}(\widehat{K}^a) \to \pi_1^{\text{ét}}(R[\frac{1}{a}])$, we get the induced Galois representation $\widehat{\rho}^a\colon \pi_1^{\text{ét}}(\widehat{K}^a) \to \mathrm{GL}_n(\mathbb{Z})$ associated to $T_{\widehat{K}^a}$ under the equivalence of loc. cit. In a similar vein, the minimality of $L_i$ is equivalent to the fact that $\pi_1^{\text{ét}}(L_i) = \ker(\widehat{\rho}^a)$, which follows from the functoriality of the étale fundamental group [SGA 1]Exposé V, Proposition 6.1. Finally, we remark that throughout we have omitted the necessary base point from the notations of the étale fundamental groups, because all the schemes whose étale fundamental groups are written are spectra of integral domains for which any geometric point can be chosen.

(b): We follow the proof of [Guo22, Lemma 3.9(ii)]. To show that $U \subseteq \overline{T(R[\frac{1}{a}])}$, it is equivalent to produce, for every $u \in U$, a sequence $(r_n)_{n \in \mathbb{N}} \in T(R[\frac{1}{a}])$ whose image in $T(\widehat{K}^a)$ converges to $u$. By construction, there is a $v \in T(L_0)$ so that $N_{L_0/\widehat{K}^a}(v) = u$. Thanks to part (a), there is a minimal Galois, finite étale extension $R[\frac{1}{a}] \to S$ such that $S \otimes_{R[\frac{1}{a}]} \widehat{K}^a \cong \prod L_i$, where each $L_i$ is a minimal splitting field of $T_{\widehat{K}^a}$, which is unique up to isomorphism by the proof of part (a); consequently, $L_i \simeq L_0$ as field extensions of $\widehat{K}^a$, for each $i$. Since $R[\frac{1}{a}] \hookrightarrow \widehat{K}^a$ is dense, by tensoring it with the finite étale $R[\frac{1}{a}]$-module $S$, we obtain a dense injection

$$S \hookrightarrow S \otimes_{R[\frac{1}{a}]} \widehat{K}^a \cong \prod L_i.$$

This implies that for the split torus $T_S$, the induced morphism $T(S) \hookrightarrow \bigoplus T(L_i)$ is dense, namely, there is a sequence $(s_n)_{n\in\mathbb{N}} \in T(S)$ whose image under the above morphism converges to $(v, 0, \ldots, 0) \in \bigoplus T(L_i)$. From the following diagram

$$\begin{tikzcd}
T(S) \arrow[r, "\text{dense}"] \arrow[d, "\mathrm{Norm}_{S/R[\frac{1}{a}]}"'] & \bigoplus T(L_i) \arrow[d] & T(L_0) \arrow[l, hook'] \arrow[dl, "\mathrm{Norm}_{L_0/\widehat{K}^a}"] \\
T(R[\frac{1}{a}]) \arrow[r] & T(\widehat{K}^a), &
\end{tikzcd}$$

we get that the image of $(r_n := \mathrm{Norm}_{S/R[\frac{1}{a}]}(s_n))_{n\in\mathbb{N}}$ in $T(\widehat{K}^a)$ converges to $u$. $\square$

Before stating Lemma 5.8, we record a statement below that will be required for its proof. Assuming the notations of 5.2, let $T \subset G_{\widehat{K}^a}$ be a maximal torus. The morphism

$$\phi\colon G(\widehat{K}^a) \to \underline{\mathrm{Tor}}(G)(\widehat{K}^a) \text{ given by } g \mapsto gTg^{-1} \text{ is open in the } a\text{-adic topology.} \tag{5.7.1}$$

Indeed, since $G_{\widehat{K}^a}/N_{G_{\widehat{K}^a}}(T) \cong \underline{\mathrm{Tor}}(G_{\widehat{K}^a})$ and since $N_{G_{\widehat{K}^a}}(T) \subset G_{\widehat{K}^a}$ is a smooth, affine $\widehat{K}^a$-subgroup, we get that $G_{\widehat{K}^a} \to \underline{\mathrm{Tor}}(G_{\widehat{K}^a})$ is a smooth morphism of affine, finite type $\widehat{K}^a$-schemes, whence an argument in a similar vein as the proof of Lemma 5.6 shows that $\phi$ is open (cf. [Čes15, Proposition 4.3(a)]). We are now ready to state the lemma.

**Lemma 5.8.** *Assuming the notations of 5.2, let $T \subset G_{\widehat{K}^a}$ be a maximal torus. The closure $\overline{G(R[\frac{1}{a}])}$ of the image of $G(R[\frac{1}{a}])$ in $G(\widehat{K}^a)$ contains the image $U$ of the norm map (5.6.1).*

*Proof.* We follow the proof of [Guo22, Lemma 3.10]. It suffices to construct a sequence $(g_n)_{n\in\mathbb{N}} \in G(\widehat{K}^a)$ that $a$-adically converges to the identity and a sequence $(T_n)_{n\in\mathbb{N}} \in \underline{\mathrm{Tor}}(G)(R[\frac{1}{a}])$ such that $(T_n)_{\widehat{K}^a} = g_nTg_n^{-1}$, for all $n \in \mathbb{N}$. By (5.7.1), the morphism $\phi\colon G(\widehat{K}^a) \to \underline{\mathrm{Tor}}(G)(\widehat{K}^a)$ given by $g \mapsto gTg^{-1}$ is open in the $a$-adic topology. Choosing a basis $(U_n)_{n\in\mathbb{N}}$ of open neighbourhoods of $\mathrm{id} \in G(\widehat{K}^a)$, the density Lemma 5.3 implies that, for each $n$,

$$\phi(U_n) \cap \mathrm{Im}(\underline{\mathrm{Tor}}(G)(R[\tfrac{1}{a}]) \to \underline{\mathrm{Tor}}(G)(\widehat{K}^a)) \neq \emptyset,$$

as a result, there exist a $T_n \in \underline{\mathrm{Tor}}(G)(R[\frac{1}{a}])$ and a $g_n \in U_n$ such that $g_nTg_n^{-1} = (T_n)_{\widehat{K}^a}$.

Let $L_0$ be the minimal finite Galois extension of $\widehat{K}^a$ that splits $T$ (or equivalently, any $(T_n)_{\widehat{K}^a}$ because they are pairwise $\widehat{K}^a$-conjugate to each other) and let $\mathrm{Norm}_{L_0/\widehat{K}^a}\colon T_n(L_0) \to T_n(\widehat{K}^a)$ be the associated norm map (5.6.1). By Lemma 5.7(b), we have that

$$U_n := \mathrm{Norm}_{L_0/\widehat{K}^a}(T_n(L_0)) \subseteq \overline{T_n(R[\tfrac{1}{a}])}.$$

The conjugate relation implies that $U_n = g_n U g_n^{-1}$. The convergence of $(g_n)$ ensures that the sequence $(g_n^{-1} u g_n)_{n\in\mathbb{N}}$ converges to $u$. Hence, any open neighbourhood $u \in B_u \subseteq U$ contains $g_n^{-1} u g_n$, for all large enough $n$. Finally, the result follows from the chain of containments below

$$u \in g_n B_u g_n^{-1} \subseteq g_n U g_n^{-1} = U_n \subseteq \overline{T_n(R[\tfrac{1}{a}])} \subseteq \overline{G(R[\tfrac{1}{a}])}.$$

$\square$

**Lemma 5.9.** *Assuming the notations of Lemma 5.8, the closure $\overline{G(R[\frac{1}{a}])}$ contains*

(a) *a normal open subgroup $N \subseteq G(\widehat{K}^a)$, and*

(b) *$\mathscr{R}_u(P)(\widehat{K}^a)$, where $P \subseteq G_{\widehat{R}^a}$ is a minimal parabolic subgroup.*

*Proof.* (a): We follow the proof of [Guo22, Proposition 3.11]. Since the claim only involves $\widehat{K}^a$, for our purposes, it suffices to apply the conclusion of part (iii) of the proof of loc. cit to show that there exists an open $U_0 \subseteq U = \mathrm{Im}(\mathrm{Norm}_{L_0/\widehat{K}^a}\colon T(L_0) \to T(\widehat{K}^a))$ such that the set $E = \bigcup_{g\in G(\widehat{K}^a)} gU_0g^{-1} \subseteq G(\widehat{K}^a)$ is open. Therefore, the subgroup $N \subseteq G(\widehat{K}^a)$ generated by $E$ is open, since it contains an open subset. Moreover, since by construction, $E$ is closed under conjugation, $N$ is normal. Given an element $g \in G(\widehat{K}^a)$, define $T^g := gTg^{-1}$ for which $U^g := \mathrm{Norm}_{L_0/\widehat{K}^a}(T^g(L_0)) = g\mathrm{Norm}_{L_0/\widehat{K}^a}(T(L_0))g^{-1} = gUg^{-1}$ is contained in $\overline{G(R[\frac{1}{a}])}$ (thanks to Lemma 5.8). To demonstrate the containment for $N$, it suffices to show the same for $E$, which follows from the formula

$$E = \bigcup\nolimits_{g\in G(\widehat{K}^a)} gU_0g^{-1} \subseteq \bigcup\nolimits_{g\in G(\widehat{K}^a)} U^g \subseteq \overline{G(R[\tfrac{1}{a}])}.$$

(b): Let $N \subset G(\widehat{K}^a)$ be the subgroup constructed in (a) (which matches with the subgroup $N$ constructed in [Guo22, Proposition 3.11]). By (a), it is enough to show that $\mathscr{R}_u(P)(\widehat{K}^a) \subseteq N$. Since the claim only involves $\widehat{K}^a$, for our purposes, it suffices to apply the conclusion of Step 2 in the proof of [Guo22, Proposition 4.7]. $\square$

Proposition 5.10 is a patching result for reductive groups over semilocal Prüfer domains. For its proof, we shall require a gluing result that we record below. Assuming the notations of 5.2,

$$R \xrightarrow{\sim} R[\tfrac{1}{a}] \times_{\widehat{K}^a} \widehat{R}^a. \tag{5.9.1}$$

Indeed, since the rings are integral domains, in particular, they are $a$-torsion free, the injectivity follows. To establish surjectivity, we use the fact that $R/a^nR \cong \widehat{R}^a/a^n\widehat{R}^a$ and argue as in the first part of the proof of [Sta22, Tag 0BNR]. We now state the proposition below.

**Proposition 5.10.** *Given a semilocal Prüfer domain $R$ of finite Krull dimension, a maximal ideal $\mathfrak{m} \subset R$, an element $a \in R$ such that $V(a) = \{\mathfrak{m}\}$, let $\widehat{R}^a$ be the $a$-adic completion of $R$ and $\widehat{K}^a := \mathrm{Frac}(\widehat{R}^a)$. Then,*

(*a*) *for an $R$-torus $T$, we have an equality*

$$T(\widehat{K}^a) = \mathrm{Im}(T(R[\tfrac{1}{a}]) \to T(\widehat{K}^a)) \cdot T(\widehat{R}^a). \tag{5.10.1}$$

*Moreover, given a reductive $R$-group $G$,*

(*b*) *for a maximal split torus $T_{\mathrm{sp}} \subseteq G_{\widehat{R}^a}$,*

$$T_{\mathrm{sp}}(\widehat{K}^a) \subseteq \mathrm{Im}(G(R[\tfrac{1}{a}]) \to G(\widehat{K}^a)) \cdot G(\widehat{R}^a), \tag{5.10.2}$$

(*c*) *for a minimal parabolic subgroup $P \subsetneq G_{\widehat{R}^a}$,*

$$P(\widehat{K}^a) \subseteq \mathrm{Im}(G(R[\tfrac{1}{a}]) \to G(\widehat{K}^a)) \cdot G(\widehat{R}^a), \tag{5.10.3}$$

(*d*) *we have an equality*

$$G(\widehat{K}^a) = \mathrm{Im}(G(R[\tfrac{1}{a}]) \to G(\widehat{K}^a)) \cdot G(\widehat{R}^a). \tag{5.10.4}$$

*Proof.* We recall that $\widehat{R}^a$ is an $a$-adically complete valuation ring of rank $\leq 1$ (see Remark 3.5).

(a): Let $R^h$ be the $a$-adic Henselisation of $R$, which is an $a$-Henselian valuation ring (see Remark 3.5). Following the proof of [Guo22, Lemma 4.6], we construct the following diagram

$$\begin{tikzcd}
1 \arrow[r] & T(R) \arrow[r] \arrow[d] & T(R[\frac{1}{a}]) \arrow[r] \arrow[d] & H^1_{V(a)}(R,T) \arrow[r] \arrow[d, "\wr"] & H^1(R,T) \arrow[r] \arrow[d] & H^1(R[\frac{1}{a}],T) \arrow[d] \\
1 \arrow[r] & T(R^h) \arrow[r] \arrow[d] & T(R^h[\frac{1}{a}]) \arrow[r] \arrow[d] & H^1_{V(a)}(R^h,T)) \arrow[r] \arrow[d] & H^1(R^h,T) \arrow[r] \arrow[d] & H^1(R^h[\frac{1}{a}],T) \arrow[d] \\
1 \arrow[r] & T(\widehat{R}^a) \arrow[r] & T(\widehat{K}^a) \arrow[r] & H^1_{V(a)}(\widehat{R}^a,T) \arrow[r] & H^1(\widehat{R}^a,T) \arrow[r] & H^1(\widehat{K}^a,T).
\end{tikzcd}$$

The rows are long exact sequences associated to cohomology with supports [Sta22, Tag 09XP]. Thanks to the toral case of the Grothendieck–Serre conjecture (Theorem 4.7), the rightmost horizontal morphisms are injective. By excision [Mil80, Chapter III, Proposition 1.27], the top vertical arrow in the middle is an isomorphism, from which we have

$$T(R^h[\tfrac{1}{a}]) = \mathrm{Im}(T(R[\tfrac{1}{a}]) \to T(R^h[\tfrac{1}{a}])) \cdot T(R^h). \tag{5.10.5}$$

The $a$-adic completion of $R^h$ is $\widehat{R}^a$ (see, for example, [FK18, Chapter 0, Proposition 7.3.5(2)]), consequently, by [BČ20, Theorem 2.2.17], the image of $T(R^h[\frac{1}{a}])$ in $T(\widehat{K}^a)$ is dense. Indeed, the application of loc. cit. is valid because we put $A = R^h, t = a, I = 1, U = \mathrm{Spec}(R^h[\frac{1}{a}])$ and $X = T$. In this case, the integral domain $A$ is $t$-torsion free, showing that $(A, t, I)$ is a 'bounded Gabber–Ramero triple'[5] (see [BČ20, Section 2.1.9]) whose completion is $\widehat{A} = \widehat{R}^a$ (see [BČ20, Example 2.1.10]). Moreover, the subset $T(\widehat{R}^a) \subseteq T(\widehat{K}^a)$ is open in the $a$-adic topology (see [BČ20, Section 2.2.7]). As a consequence of the preceding arguments, we get an equality

$$T(\widehat{K}^a) = \mathrm{Im}(T(R^h[\tfrac{1}{a}]) \to T(\widehat{K}^a)) \cdot T(\widehat{R}^a). \tag{5.10.6}$$

Combining (5.10.5) and (5.10.6), we obtain the required equality (5.10.1).

(b): By [SGA $3_{\mathrm{II}}$, Exposé XXVI, Corollaire 6.11], the centraliser $L \subset G_{\widehat{R}^a}$ of $T_{\mathrm{sp}}$ is a Levi subgroup (of a parabolic subgroup $P \subset G_{\widehat{R}^a}$), which contains a maximal torus $\tilde{T} \subset L$. Consequently,

$$\tilde{T} = Z_L(\tilde{T}) \supset Z_L(L) \supset T_{\mathrm{sp}}. \tag{5.10.7}$$

Firstly, we shall follow the arguments of [Guo22, Corollary 3.12] to produce a torus $T \in \underline{\mathrm{Tor}}(G)(R)$ and an element $g \in \overline{G(R[\frac{1}{a}])} \cap G(\widehat{R}^a) \subseteq G(\widehat{K}^a)$ such that $T_{\widehat{K}^a} = g\tilde{T}_{\widehat{K}^a}g^{-1}$. Since, by definition, $\widehat{R}^a \subset \widehat{K}^a$ is open and since $G$ and $\underline{\mathrm{Tor}}(G)$ are affine schemes of finite type, the subsets

$$G(\widehat{R}^a) \subseteq G(\widehat{K}^a) \qquad \text{and} \qquad \underline{\mathrm{Tor}}(G)(\widehat{R}^a) \subseteq \underline{\mathrm{Tor}}(G)(\widehat{K}^a)$$

are open (see [Čes15, Page 8, Property (ix)]). Thanks to (5.7.1), the morphism $\phi\colon G(\widehat{K}^a) \to \underline{\mathrm{Tor}}(G)(\widehat{K}^a)$ given by $g \mapsto g\tilde{T}_{\widehat{K}^a}g^{-1}$ is open in the $a$-adic topology (by the definition of a maximal torus, $\tilde{T}_{\widehat{K}^a} \in \underline{\mathrm{Tor}}(G)(\widehat{K}^a)$). Considering a normal open subgroup $N \subseteq \overline{G(R[\frac{1}{a}])}$ (Lemma 5.9(a)), we take an open neighbourhood $\mathrm{id} \in W \subseteq N \cap G(\widehat{R}^a)$, from which we get an open $\phi(W) \subseteq \underline{\mathrm{Tor}}(G)(\widehat{K}^a)$ (since $\phi$ is open). Thanks to Lemma 5.3, the image of $\underline{\mathrm{Tor}}(G)(R[\frac{1}{a}])$ in $\underline{\mathrm{Tor}}(G)(\widehat{K}^a)$ is dense, consequently, we get $\phi(W) \cap \underline{\mathrm{Tor}}(G)(\widehat{R}^a) \cap \underline{\mathrm{Tor}}(G)(R[\frac{1}{a}]) \neq \emptyset$. Therefore, thanks to the diagram (5.9.1) and the fact that $\underline{\mathrm{Tor}}(G)$ is a finite type, affine $R$-scheme,

$$\underline{\mathrm{Tor}}(G)(R) \xrightarrow{\sim} \underline{\mathrm{Tor}}(G)(R[\tfrac{1}{a}]) \times_{\underline{\mathrm{Tor}}(G)(\widehat{K}^a)} \underline{\mathrm{Tor}}(G)(\widehat{R}^a).$$

The displayed bijection shows that there is a torus $T \in \underline{\mathrm{Tor}}(G)(R)$ such that $T \in \phi(W)$, i.e., there exists a $g \in W \subset N \cap G(\widehat{R}^a) \subseteq \overline{G(R[\frac{1}{a}])} \cap G(\widehat{R}^a)$ so that

$$T_{\widehat{K}^a} = g\tilde{T}_{\widehat{K}^a}g^{-1}, \text{ as desired.} \tag{5.10.8}$$

Lastly, we use the patching for a torus (5.10.1) to prove the containment (5.10.2). The patching (5.10.1) for $T$ produces

$$T(\widehat{K}^a) = \mathrm{Im}(T(R[\tfrac{1}{a}]) \to T(\widehat{K}^a)) \cdot T(\widehat{R}^a) \subseteq \overline{G(R[\tfrac{1}{a}])} \cdot G(\widehat{R}^a).$$

Keeping the relation (5.10.8) in mind, we rewrite the above as $\tilde{T}(\widehat{K}^a) \subseteq g^{-1}\overline{G(R[\frac{1}{a}])} \cdot G(\widehat{R}^a)g$. However, since $g \in W \subseteq \overline{G(R[\frac{1}{a}])} \cap G(\widehat{R}^a)$, we obtain $\tilde{T}(\widehat{K}^a) \subseteq \overline{G(R[\frac{1}{a}])} \cdot G(\widehat{R}^a)$. Using the containment (5.10.7), we further rewrite to produce the requisite formula

$$T_{\mathrm{sp}}(\widehat{K}^a) \subseteq \overline{G(R[\tfrac{1}{a}])} \cdot G(\widehat{R}^a) = \mathrm{Im}(G(R[\tfrac{1}{a}]) \to G(\widehat{K}^a)) \cdot G(\widehat{R}^a)$$

[5] The requirement of being a 'Henselian triple' is satisfied due to the Henselian property of $R^h$.

(the last equality follows because we can argue as in [Guo22, Corollary 3.13] using the fact that $\mathrm{Im}(G(R[\frac{1}{a}]) \to G(\widehat{K}^a))$ contains an open subgroup $N$).

(c): We follow Step 3 of the proof of [Guo22, Proposition 4.7]. Let $T_{\mathrm{sp}} \subset L$ be a maximal split torus. By [Guo22, Proposition 4.4], since $H := L/T_{\mathrm{sp}}$ is anisotropic, $H(\widehat{R}^a) = H(\widehat{K}^a)$. Consequently, we obtain the following morphism of long exact sequences

$$\begin{tikzcd} 0 \arrow[r] & T_{\mathrm{sp}}(\widehat{R}^a) \arrow[r] \arrow[d] & L(\widehat{R}^a) \arrow[r] \arrow[d] & H(\widehat{R}^a) \arrow[r] \arrow[d, equal] & H^1(\widehat{R}^a, T_{\mathrm{sp}}) \arrow[d] \\ 0 \arrow[r] & T_{\mathrm{sp}}(\widehat{K}^a) \arrow[r] & L(\widehat{K}^a) \arrow[r] & H(\widehat{K}^a) \arrow[r] & H^1(\widehat{K}^a, T_{\mathrm{sp}}), \end{tikzcd}$$

where the rightmost term of each row vanishes due to the Hilbert theorem 90. A diagram chase yields $L(\widehat{K}^a) = T_{\mathrm{sp}}(\widehat{K}^a) \cdot L(\widehat{R}^a)$. Combining this with (5.10.2) and Lemma 5.9(b), we obtain the requisite containment (5.10.3).

(d): We follow the proof of [Guo22, Proposition 4.7]. The containment on one side being obvious, to prove the displayed equality (5.10.4), it is enough to show that

$$G(\widehat{K}^a) \subseteq \mathrm{Im}(G(R[\tfrac{1}{a}]) \to G(\widehat{K}^a)) \cdot G(\widehat{R}^a). \tag{5.10.9}$$

The case when $G_{\widehat{R}^a}$ is anisotropic follows from [Guo22, Proposition 4.4(c)], which implies that $G(\widehat{R}^a) = G(\widehat{K}^a)$. The case when $G_{\widehat{R}^a}$ contains no proper parabolic subgroup and $G_{\widehat{R}^a}$ contains a nontrivial split torus, say $T_{\mathrm{sp}}$, follows from the arguments in the proof of (c). Indeed, replacing $L$ by $G$ in the proof of (c), we obtain $G(\widehat{K}^a) = T_{\mathrm{sp}}(\widehat{K}^a) \cdot G(\widehat{R}^a)$. Combining this with (5.10.2), we obtain the requisite containment (5.10.9).

Therefore, we may assume that $G_{\widehat{R}^a}$ contains a proper parabolic subgroup. Let $P \subset G_{\widehat{R}^a}$ be a minimal parabolic subgroup and let $Q$ be its 'opposite' parabolic subgroup (see [SGA $3_{\mathrm{III}}$, Exposé XXVI, Théorème 4.3.2]). By [SGA $3_{\mathrm{III}}$, Exposé XXVI, Corollaire 5.2], there is a surjection

$$\mathscr{R}_u(P)(\widehat{K}^a) \cdot \mathscr{R}_u(Q)(\widehat{K}^a) \twoheadrightarrow G(\widehat{K}^a)/P(\widehat{K}^a),$$

which combines with Lemma 5.9(b) to yield $G(\widehat{K}^a) \subseteq \overline{G(R[\frac{1}{a}])} \cdot P(\widehat{K}^a)$. Finally, the last containment along with (5.10.3) produces $G(\widehat{K}^a) \subseteq \overline{G(R[\frac{1}{a}])} \cdot P(\widehat{K}^a) \subseteq \mathrm{Im}(G(R[\frac{1}{a}]) \to G(\widehat{K}^a)) \cdot G(\widehat{R}^a)$ (where the last equality follows because we can argue as in [Guo22, Corollary 3.13], since $\mathrm{Im}(G(R[\frac{1}{a}]) \to G(\widehat{K}^a))$ contains a normal open subgroup). □

We conclude this chapter with the proof of the Grothendieck–Serre Conjecture in the case of semilocal Prüfer domains. The argument is inspired from the proof of [Guo22, Proposition 4.10]. By an induction argument, an application of Proposition 5.10 reduces the proof to the case of complete valuation rings of rank 1.

**Theorem 5.11.** *Given a semilocal Prüfer domain $R$ and a reductive $R$-group $G$, a generically trivial $G$-torsor $E$ over $R$ is trivial.*

*Proof.* The fraction field $F$ of $R$ can be written as $F = \bigcup F'$, where the filtered union is taken over finitely generated sub-extensions $F' \subset F$ over the prime subfield $\mathbb{F}$. The previous equality implies that $R = \bigcup (R \cap F')$. By Lemma 3.3, the fraction field of the Prüfer domain $R' := R \cap F'$ is $F'$, therefore, the fact that $\mathrm{tr.deg}_{\mathbb{F}}(F') < \infty$ ensures that $R'$ has finite Krull dimension ([BouCA, Chapter VI, Section 10, Number 3, Corollary 1]). Thanks to [Gir71, Chapitre VII, Lemme 2.1.6], by a limit argument, we may assume, without loss of generality, that $R$ is of finite Krull dimension. We induct on

$$d(R) := \sum_{\mathfrak{m} \in \mathrm{MaxSpec}(R)} \dim(R_{\mathfrak{m}}).$$

If $d(R) = 0$, then $R$ is a field and the result is trivial. On the other hand, if $R$ is a valuation ring, then the result follows from [Guo22, Theorem 1.3]. Hence, we may assume that $d(R) \geq 2$. Assuming that the result is true for all semilocal Prüfer domains $R'$ such that $d(R') \leq d$, where $d \geq 1$, we shall prove that it is true for a Prüfer domain $R$ such that $d(R) = d + 1 \geq 2$.

We choose a maximal ideal $\mathfrak{m} \subset R$ and an element $a \in R$ such that $V(a) = \{\mathfrak{m}\}$ (such an element $a \in R$ exists by Remark 3.5), and let $\widehat{R}^a$ be the $a$-adically complete valuation ring of rank $\leq 1$ (see Remark 3.5) obtained as the $a$-adic completion of $R$ and let $\widehat{K}^a := \mathrm{Frac}(\widehat{R}^a)$. By Lemma 3.2(a), the localisation $R[\frac{1}{a}]$ is a Prüfer domain that has $d(R[\frac{1}{a}]) < d(R)$ and therefore, by the induction hypothesis, $E \mid_{R[\frac{1}{a}]}$ is a trivial torsor, i.e., it has a section $s_a \in E(R[\frac{1}{a}])$. In a similar vein, by the induction hypothesis, there is a section $\widehat{s}^a \in E(\widehat{R}^a)$. When restricted to $\widehat{K}^a$, the definition of a $G$-torsor ensures that there is an element $g \in G(\widehat{K}^a)$ such that $s_a \mid_{\widehat{K}^a} = \widehat{s}^a \mid_{\widehat{K}^a} \cdot g$. Thanks to the patching formula (5.10.4), there are elements $g_a \in G(R[\frac{1}{a}])$ and $\widehat{g}^a \in G(\widehat{R}^a)$ such that $g = \widehat{g}^a \mid_{\widehat{K}^a} \cdot g_a^{-1} \mid_{\widehat{K}^a}$, from which the previous equality that equates the sections of $E$ can be rewritten as $(s_a \cdot g_a) \mid_{\widehat{K}^a} = (\widehat{s}^a \cdot \widehat{g}^a) \mid_{\widehat{K}^a}$. Therefore, thanks to the diagram (5.9.1) and the fact that $E$ is a finite type, affine $R$-scheme,

$$E(R) \xrightarrow{\sim} E(R[\tfrac{1}{a}]) \times_{E(\widehat{K}^a)} E(\widehat{R}^a).$$

The displayed bijection produces a section $s \in E(R)$ that is a gluing of $s_a$ and $\widehat{s}^a$. The existence of a global section means that $E$ is a trivial torsor over $R$. The induction step is thus complete and we are done. $\square$

# Chapter 6

# Presentation Lemma for Valuation Rings of rank at most 1

In this chapter, following the arguments of [Čes22], we prove a presentation lemma over valuation rings of rank 1 in the style of Gabber (Presentation Lemma 6.5). Our version differs from the ones proved in various parts of the literature, notably [Qui73, Lemma 5.12], [Gab94, Lemma 3.1] and [CHK97, Theorem 3.1.1], by adding a codimension at least 2 requirement. This is the price that we have to pay in the mixed characteristic setting.

We start with the following result, which will be useful in the proof of Proposition 6.4.

**Lemma 6.1.** *A field $K$ which is finitely generated over its prime subfield is the fraction field of an integral smooth scheme over $\mathbb{F}_p$ or $\mathbb{Z}$.*

*Proof.* Let $R$ be a polynomial algebra over the prime subring ($\mathbb{F}_p$ or $\mathbb{Z}$) such that $K$ is a finite algebraic extension over the fraction field of $R$. The integral closure $R'$ of $R$ in $K$ being finite type over a field or $\mathbb{Z}$, is an excellent ring by [Sta22, Tag 07QS], and hence the regular locus $X \subset \operatorname{Spec} R'$ is a nonempty open ([Sta22, Tag 07P7]). It remains to show that there is a non-empty open $U \subset X$ that is smooth over $\mathbb{F}_p$ or $\mathbb{Z}$. In the positive characteristic case, the smoothness of $X$ follows from [Sta22, Tag 0B8X]. In the mixed characteristic, there exists an open subset of $X$ containing the generic fibre of $R'$ which is smooth over $\mathbb{Z}$ by [Sta22, Tag 01V9] . $\square$

The following lemma is inspired by [EGA $\text{III}_1$, Théorème 2.2.1 and Corollaire 2.2.4]. It will also be required in the proof of Proposition 6.4.

**Lemma 6.2.** *Let $R$ be a semilocal Prüfer domain of finite Krull dimension and let $X$ be a flat, projective $R$-scheme with a closed immersion $\iota\colon X \hookrightarrow \mathbb{P}^{m-1}_R$, for some $m \geq 1$. Set $\mathcal{O}_X(1) := \iota^*(\mathcal{O}_{\mathbb{P}^{m-1}_R}(1))$. For any quasi-coherent $\mathcal{O}_X$-module $\mathscr{G}$, set $\mathscr{G}(n) := \mathscr{G} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$. Then given a coherent $\mathcal{O}_X$-module $\mathscr{F}$, and a surjection $\varphi\colon \mathscr{F} \twoheadrightarrow \mathscr{G}$ of coherent $\mathcal{O}_X$-modules,*

*(i) we have that $H^q(X, \mathscr{F}) = 0$, for each $q > m$,*

*(ii) there exists an integer $N$ such that for all $n \geq N$, we have*

$$H^q(X, \mathscr{F}(n)) = 0, \text{ for any } q \geq 1 \text{ and}$$

*(iii) there exists an integer $N$ such that for all $n \geq N$, we have*

$$\varphi(X) \colon \Gamma(X, \mathscr{F}(n)) \twoheadrightarrow \Gamma(X, \mathscr{G}(n)).$$

*Proof.* Given a quasi-coherent $\mathcal{O}_X$-module $\mathscr{K}$, by [EGA II, Corollaire 3.4.5 and Proposition 3.5.2], we have $\iota_*(\mathscr{K}(n)) \cong (\iota_*\mathscr{K})(n)$, for all $n \geq 0$. We note that $X$ and $\mathbb{P}_R^{m-1}$ are actually locally coherent schemes (see §4.1). Indeed, they are $R$-schemes that are locally of finitely presentation (coherence of $R$ is used here).

(i): Since $X$ is a closed subscheme of $\mathbb{P}_R^{m-1}$, it can be covered by $m$ affines, say $\{U_i\}$. Consequently, thanks to [Sta22, Tag 01XD] or [EGA III$_1$, Proposition 1.4.1], since $X$ is separated, the $q$-th Čech cohomology group $\check{H}^q(\{U_i\}, \mathscr{F})$ of $\mathscr{F}$ with respect to $\{U_i\}$ identifies itself with $H^q(X, \mathscr{F})$, for each $q$. The claim follows because $\check{H}^q(\{U_i\}, \mathscr{F}) = 0$, for all $q > m$.

(ii): We follow the proof of [EGA III$_1$, Proposition 2.2.2]. By definition, $\mathcal{O}_X$ is $R$-flat, and therefore, by [RG71, Première partie, Théorème 3.4.6], the $\mathcal{O}_{\mathbb{P}_R^{m-1}}$-module $\iota_*\mathcal{O}_X$ is of finite presentation. Given that $\mathbb{P}_R^{m-1}$ is locally coherent, this means that $\iota_*(\mathcal{O}_X)$ is automatically a coherent $\mathcal{O}_{\mathbb{P}_R^{m-1}}$-module ([Sta22, Tag 01BZ]). Similarly, $\iota_*(\mathscr{F})$ is a coherent $\mathcal{O}_{\mathbb{P}_R^{m-1}}$-module. Since higher direct images under a closed immersion vanish ([Sta22, Tag 01QY]), it is enough to show that exists an integer $N$ such that for any $n \geq N$, we get $H^q(\mathbb{P}_R^{m-1}, \iota_*(\mathscr{F}(n))) = 0$, for all $q \geq 1$. Therefore, it suffices to assume that $X = \mathbb{P}_R^{m-1}$.

Thanks to [EGA II, Corollaire 2.7.10], there exists a surjection $j \colon \mathscr{L} := \mathcal{O}_X(r)^{\oplus s} \twoheadrightarrow \mathscr{F}$, for some $r \in \mathbb{Z}$ and $s \geq 0$. Letting $\mathscr{K} := \ker j$, we get a short exact sequence $0 \to \mathscr{K} \to \mathscr{L} \to \mathscr{F} \to 0$ of coherent sheaves ([Sta22, Tag 01BY]). Since $\mathcal{O}_X(n)$ is a locally free $\mathcal{O}_X$-module for any $n$, we get a short exact sequence

$$0 \to \mathscr{K}(n) \to \mathscr{L}(n) \to \mathscr{F}(n) \to 0, \tag{6.2.1}$$

of coherent modules ([Sta22, Tag 01CE]), for each $n$. We shall show (ii) by the method of descending induction. For $q > m$, the result follows from (i). Suppose that for $d \geq 2$ and for any coherent $\mathcal{O}_X$-module $\mathscr{M}$, there exists an integer $N$ such that $H^q(X, \mathscr{M}(n)) = 0$, for all $q \geq d$ and for any $n \geq N$. We shall show that there exists an integer $N$ such that $H^q(X, \mathscr{F}(n)) = 0$, for all $q \geq d-1$ and for any $n \geq N$. Thanks to [EGA III$_1$, Corollaire 2.1.13], we have $H^q(\mathcal{O}_X(t)) = 0$, for all $q \geq 1$ and for any $t \geq 0$; consequently,

$$H^q(X, \mathscr{L}(n)) = 0, \text{ for all } q \geq 1 \text{ and } n \gg 0.$$

We choose $N$ such that for any $n \geq N$, we get $H^q(X, \mathscr{L}(n)) = 0$, for every $q \geq 1$, and $H^q(X, \mathscr{K}(n)) = 0$, for every $q \geq d$. With this choice, writing the associated long exact sequence of cohomology of (6.2.1), we get isomorphisms $H^q(X, \mathscr{F}(n)) \cong H^{q+1}(X, \mathscr{K}(n))$, for all $q \geq 1$ and for any $n \geq N$. This implies that $H^q(X, \mathscr{F}(n)) = 0$, for every $q \geq d-1$ and for any $n \geq N$, and the induction step is complete. Thus, we are done.

(iii): Letting $\mathscr{K} := \ker \varphi$, we get a short exact sequence $0 \to \mathscr{K} \to \mathscr{F} \to \mathscr{G} \to 0$ of coherent sheaves ([Sta22, Tag 01BY]). In a similar vein as above, since $\mathcal{O}_X(n)$ is a locally free $\mathcal{O}_X$-module for any $n$, we get a short exact sequence

$$0 \to \mathscr{K}(n) \to \mathscr{F}(n) \to \mathscr{G}(n) \to 0 \tag{6.2.2}$$

of coherent modules ([Sta22, Tag 01CE]), for each $n$. By (ii), there exists an integer $N$ such that $H^1(X, \mathscr{K}(n)) = 0$, for any $n \geq N$. Writing the long exact sequence of cohomology associated to (6.2.2), we get the requisite surjection, and we are done. □

The following is [Čes22, Proposition 3.6]. For a review of weighted projective spaces and weighted blow-ups see op. cit. Section 3.4.

**Proposition 6.3.** *Let $k$ be a field, let $X$ be a projective $k$-scheme of pure dimension $d$, let $\mathcal{O}_X(1)$ be a very ample line bundle on $X$, let $W \subset X^{\mathrm{sm}}$ be an open, let $x_1, \dots, x_n \in W$ and let $Y \subset X$ be a closed subscheme such that $Y \setminus W$ is of codimension $\geq 2$ in $X$. Letting $w_1 := 1$, upon replacing $\mathcal{O}_X(1)$ by any large power, there exist integers $w_2, \dots, w_d \geq 1$ and nonzero sections $h_k \in \Gamma(X, \mathcal{O}_X(w_k))$ for each $k = 1, \dots, d$ such that*

*(i) the hypersurface $H_1 := V(h_1)$ does not contain any $x_i$,*

*(ii) the hypersurfaces $H_i := V(h_i)$ satisfy $Y \cap H_1 \cap \ldots \cap H_d = \emptyset$,*

*(iii) in the following commutative diagram with vertical morphisms determined by the sections $h_i$:*

$$\begin{tikzcd}
X \setminus H_1 \arrow[r, hook] \arrow[d, "\pi"] & X \setminus H_1 \cap \ldots \cap H_d \arrow[r, hook] \arrow[d, "\overline{\pi}"] & \overline{X} := \mathrm{Bl}_X(h_1, \dots, h_d) \arrow[d, "\overline{\pi}"] \\
\mathbb{A}^{d-1}_k \arrow[r, hook] & \mathbb{P}_k(w_1, \dots, w_d) \arrow[r, equal] & \mathbb{P}_k(w_1, \dots, w_d)
\end{tikzcd}$$

*the morphism $\pi$ is smooth of relative dimension 1 at $x_i$, for each $i$,*

*(iv) for every $i$, we have $Y \cap H_1 \cap \overline{\pi}^{-1}(\pi(x_i)) = \emptyset$,*

*(v) for every $i$, we have $(Y \setminus W) \cap \overline{\pi}^{-1}(\pi(x_i))) = \emptyset$,*

*(vi) for every $i$, the morphism $\pi$ is smooth at each $Y \cap \overline{\pi}^{-1}(\pi(x_i))$,*

*(vii) there are affine opens*

$$S \subseteq \mathbb{A}^{d-1}_k \qquad \text{and} \qquad x_1, \dots, x_n \in U \subseteq W \cap \pi^{-1}(S) \subseteq X \setminus H_1$$

*such that $\pi\colon U \to S$ is smooth of relative dimension 1 and $Y \cap U = Y \cap \pi^{-1}(S)$ is $\pi$-finite.*

*Moreover, the sections can be chosen iteratively: for each $i \geq 1$ and suitable sections $h_1, \dots, h_{i-1}$, we can choose $h_i$ to be of any sufficiently large degree divisible by the characteristic exponent of $k$.*

The idea of the proof (due to Česnavičius) of the above is to slice $\overline{X}$ by $d-1$ hyperplanes $H_1, \dots, H_{d-1}$ in "generic positions". The hyperplanes are chosen so that they intersect $\overline{X} \setminus W$ in a nice fashion, for example, we ensure that

$$Y \cap H_1 \cap \cdots \cap H_{d-1} \text{ is a finite set and that } (Y \setminus W) \cap H_1 \cap \cdots \cap H_{d-1} \text{ is empty.}$$

The morphism $\pi\colon X \to \mathbb{A}^{d-1}_V$ is defined by projections onto the hypersurfaces $H_1, \dots, H_{d-1}$. By making sure that $\pi$ is smooth at the points of $x_1, \dots, x_n$, we shrink to a neighbourhood $x_1 \dots, x_n \in U \subset X$ where $\pi$ is smooth. The finiteness at $Y \cap U$ is confirmed by the properness of $Y$ and the quasi-finiteness of $\pi$ at $Y$.

The next statement is a version of the presentation lemma over possibly mixed-characteristic valuation rings of finite rank. It is inspired from [Čes22, Variant 3.7], where it was proved in the case of semilocal Dedekind rings. Following the arguments of the proof of loc. cit., the idea is to bootstrap from the special fibre.

**Proposition 6.4.** *Given*

- *a semilocal Prüfer domain $R$ of finite Krull dimension,*
- *a flat, projective $R$-scheme $X$ which is $R$-fibrewise of pure dimension $d$,*
- *an $R$-relatively very ample line bundle $\mathcal{O}_X(1)$ on $X$,*
- *an open $W \subseteq X^{\mathrm{sm}}$ containing points $x_1, \ldots, x_n \in X$, and*
- *a closed subscheme $Y \subset X$ such that $Y \setminus W$ is $R$-fibrewise of codimension $\geq 2$;*

*letting $w_1 := 1$, upon replacing $\mathcal{O}_X(1)$ by any large power, there exist integers $w_2, \ldots, w_d \geq 1$ and nonzero sections $s_k \in \Gamma(X, \mathcal{O}(w_k))$ for each $k = 1, \ldots, d$ such that*

*(i) the hypersurface $H_1 := V(s_1)$ does not contain any $x_i$,*

*(ii) the hypersurfaces $H_i := V(s_i)$ satisfy $Y \cap H_1 \cap \ldots \cap H_d = \emptyset$,*

*(iii) in the following commutative diagram with vertical morphisms determined by the sections $s_i$:*

$$\begin{tikzcd} X \setminus H_1 \arrow[r, hook] \arrow[d, "\pi"] & X \setminus H_1 \cap \ldots \cap H_d \arrow[r, hook] \arrow[d, "\pi"] & \overline{X} := \mathrm{Bl}_X(s_1, \ldots, s_d) \arrow[d, "\overline{\pi}"] \\ \mathbb{A}^{d-1}_R \arrow[r, hook] & \mathbb{P}_R(w_1, \ldots, w_d) \arrow[r, equal] & \mathbb{P}_R(w_1, \ldots, w_d) \end{tikzcd}$$

*the morphism $\pi$ is smooth at each $x_i$, for each $i$,*

*(iv) for every $i$, we have $Y \cap H_1 \cap \overline{\pi}^{-1}(\pi(x_i)) = \emptyset$,*

*(v) for every $i$, the morphism $\pi$ is smooth at each $Y \cap \overline{\pi}^{-1}(\pi(x_i))$, and*

*(vi) there are affine opens*

$$S \subseteq \mathbb{A}^{d-1}_R \qquad \text{and} \qquad x_1, \ldots, x_n \in U \subseteq W \cap \pi^{-1}(S)$$

*such that $\pi \colon U \to S$ is smooth of relative dimension 1 and $Y \cap U = Y \cap \pi^{-1}(S)$ is $\pi$-finite.*

*Proof.* If $R$ is a field, then the claim follows from Proposition 6.3. Therefore, we may assume that $R$ is of finite, positive Krull dimension. Firstly, we assume that each of the points $x_1, \ldots, x_n$ specialises to a point in a special $R$-fibre of $W$. In this case, each of the points $x_1, \ldots, x_n$ specialises to a closed point $x_i'$ in a special $R$-fibre of $W$. We shall, without loss of generality, specialise each point $x_i$ to $x_i'$ and assume that $x_i = x_i'$, i.e., we assume that $x_i$ is a closed point in a special $R$-fibre of $W$.

Let $C$ be the subscheme of closed points of $\operatorname{Spec} R$ and let $I \subset R$ be the ideal of vanishing of $C$. We write $I = \bigcup I_\lambda$, where the filtered union is taken over finitely generated sub-ideals $I_\lambda \subset I$, and set $C_\lambda := \operatorname{Spec}(R/I_\lambda)$, for each $\lambda$. Letting $\iota_\lambda \colon X_{C_\lambda} := X \times_{\operatorname{Spec} R} C_\lambda \hookrightarrow X$ be the inclusion, since $I_\lambda$ is finitely generated, the $\mathcal{O}_X$-module $\iota_{\lambda *}\mathcal{O}_{X_{C_\lambda}}$ is finitely presented, and hence, the morphism $\mathcal{O}_X \to \iota_{\lambda *}\mathcal{O}_{X_{C_\lambda}}$ is a surjection of coherent $\mathcal{O}_X$-modules (§4.1 and [Sta22, Tag 01BZ]). As a consequence, there exists an integer $N$ such that

$$\Gamma(X, \mathcal{O}_X(r)) \to \Gamma(X, (\iota_{\lambda *}\mathcal{O}_{X_{C_\lambda}})(r)) = \Gamma(X_{C_\lambda}, \mathcal{O}_{X_{C_\lambda}}(r)) \text{ is a surjection, for all } r \geq N. \tag{6.4.1}$$

Upon raising $\mathcal{O}_X(1)$ to any large enough power, without loss of generality, we may assume that $N = 1$ in (6.4.1). Since, by our assumption, the points $x_1, \ldots, x_n$ lie over $C$, we use Proposition 6.3 to find sections $h_i \in \mathcal{O}_{X_C}(w_i)$, for each $i$, (the last aspect of Proposition 6.3 ensures that these $h_i$ may be chosen to have constant degrees on $C$) that satisfy the claim in Proposition 6.3. By a limit argument, there exist a $\lambda$ and sections $h_{i,\lambda} \in \mathcal{O}_{X_{C_\lambda}}(w_i)$ that lift $h_i$, for all $i$. Finally, (6.4.1) implies that there exist sections $s_i \in \mathcal{O}_X(w_i)$ that lift $h_{i,\lambda}$, for each $i$.

Since $Y$ and $H_i$ are closed subschemes of the projective $R$-scheme $X$, for all $i$, they are $R$-projective (see [EGA II, Définition 5.5.2]), in particular, $R$-proper. As a consequence, their images along the respective structure morphisms to $\operatorname{Spec} R$ are closed, whence (i) and (ii) follows by using their respective counterparts in Proposition 6.3. We note that in (iii) the weighted blowup need not commute with base change, however, the formation of the morphism $\overline{\pi}\colon X \setminus (H_1 \cap \ldots \cap H_d) \to \mathbb{P}_R(w_1, \ldots, w_d)$ does and this suffices for our purposes. By [RG71, Première partie, Corollaire 3.4.7], the flat, finite type $R$-scheme $X \setminus H_1$ is $R$-finitely presented, which, in turn, implies that $\pi\colon X \setminus H_1 \to \mathbb{A}^{d-1}_R$ is finitely presented. Therefore, thanks to the fibrewise criterion of flatness [Sta22, Tag 039C], $\pi$ is flat at $x_i$, for each $i$, whence, thanks to the fibrewise criterion of smoothness [Sta22, Tag 01V8], it is smooth at $x_i$, for each $i$. This proves (iii). In order to prove (iv), we use the fact that $\overline{\pi}$ is proper to argue that $\pi(x) = \overline{\pi}(x) \in \mathbb{P}_R(w_1, \ldots, x_d)$ is a closed point, which implies $\overline{\pi}^{-1}(\pi(x)) \subset \overline{X}$ is a closed subset. To finish this proof, it suffices to use the counterpart of (iv) in Proposition 6.3 and the fact that images of proper morphisms are closed. In a similar vein to the proof of (iii), the proof of (v) follows from an application of the fibrewise criterion of flatness [Sta22, Tag 039C] followed by an application of the fibrewise criterion of smoothness [Sta22, Tag 01V8]. It remains to show (vi). First, we claim that the morphism $\overline{\pi}$ when restricted to $Y \cap \overline{\pi}^{-1}(\pi(x_i))$ has finite $R$-fibres (and hence, by [Sta22, Tag 02NH], it is quasi-finite), for each $i$. In fact, given that $R$ has a finite spectrum, the claim even shows that $Y \cap \overline{\pi}^{-1}(\pi(x_i))$ is a finite set, for each $i$. Combing the facts that $(Y \cap \overline{\pi}^{-1}(\pi(x_i))) \cap H_1 = \emptyset$ and that $H_1$ is a hypersurface, this claim is a consequence of Krull's principal ideal theorem. The openness of the quasi-finite locus ([Sta22, Tag 01TI]) implies that there exists an open subset $U_1 \subseteq Y$ containing $Y \cap \pi^{-1}(\pi(x_i))$, for all $i$, such that $\pi|_{U_1}$ is quasi-finite. Since $Y$ is proper, taking any open subset

$$\pi(x_1), \ldots, \pi(x_n) \in S_0 \subseteq (\mathbb{A}^{d-1}_R \setminus \pi(Y \setminus U_1)),$$

we observe that $\pi|_{Y \cap \pi^{-1}(S_0)}$ is quasi-finite, which implies that it is even finite ([Sta22, Tag 02OG]). We choose an affine open $\pi(x_1), \ldots, \pi(x_n) \in S_0 \subseteq (\mathbb{A}^{d-1}_R \setminus \pi(Y \setminus U_1))$. By the definition of a smooth morphism [Sta22, Tag 01V5], there exists an affine open

$$U_0 \subseteq \pi^{-1}(S_0) \cap W$$

containing $x_1, \ldots, x_n$ and the points of $Y \cap \overline{\pi}^{-1}(\pi(x_i))$, for all $i$, such that $\pi|_{U_0}\colon U_0 \to S_0$ is smooth. A dimension count shows that $\pi|_{U_0}$ is of relative dimension 1. Finally, it remains to find affine opens $\pi(x_1), \ldots, \pi(x_n) \in S \subseteq S_0$ and $U \subseteq U_0$ containing $x_1, \ldots, x_n$ and $Y \cap \overline{\pi}^{-1}(\pi(x_i))$, for all $i$, such that $Y \cap U = Y \cap \pi^{-1}(S)$. For this, we can choose any principal affine open

$$\pi(x_1), \ldots, \pi(x_n) \in S \subseteq S_0 \setminus \pi(Y \setminus U_0) \text{ and set } U := U_0 \cap \pi^{-1}(S).$$

In general, $x_1, \ldots, x_n$ might not have a specialisation in a special fibre of $W$. In this regard, thanks to Lemma 3.3(c) and a limit argument, without loss of generality, we may assume that $K$ is finitely generated over the prime subfield. By Lemma 6.1, the field $K$ is a fraction field of a regular domain $A$ that is smooth over $\mathbb{F}_p$ or $\mathbb{Z}$. Moreover, this $A$ is of positive Krull dimension, since otherwise $K$ is a finite field, in which case, it contradicts our assumption that $R$ is not a field. By localising $A$, we may assume that

1. the scheme $X_K$ spreads out to a projective, flat $A$-scheme $X_A$ that is fibrewise of pure dimension $d$ by [EGA IV$_3$, Théorème 12.2.1 (ii) and (v)],
2. the relative $K$-very ample line bundle $\mathcal{O}_{X_K}(1)$ spreads out to a relative $A$-very ample line bundle,
3. there is an open $W_A \subset X_A^{\mathrm{sm}}$ which intersects the $K$-fibre at $W_K$ by [Sta22, Tag 01V9],
4. each point $x_i$ that lies in $W_K$ spreads out to an $A$-finite closed subscheme in $W_A$,
5. and the closed subscheme $Y_K$ spreads out to an $A$-flat closed subscheme $Y_A$ such that $Y_A \setminus W_A$ is $A$-fibrewise of codimension $\geq 2$ in $X_A$ (see [EGA IV$_3$, Corollaire. 12.2.2 (i)]).

Given that $A$ is of positive Krull dimension, it has infinitely many primes of height 1, permitting us to choose such a prime $\mathfrak{p} \subset A$ so that the discrete valuation subring $A_{\mathfrak{p}}$ of $K$ is different from each of the localisations of $R$. Localising at one such height 1 prime of $A$, we can assume that it is a discrete valuation ring, and consider $\tilde{R} := R \cap A$ which is a Prüfer domain by Lemma 3.3. Over the open cover $\operatorname{Spec} R$ and $\operatorname{Spec} A$ of $\operatorname{Spec} \tilde{R}$

1. we glue $X$ and $X_A$ along $X_K$ to obtain a projective $\tilde{R}$-scheme $X_{\tilde{R}}$ fibrewise of pure dimension $d$ with a relative very ample line bundle,
2. we glue $W$ and $W_A$ along $W_K$ to obtain an open $W_{\tilde{R}} \subset X_{\tilde{R}}^{\mathrm{sm}}$,
3. we glue $Y$ and $Y_A$ along $Y_K$ to obtain a closed subscheme $Y_{\tilde{R}} \subset X_{\tilde{R}}$ such that the special fibres of $Y_{\tilde{R}} \setminus W_{\tilde{R}}$ are of codimension $\geq 2$.

By construction, $x_1, \dots, x_n$ specialise to points in the special fibre, and therefore the previous case applies and we are done. □

As a point of departure of the proof of Theorem A (see Section 2.1.1), we establish Theorem B from the introduction. Presentation Lemma 6.5 is inspired from [Česnavičius22, Proposition 4.1], where it was proved in the case of semilocal Dedekind rings. We note that the codimension $\geq 2$ hypothesis on $Y$ is to ensure the finiteness of the morphism $\pi$ when restricted to $Y \cap U$ (which, as we mentioned in the introduction, is important in the proof of Theorem A).

**Presentation Lemma 6.5.** *For*

- *a semilocal Prüfer domain $R$ of Krull dimension at most 1,*
- *a smooth $R$-scheme $X = \operatorname{Spec} A$ fibrewise of pure relative dimension $d > 0$,*
- *a closed subscheme $Y \subset X$ that is of codimension at least 2, and*
- *points $x_1, \dots, x_n \in X$;*

*there are an affine open $x_1, \dots, x_n \in U \subset X$, an affine open $S \subset \mathbb{A}_R^{d-1}$ and a smooth $R$-morphism $\pi\colon U \to S$ of pure relative dimension 1 such that $Y \cap U$ is $\pi$-finite.*

*Proof.* Choosing an embedding of $X$ into some $R$-affine space, let $\overline{X}$ be the schematic image of the corresponding morphism from $X$ to the $R$-projective space. Letting $j\colon X \hookrightarrow \overline{X}$ be the inclusion, by [Sta22, Tag 01RE], the canonical morphism $\mathcal{O}_{\overline{X}} \hookrightarrow j_*\mathcal{O}_X$ is injective. Therefore, since $X$ is $R$-flat, in particular, since $\mathcal{O}_X$ is $R$-torsion-free, $\mathcal{O}_{\overline{X}}$ is $R$-torsion-free, and hence, $\overline{X}$ is $R$-flat (it follows from the fact that flatness can be checked locally on the target and from [BouCA, Chapter I, §2.4, Proposition 3(ii)]). Moreover, since $\overline{X}$ is $R$-flat and of $R$-finite type, thanks to [RG71, Première partie, Corollaire 3.4.7], it is of $R$-finite presentation; whence, by [Sta22, Tag 02FZ and Tag 0D4J], it is also of $R$-fibrewise of pure dimension $d$. The flatness of $X$ and the constancy of fibrewise dimension ensures that the special $R$-fibres of $X$ are of codimension 1 in X ([Sta22, Tag 0D4H, cf. Tag 054L]). By [Sta22, Tag 081I], the generic fibre of $X$ is dense in $\overline{X}$. This implies that the special fibres of $\overline{X}$ are of codimension $\geq 1$ in $\overline{X}$, and on the other hand, they are of codimension $\leq 1$ thanks to [Sta22, Tag 0D4I]; showing that they are of codimension 1.

We define $\overline{Y}$ to be the schematic closure of $Y$ in $\overline{X}$. Given that the points of $Y$ are of height $\geq 2$, the points of $\overline{Y} \setminus Y$ are of height $\geq 3$. The generic fibre of $\overline{Y} \setminus Y$ is of codimension $\geq 3$, and since the special fibres of $\overline{X}$ are of codimension 1 in $\overline{X}$, the special fibres of $\overline{Y} \setminus Y$ are of codimension $\geq 2$ in the corresponding special fibres of $\overline{X}$. The preceding arguments needed that $X$ is catenary, which is the content of Proposition 3.8. The rest follows by applying Proposition 6.4 to $X \hookrightarrow \overline{X}$, i.e., by inputting our $X$ as the $W$ of the proposition and our $\overline{X}$ as the $X$. $\square$

# Chapter 7

# Lifting $E$ to a Torsor over the Relative Affine Line

In this chapter, the main result, i.e., Proposition 7.4.4, asserts that, starting with a generically trivial torsor $E$ on $\operatorname{Spec} A$ (as in Theorem A), we can produce a torsor $\mathscr{E}$ on $\mathbb{A}^1_A$ that pulls-back to $E$ along the zero section and that trivialises away from an $A$-finite closed subscheme $Z \subset \mathbb{A}^1_A$. This is enough to show that $E$ is trivial (see Chapter 8). Indeed, Proposition 8.2 implies that the pullback of $\mathscr{E}$ along the zero section is trivial. We arrive at Proposition 7.4.4 by taking a couple of steps (discussed in Section 2.1.1), for which Proposition 7.1.1 is the first one. We start with the first step.

## 7.1 Lifting the Torsor $E$ to a Relative Curve $C$

The first step, i.e., Proposition 7.1.1, utilises Presentation Lemma 6.5 to produce a relative curve $C \to \operatorname{Spec} A$ with a section $s \in C(A)$ and lifts (with respect to $s$) the generically trivial $G$-torsor $E$ over $A$ to a generically trivial torsor $\mathscr{E}$ over $C$ under a quasi-split reductive group $\mathscr{G}$ (which itself is a lift of $G$ to $C$). It helps us reduce the problem of studying a torsor over the ring $A$ to studying torsors over a relative curve over $A$.

**Proposition 7.1.1.** *For a semilocal Prüfer domain $R$ of dimension at most $1$, a ring $A$ that is obtained as the semilocalisation of a smooth $R$-domain at finitely many primes, a quasi-split reductive $A$-group scheme $G$ with a maximal torus $T$, a Borel subgroup $T \subset B \subset G$ and a generically trivial $G$-torsor $E$, there are*

(*i*) *a smooth, affine $A$-scheme $C$ of pure relative dimension $1$,*

(*ii*) *a section $s \in C(A)$,*

(*iii*) *a quasi-split reductive $C$-group scheme $\mathscr{G}$ with a maximal torus $\mathscr{T} \subset \mathscr{G}$ whose $s$-pullback is $T \subset G$,*

(*iv*) *a Borel subgroup $\mathscr{T} \subset \mathscr{B} \subset \mathscr{G}$ whose $s$-pullback is $T \subset B \subset G$,*

(*v*) *a $\mathscr{G}$-torsor $\mathscr{E}$ whose $s$-pullback is $E$, and*

(*vi*) *an $A$-finite closed subscheme $Z \subset C$*

*such that $\mathscr{E}_{C \setminus Z}$ reduces to an $\mathscr{R}_u(\mathscr{B})_{C \setminus Z}$-torsor.*

*Proof.* We follow the proof of [Čes22, Proposition 4.2]. Let $\mathcal{A}$ be a smooth $R$-domain such that $A$ is the semilocalisation of $\mathcal{A}$ at finitely many primes $\mathfrak{p}_i$. Possibly by shrinking $\operatorname{Spec}\mathcal{A}$, we may assume that $G, B, T$ and $E$ begin life over $\mathcal{A}$ (see [Con14, Proposition 3.1.9 for the spreading out argument for $G$, Theorem 5.2.11(1) for $B$, Theorem 3.2.6 for $T$]). If $\mathcal{A}$ is of $R$-relative dimension 0, then $\mathcal{A}$ is $R$-étale, and hence a semilocal Prüfer domain by [Sta22, Tag 092S, Tag 092D, Tag 092N, and Tag 092E] and by the fact that étale morphisms are quasi-finite. By Theorem 5.11, the torsor $E_A$ is trivial, and we can take $C = \mathbb{A}^1_A$, the zero section as $s$, $Z = \emptyset$, and the group schemes to be the base changes of their counterparts over $A$ along the structure map of $C$.

Henceforth, we assume that $\mathcal{A}$ is of $R$-relative dimension $d \geq 1$ and let $K$ be its fraction field. In view of [SGA $3_{\text{III}}$, Exposé XXVI, Corollaire 3.6, Lemme 3.20], the quotient $E/B$ is representable by a projective $\mathcal{A}$-scheme. Thanks to Lemma 3.10, the localisation $\mathcal{A}_\mathfrak{p}$ of $\mathcal{A}$ at any height 1 prime $\mathfrak{p} \in \operatorname{Spec}\mathcal{A}$ is a valuation ring, consequently, the valuative criterion of properness lifts a generic section $t \in (E/B)(K)$ that is induced by a generic trivialisation of $E$, to a unique section $t'$ as in the diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "t"] \arrow[d] & E/B \arrow[d] \\ \operatorname{Spec}\mathcal{A}_\mathfrak{p} \arrow[r] \arrow[ur, dashed, "t'"] & \operatorname{Spec}\mathcal{A}. \end{tikzcd} \tag{7.1.1.1}$$

By spreading out and gluing the sections we find a section $\tilde{t} \in (E/B)(U)$ over an open $U \subset \operatorname{Spec}\mathcal{A}$ containing all the height 1 primes of $\operatorname{Spec}\mathcal{A}$. Indeed, the gluing can be done because, due to irreducibility of $\operatorname{Spec}\mathcal{A}$, any two sections $t_1 \in (E/B)(U_1)$ and $t_2 \in (E/B)(U_2)$ lifting $t$ agree on an open $U_{12} \subset U_1 \cap U_2$ which is dense; and as a consequence, $t_1$ and $t_2$ agree on $U_1 \cap U_2$, making it suitable to glue them to a section $t_0 \in (E/B)(U_1 \cup U_2)$. Thus, the $G_U$-torsor $E_U$ reduces to a generically trivial $B_U$-torsor $E^t$ obtained by base changing $E \to E/B$ along $t\colon U \to E/B$. Consider

$$\text{the torus } T := B/\mathscr{R}_u(B) \text{ and the induced } T_U\text{-torsor } E^T := E^t/\mathscr{R}_u(B).$$

The fact that $T$ is a torus can be checked at the geometric fibres, and the geometric fibres are tori thanks to [Bor91, Theorem 10.6(4)]. Since $Y := \operatorname{Spec}\mathcal{A} \setminus U$ is of codimension $\geq 2$, by the purity result for tori (4.5.3), we can lift $E^T$ to a $T$-torsor $\widetilde{E^T}$ over $\operatorname{Spec}\mathcal{A}$. The torsor $\widetilde{E^T}$ is generically trivial, whose generic section is induced by the one of $E$. Then, the torus case of the Grothendieck–Serre conjecture (Theorem 4.7) implies that $\widetilde{E^T} \times_{\operatorname{Spec}\mathcal{A}} \operatorname{Spec} A$ is trivial. As a result, possibly by shrinking $\operatorname{Spec}\mathcal{A}$ around the maximal ideals of $A$, we can assume that $\widetilde{E^T}$ is trivial and that $E_U$ reduces to an $\mathscr{R}_u(B)$-torsor.

Finally, we apply the Presentation Lemma 6.5 to produce

- an affine open $U' \subset \operatorname{Spec}\mathcal{A}$ containing the $\mathfrak{p}_i$,
- an affine open $S \subset \mathbb{A}^{d-1}_R$ and
- a smooth morphism $\pi\colon U' \to S$ of pure relative dimension 1 such that $Y \cap U'$ is $\pi$-finite.

Let $f\colon \operatorname{Spec} A \to U'$ be the natural morphism induced by localisation, and let $C$ and $Z$ be defined the by the respective Cartesian diagrams

$$\begin{tikzcd}
Z \arrow[rr, hook] \arrow[dr, hook] \arrow[dddr] & & Y \cap U' \arrow[dr, hook] \arrow[dddr] & \\
& C \arrow[rr, hook] \arrow[dd] & & U' \arrow[dd, "\pi"] \\
& & & \\
& \operatorname{Spec} A \arrow[uu, dashed, "s"] \arrow[uurr, dashed, "f"] \arrow[rr] & & S,
\end{tikzcd}$$

where the section $s \in C(A)$ is induced by $f$. To satisfy the conditions (iii)-(v), we take $\mathscr{G}, \mathscr{T}, \mathscr{B}$ and $\mathscr{E}$ to be the base changes of $G, T, B$ and $E$ along the morphism $C \to U'$, so that, by construction, their $s$-pullbacks are the respective objects over $A$. The remaining condition (vi) is also satisfied. Indeed, since $E_U$ reduces to an $\mathscr{R}_u(B)_U$-torsor and $C \setminus Z = (U \cap U') \times_S \operatorname{Spec} A$, the torsor $\mathscr{E}_{C\setminus Z}$ reduces to an $\mathscr{R}_u(\mathscr{B})_{C\setminus Z}$-torsor. □

## 7.2 Replacing $C$ by an Étale Cover to Equate $\mathscr{G}$ and $G_C$

As a second step towards proving Theorem A (see Section 2.1.1), we show that the reductive group $\mathscr{G}$ in Proposition 7.1.1 on $C$ may be assumed to be isomorphic to the constant group $G_C$. The main result, Proposition 7.2.5, is proved by following arguments in [Čes22$_{\mathrm{Surv}}$, Section 6.3]. In this regard, the theory of toric varieties is employed to find an equivariant compactification of the torus (Proposition 7.2.1) over an arbitrary scheme $S$ with finitely many connected components. This extends the toroidal compactification results of Colliot-Thélène, Harari, and Skorobogatov in [CHS05] (which were generalised in [Čes22$_{\mathrm{Surv}}$, Section 6.3]). Equivariant compactifications $G \hookrightarrow \overline{G}$ of reductive groups make it possible to utilise techniques from projective geometry, for example, the Bertini theorem could be used to slice equivariant compactifications of $G$-torsors (see [Čes22$_{\mathrm{Surv}}$, Proposition 6.2.4]) by projective hypersurfaces to obtain a finite étale section (cf. [Čes22$_{\mathrm{Surv}}$, Lemma 6.2.2]). In particular, this argument shows that $G$-torsors can be trivialised by passing to a finite étale cover (see [Čes22$_{\mathrm{Surv}}$, Proposition 6.2.5]).

Toric varieties over valuation rings have been previously studied by Gubler and Soto in [GS15], and similar compactification results of normal varieties over valuation rings of rank 1 have been studied by Soto in [Sot18]. The Noetherian version of Proposition 7.2.1(a) (resp., (b)) was proved in [Čes22$_{\mathrm{Surv}}$, Theorem 6.3.1] (resp., op. cit. Proposition 6.2.4). We shall use the arguments of op. cit. to generalise to our setting. We begin with Proposition 7.2.1, which is the key result of this section. Although the assumption on $S$ does not seem to be necessary, we make it for simplicity.

**Proposition 7.2.1.** *Let $S$ be a scheme with finitely many connected components and let $T$ be an isotrivial $S$-torus.*

(*a*) *Then there are a projective, smooth $S$-scheme $\overline{T}$ with commuting left and right actions of $T$ and an $S$-fibrewise dense open immersion*

$$T \hookrightarrow \overline{T}$$

*that is equivariant with respect to both left and right actions of $T$.*

(*b*) *Moreover, if $S$ is quasi-compact and quasi-separated, any isotrivial $T$-torsor $E$ admits an $S$-fibrewise dense open immersion*

$$E \hookrightarrow \overline{E}$$

*into a finitely presented, projective $S$-scheme $\overline{E}$.*

*Proof.* (a): We shall use the arguments in the proof of [Čes22$_{\mathrm{Surv}}$, Theorem 6.3.1] (which uses techniques from [CHS05]). Writing $S$ as a union of its connected components, we may assume that it is connected. Let $S' \to S$ be a connected, Galois, finite étale cover that splits $T$. Arguing as in [Čes22$_{\mathrm{Surv}}$, Theorem 6.3.1], we use the work of Danilov in [Dan78] (more precisely, op. cit. Section 5) on toroidal compactifications to construct an analogous $\mathrm{Gal}(S'/S)$-equivariant $S'$-compactification $T_{S'} \hookrightarrow \overline{T}'$ such that $\overline{T}'$ is equipped with a $\mathrm{Gal}(S'/S)$-action that is compatible with the $\mathrm{Gal}(S'/S)$-action on $S'$ and that commutes with the left and the right actions of $T_{S'}$. In fact, we construct $\overline{T}'$ by base changing a scheme over $\mathrm{Spec}(\mathbb{Z})$ to $S'$. Using the dictionary between fans and toric varieties, the projectivity and the smoothness of $\overline{T}'$ translates to some combinatorial fact, which is checked by hand. Following this, we show that the induced morphism on the quotients

$$T \cong T_{S'} / \mathrm{Gal}(S'/S) \longleftrightarrow \overline{T} := \overline{T}' / \mathrm{Gal}(S'/S)$$

satisfies the hypothesis of (a). In effect, the $\mathrm{Gal}(S'/S)$-invariant left (resp., right) equivariant action of $T_{S'}$ on $\overline{T}'$ automatically descends to left (resp., right) equivariant action of $T$ on $\overline{T}$. Also, the $S$-fibrewise density of $T \hookrightarrow \overline{T}$ is ensured by the $S'$-fibrewise density of $T_{S'} \hookrightarrow \overline{T}'$. Thus, it suffices to check that $\overline{T}$ is a smooth and projective $S$-scheme. Thanks to [Sta22, Tag 07S7], the projectivity of $\overline{T}'$ and the freeness of the $\mathrm{Gal}(S'/S)$-action on $\overline{T}'$, which is guaranteed by the fact that the $\mathrm{Gal}(S'/S)$-action on $S'$ is free, ensures that $\overline{T}$ is an $S$-scheme and that the quotient morphism $q\colon \overline{T}' \to \overline{T}$ is a $\mathrm{Gal}(S'/S)$-torsor. In particular, $q$ is surjective and finite locally free ([Sta22, Tag 02VO]). Therefore, by [Sta22, Tag 05B5], $\overline{T}$ is a smooth $S$-scheme, in particular, it is locally of finite type over $S$. Consequently, thanks to [Sta22, Tag 0AH6], $\overline{T}$ is a proper $S$-scheme. Since $q$ is finite locally free, it has a norm ([Sta22, Tag 0BD2]), whence, by [Sta22, Tag 0BD0], $\overline{T}$ has a relative $S$-ample line bundle. This finishes the proof.

(b): Let $T \hookrightarrow \overline{T}$ be a compactification as in (a). Writing $S$ as a union of its connected components, we may assume that it is connected. Following [Čes22$_{\mathrm{Surv}}$, Proposition 6.2.4], we show that the canonical morphism $E \hookrightarrow \overline{E} := E \times^T \overline{T}$[6] satisfies the requisite properties. By [Sta22, Tag 06PH], $\overline{E}$ is represented by an algebraic space.

To verify that $\overline{E}$ is a projective $S$-scheme, we pass to a Galois, finite étale cover $\tilde{S} \to S$ (say, of degree $d$) which trivialises $E$ (which exists by the isotriviality of $E$) to get a finitely presented, projective $\tilde{S}$-scheme $\overline{E}_{\tilde{S}} \cong \overline{T}_{\tilde{S}}$. A limit argument and the fact that Weil restriction commutes with limits of schemes (as a right adjoint to the base change functor, see [BLR90, Section 7.6, Lemma 1]), reduces us to the Noetherian case so that [CGP15, Proposition A.5.8] shows that

$$f\colon \mathrm{Res}_{\tilde{S}/S}(\overline{E}_{\tilde{S}}) \to S$$

is a quasi-projective morphism that is equipped with a relatively ample line bundle, say $\mathscr{L}$, which is even relatively very ample because $S$ is quasi-compact. In fact, given the following canonical morphisms

$$\overline{E}_{\tilde{S}} \xleftarrow{\psi} \mathrm{Res}_{\tilde{S}/S}(\overline{E}_{\tilde{S}}) \times_S \tilde{S} \xrightarrow{\pi} \mathrm{Res}_{\tilde{S}/S}(\overline{E}_{\tilde{S}}),$$

[6] We define the *contracted product* $E \times^T \overline{T}$ as the quotient (as a stack [Sta22, Tag 044O]) of $E \times_S \overline{T}$ by the diagonal action of $T$.

by the proof of loc. cit., $\mathscr{L}$ can be chosen to be $N_\pi(\mathscr{M})$, where $\mathscr{M}$ is the pullback along $\psi$ of an $\tilde{S}$-relatively very ample line bundle on $\overline{E}_{\tilde{S}}$ and $N_\pi(\mathscr{M})$ is the norm (see [EGA II, §6.5] or [Sta22, Tag 0BCX]) of $\mathscr{M}$ along the finite étale morphism $\pi$. According to [EGA II, Remarques 5.5.4(i)], to verify that $f$ is projective, it remains to check that it is proper and that $f_*(\mathscr{L})$ is a finite-type quasi-coherent $\mathcal{O}_S$-module. Since the property of properness can be checked fpqc locally [Sta22, Tag 02L1], it is sufficient to show that $\mathrm{Res}_{\tilde{S}/S}(\overline{E}_{\tilde{S}}) \times_S \tilde{S}$ is projective. We have

$$\mathrm{Res}_{\tilde{S}/S}(\overline{E}_{\tilde{S}}) \times_S \tilde{S} \xrightarrow{\sim} \mathrm{Res}_{\tilde{S}\times_S\tilde{S}/\tilde{S}}(\overline{E}_{\tilde{S}} \times_S \tilde{S}) \xrightarrow{\sim} \overline{E}_{\tilde{S}}^{\times d},$$

where the isomorphisms follow from [JLMMS17, Proposition 2.2(2) and Lemma 2.3 respectively]. Therefore, the properness of $\overline{E}_{\tilde{S}}$ implies the same for the morphism $f$. Lastly, we need to verify that $f_*(\mathscr{L})$ is a finite-type $\mathcal{O}_S$-module. Since it is enough to verify this Zariski locally on $S$, we may assume that $S = \operatorname{Spec} A$ and $\tilde{S} = \operatorname{Spec} \tilde{A}$. Furthermore, since $f_*(\mathscr{L})$ is a quasi-coherent $\mathcal{O}_S$-module (see, for example, [Sta22, Tag 03M9]), it suffices to check that

$$\Gamma(S, f_*(\mathscr{L})) = \Gamma(\mathrm{Res}_{\tilde{S}/S}(\overline{E}_{\tilde{S}}), \mathscr{L}) \text{ is a finite } A\text{-module.} \tag{7.2.1.1}$$

By the affine base change theorem [Sta22, Tag 0CKW] and the Galois descent of finite modules, it is enough to show (7.2.1.1) after taking a finite étale cover of $S$. This permits us to show (7.2.1.1) after base changing along $\tilde{S} \to S$. By abuse of notation, we replace $S$ by $\tilde{S}$ and $\tilde{S}$ by $\tilde{S} \times_S \tilde{S}$ to assume, without loss of generality, that the finite étale cover $\tilde{S} \to S$ is split, say of degree $n$. Let $\tilde{S} = \coprod S_i$ be a decomposition into connected components and let $\mathscr{M}_i := \mathscr{M}\,|_{S_i}$, for each $i = 1, \dots, n$. We identify $S_i$ with $S$ and $\mathscr{M}_i$ with a line bundle on $\mathrm{Res}_{\tilde{S}/S}(\overline{E}_{\tilde{S}})$, for each $i = 1, \dots, n$, and therefore, we identify $\tilde{S}$ with a disjoint union of $n$ copies of $S$, or equivalently, we identify $\tilde{A}$ with a product of $n$ copies of $A$. Using this identification, the norm $N_{\tilde{A}/A}$ is the morphism $(a_1, \dots, a_n) \mapsto \prod a_1 \cdots a_n$, forcing the equality $N_\pi(\mathscr{M}_1, \dots, \mathscr{M}_n) = \mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_n$ (we use the fact that norms commute with base change [Sta22, Tag 0BD2]). Since each $\Gamma(\mathrm{Res}_{\tilde{S}/S}(\overline{E}_{\tilde{S}}), \mathscr{M}_i)$ is a finitely type $A$-module, it follows that $\Gamma(\mathrm{Res}_{\tilde{S}/S}(\overline{E}_{\tilde{S}}), \mathscr{L}) = \otimes_i \Gamma(\mathrm{Res}_{\tilde{S}/S}(\overline{E}_{\tilde{S}}), \mathscr{M}_i)$ is a finite $A$-module. Thus, we are done.

The above argument shows that $\overline{E}'$ is a projective $S$-scheme. To argue that $\overline{E}$ is a projective $S$-scheme, we check that the canonical morphism $\overline{E} \to \overline{E}'$ is a closed immersion by base changing along $\tilde{S} \to S$ (being a closed immersion is fpqc local on the base [Sta22, Tag 02L6]). Since schematic image commutes with flat base change in the quasi-compact case [Sta22, Tag 081I], the $S$-fibrewise density can be checked after taking an fpqc cover, enabling us to assume that $E$ is trivial, in which case the result is equivalent to the $S$-fibrewise density of $T \hookrightarrow \overline{T}$. □

The following Corollary 7.2.2, which is a generalisation of [Čes22$_{\mathrm{Surv}}$, Corollary 6.3.2] (cf. [Fed22, Proposition 4.3]), is a formal consequence of Proposition 7.2.1 (see [Čes22$_{\mathrm{Surv}}$, Lemma 6.2.2]). It shows that sections of torsors under isotrivial tori over closed subschemes can be extended after passing to finite étale covers.

**Corollary 7.2.2.** *For a normal, semilocal domain $A$, an ideal $I \subset A$, an $A$-torus $T$, a $T$-torsor $E$ on $A$, and an $e \in E(A/I)$, there are a faithfully flat, finite, étale $A$-algebra $\tilde{A}$ with an $A/I$-point $a\colon \tilde{A} \twoheadrightarrow A/I$ and an $\tilde{e} \in E(\tilde{A})$ whose $a$-pullback is $e$.*

*Proof.* The ring $A$ can be written as a filtered colimit of Noetherian, normal, semilocal domains $A_\lambda$, where $\lambda \in \Lambda$. By a limit argument, there exists a $\lambda \in \Lambda$ such that $T$ (resp., $E$) descends to an $A_\lambda$-torus

$T_\lambda$ (resp., $T_\lambda$-torsor $E_\lambda$). By [SGA $3_{\mathrm{II}}$, Exposé X, Corollaire 5.14 and Théorème 5.16] (or more generally, [SGA $3_{\mathrm{III}}$, Exposé XXIV, Théorème 4.1.5 and Corollaire 4..1.6]) $T_\lambda$ and $E_\lambda$ are isotrivial, which implies that $T$ and $E$ are isotrivial. Thanks to Proposition 7.2.1(b), a toroidal compactification $T \hookrightarrow \overline{T}$ of part (a) produces an $A$-fibrewise dense open immersion $E \hookrightarrow \overline{E} := E \times^T \overline{T}$ into a projective, finitely presented $A$-scheme $\overline{E}$. This allows us to use [Čes22$_{\mathrm{Surv}}$, Lemma 6.2.2] to conclude that $\overline{E}$ has property ($\star$) of loc. cit. with respect to $I \subset A$, i.e., given $e \in E(A/I)$, there are a faithfully flat, finite, étale $A$-algebra $\tilde{A}$ with an $A/I$-point $a \colon \tilde{A} \twoheadrightarrow A/I$ and an $\tilde{e} \in E(\tilde{A})$ whose $a$-pullback is $e$. □

*Remark* 7.2.3. If we remove the 'finite' hypothesis from the $A$-algebra $\tilde{A}$ in Corollary 7.2.2, the statement is a consequence of invariance under Henselian pairs [Čes22$_{\mathrm{Surv}}$, Proposition 6.1.1(a)].

The following Lemma 7.2.4 draws inspiration from [Fed22, Proposition 4.4] and [Čes22, Lemma 5.1] and we shall closely follow their proofs. This lemma which is, in principle, a non-Noetherian version of loc. cit., provides the necessary ingredient to prove Proposition 7.2.5.

**Lemma 7.2.4.** *For a normal, semilocal domain $A$ whose spectrum is a Noetherian topological space, an ideal $I \subset A$, quasi-split reductive $A$-group schemes $G$ and $G'$ that have the same type on geometric $A$-fibres, maximal tori $T \subset G$ and $T' \subset G'$, Borel subgroups $T \subset B \subset G$ and $T' \subset B' \subset G'$ and an $A/I$-group isomorphism $\iota \colon G_{A/I} \xrightarrow{\sim} G'_{A/I}$ with $\iota(T \subset B) = T' \subset B'$, there are*

- *a faithfully flat, finite, étale cover $f \colon A \to \tilde{A}$ equipped with an $A/I$-point $a \colon \tilde{A} \twoheadrightarrow A/I$ and*
- *an $\tilde{A}$-group isomorphism $\tilde{\iota} \colon G_{\tilde{A}} \xrightarrow{\sim} G'_{\tilde{A}}$ with $\tilde{\iota}(T_{\tilde{A}} \subset B_{\tilde{A}}) = T'_{\tilde{A}} \subset B'_{\tilde{A}}$ such that $a^*(\tilde{\iota}) = \iota$.*

*Proof.* Let $\underline{\mathrm{Aut}}(G, B, T)$ (resp., $\underline{\mathrm{Out}}(G)$, resp., $\underline{\mathrm{Isom}}((G, B, T), (G', B', T'))$) be the group of automorphisms of $G$ fixing $T \subset B$ (resp., the subgroup of outer automorphisms of $G$, resp., the set of isomorphisms $G \xrightarrow{\sim} G'$ sending $T \subset B$ to $T' \subset B'$). Suppose that $Z \subset G$ is the center, let $G^{\mathrm{ad}} := G/Z$ be the adjoint group and set $T^{\mathrm{ad}} := T/Z$. By [SGA $3_{\mathrm{III}}$, Exposé XXIV, Proposition 2.1 and Corollaire 2.2], there is an exact sequence

$$1 \longrightarrow T^{\mathrm{ad}} \longrightarrow \underline{\mathrm{Aut}}(G, B, T) \longrightarrow \underline{\mathrm{Out}}(G) \longrightarrow 1, \tag{7.2.4.1}$$

and $X := \underline{\mathrm{Isom}}((G, B, T), (G', B', T'))$ is an $\underline{\mathrm{Aut}}(G, B, T)$-torsor. As a consequence of the exact sequence (7.2.4.1), $X/T^{\mathrm{ad}}$ is a torsor under $\underline{\mathrm{Out}}(G)$, which is represented by a group scheme that becomes constant étale locally on $A$ (see [SGA $3_{\mathrm{III}}$, Exposé XXIV, Théorème 1.3(ii)]). Thanks to [SGA $3_{\mathrm{II}}$, Exposé X, Corollaire 5.14] (with [Sta22, Tag 04MF and Tag 04ME], see also [Čes22$_{\mathrm{Surv}}$, Example 6.2.1]), the hypothesis on $A$ ensures that the connected components of $X/T^{\mathrm{ad}}$ are open subschemes that are finite étale over $\operatorname{Spec} A$. The $A/I$-point $\iota \in (X/T^{\mathrm{ad}})(A/I)$ touches finitely many of these connected components, whose union is then a spectrum of a finite étale $A$-algebra $A_T$. By adding further components, we may assume that the closed fibres of $A_T$ is nonempty so that $A_T$ is a faithfully flat $A$-algebra. Lastly, since

$$X \longrightarrow X/T^{\mathrm{ad}}$$

is a $T^{\mathrm{ad}}$-torsor, Corollary 7.2.2 produces a faithfully flat, finite étale $A_T$-algebra $\tilde{A}$ with a section $s$ lifting the $A/I$-point $\iota$. □

We are now ready to prove the main result of this section. We follow the proof of [Čes22, Proposition 5.2] (cf. [Fed22, Proposition 4.5(a)-(d)]).

**Proposition 7.2.5.** *For a semilocal Prüfer domain $R$ of dimension at most 1, a ring $A$ that is obtained as the semilocalisation of a smooth $R$-domain at finitely many primes, a quasi-split reductive $A$-group scheme $G$ with a maximal torus $T$, a Borel subgroup $T \subset B \subset G$ and a generically trivial $G$-torsor $E$, there are*

(i) *a smooth, affine $A$-scheme $C$ of pure relative dimension 1,*

(ii) *a section $s \in C(A)$,*

(iii) *an $A$-finite closed subscheme $Z \subset C$, and*

(iv) *a $G_C$-torsor $\mathscr{E}$ whose $s$-pullback is $E$ such that $\mathscr{E}$ reduces to an $\mathscr{R}_u(B)$-torsor over $C \setminus Z$.*

*Proof.* By Proposition 7.1.1, there are a smooth, affine $A$-scheme $C$ of pure relative dimension 1 with a section $s \in C(A)$, an $A$-finite closed subscheme $Z \subset C$, a quasi-split reductive $C$-group scheme $\mathscr{G}$ whose $s$-pullback is $G$ along with a maximal torus $\mathscr{T} \subset \mathscr{G}$ whose $s$-pullback is $T \subset G$, a Borel subgroup $\mathscr{T} \subset \mathscr{B} \subset \mathscr{G}$ whose $s$-pullback is $T \subset B \subset G$ and a $\mathscr{G}$-torsor $\mathscr{E}$ such that $\mathscr{E}$ reduces to an $\mathscr{R}_u(\mathscr{B})$-torsor over $C \setminus Z$. However, a priori it is not clear that the group $\mathscr{G}$ equals $G_C$. In order to obtain the equality of groups, we shall replace $C$ by the source of an étale morphism $\tilde{C} \to C$.

We replace $C$ by its connected component containing the image of $s$ to arrange that $C$ be connected. Let $\mathcal{C}$ be the coordinate ring of the semilocalisation of $C$ at the closed points of $Z$ and those of the image of $s$ and let $\mathcal{I} \subset \mathcal{C}$ be the ideal of vanishing at the closed subset determined by $s$. Since $\mathcal{C}$ is essentially smooth over $A$, an argument similar to the proof of Corollary 4.9 shows that $\mathcal{C}$ is a normal domain. Moreover, by Remark 3.9, the topological space $\mathcal{C}$ is Noetherian. By the construction of $\mathscr{G}$, there is an isomorphism $\iota \colon G_{\mathcal{C}/\mathcal{I}} \xrightarrow{\sim} \mathscr{G}_{\mathcal{C}/\mathcal{I}}$ that sends $T_{\mathcal{C}/\mathcal{I}} \subset B_{\mathcal{C}/\mathcal{I}}$ to $\mathscr{T}_{\mathcal{C}/\mathcal{I}} \subset \mathscr{B}_{\mathcal{C}/\mathcal{I}}$. The groups $G_{\mathcal{C}}$ and $\mathscr{G}_{\mathcal{C}}$ have constant type on the geometric fibres over the connected scheme $\operatorname{Spec} \mathcal{C}$. The isomorphism $\iota$ lets us conclude that they even have the same type on the geometric fibres. Thanks to Lemma 7.2.4, there are a faithfully flat, finite, étale $\mathcal{C}$-algebra $\tilde{\mathcal{C}}$ with a $\mathcal{C}/\mathcal{I}$-point $a \colon \tilde{\mathcal{C}} \twoheadrightarrow \mathcal{C}/\mathcal{I}$ and an isomorphism $\tilde{\iota} \colon G_{\tilde{\mathcal{C}}} \xrightarrow{\sim} \mathscr{G}_{\tilde{\mathcal{C}}}$ lifting $\iota$ that sends $T_{\tilde{\mathcal{C}}} \subset B_{\tilde{\mathcal{C}}}$ to $\mathscr{T}_{\tilde{\mathcal{C}}} \subset \mathscr{B}_{\tilde{\mathcal{C}}}$. We spread out $\tilde{\mathcal{C}}$ to a smooth, affine $A$-scheme $\tilde{C}$ of pure relative dimension 1. Likewise, possibly by shrinking $\tilde{C}$, the finite étale morphism $\tilde{\mathcal{C}} \to \mathcal{C}$ spreads out to a finite étale morphism $\tilde{C} \to C'$, where $C' \subset C$ is an open subscheme containing $Z$ and the image of $s$. The section $s$ lifts to a section $\tilde{s} := a \circ s \in \tilde{C}(A)$ and an $A$-finite closed subscheme $\tilde{Z} \subset \tilde{C}$ is defined by base changing $Z \subset C$ along $\tilde{C} \to C'$. The last condition (iv) is automatically satisfied by construction. □

## 7.3 Building a Nisnevich type gluing square along $C \to \mathbb{A}^1_A$

In this section, we follow [Čes22, Section 6]. Starting with the $A$-curve $C$ and the $A$-finite closed subscheme $Z \subset C$ of Proposition 7.2.5, it is shown in Proposition 7.3.3 that there is a flat, quasi-finite morphism $C \to \mathbb{A}^1_A$ such that $Z$ maps isomorphically to a closed subset $Z' \subset \mathbb{A}^1_A$ for which it is the scheme theoretic pre-image. This produces a Nisnevich type gluing Cartesian square

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "\wr"] & C \arrow[d] \\ Z' \arrow[r, hook] & \mathbb{A}^1_A. \end{tikzcd} \tag{7.3.0.1}$$

However, a priori, if some residue fields of $A$ are finite, the closed subset $Z$ could have 'too many' rational points in the fibres, making it impossible to be embedded in the affine line. The following Lemmas 7.3.1 and 7.3.2 (see [Čes22, Lemmas 6.1 and 6.3]) show that we can modify $C$ and $Z$ to overcome this problem.

**Lemma 7.3.1** ([Čes22, Lemma 6.1]). *For a semilocal ring $A$, a quasi-projective $A$-scheme $C$ of finite presentation with an $A$-finite closed subscheme $Z \subset C$ and a section $s \in C(A)$, there are*

- *an open affine subscheme $C' \subseteq C$ containing $Z$ and the image of $s$ and*
- *a finite, étale, surjective morphism $\tilde{C} \to C'$ that lifts $s$ to a section $\tilde{s} \in \tilde{C}(A)$*

*such that for all maximal ideals $\mathfrak{m} \subset A$ with a finite residue field $\kappa(\mathfrak{m})$, the closed subscheme $\tilde{Z} := Z \times_C \tilde{C}$ satisfies the inequality*

$$\#\{z \in \tilde{Z}_{\kappa(\mathfrak{m})} \mid [\kappa(z) \colon \kappa(\mathfrak{m})] = d\} < \#\{z \in \mathbb{A}^1_{\kappa(\mathfrak{m})} \mid [\kappa(z) \colon \kappa(\mathfrak{m})] = d\}, \quad \text{for every } d \geq 1.$$

*Proof.* We restate the proof of loc. cit. Applying loc. cit. to the $A$-finite closed subscheme $Z' := Z \cup s$, such an open subscheme $C' \subseteq C$ (resp., a finite étale $C'$-scheme $\tilde{C}$) is constructed by spreading out the semilocalisation $S$ of $C$ at $Z'$ (resp., by spreading out the semilocal scheme $\tilde{S}$) as in the proof of loc. cit. □

**Lemma 7.3.2** ([Čes22, Lemma 6.3]). *Given a semilocal ring $A$, a flat, affine $A$-scheme $C$ with Cohen–Macaulay fibres of pure dimension 1 and $A$-finite closed subschemes $Y \subset C$ and $Z \subset C^{\mathrm{sm}}$ such that for every maximal ideal $\mathfrak{m} \subset A$ with finite residue field $\kappa(\mathfrak{m})$, we have*

$$\#\{z \in Z_{\kappa(\mathfrak{m})} \mid [\kappa(z) \colon \kappa(\mathfrak{m})] = d\} < \#\{z \in \mathbb{A}^1_{\kappa(\mathfrak{m})} \mid [\kappa(z) \colon \kappa(\mathfrak{m})] = d\}, \quad \text{for every } d \geq 1,$$

*there are*

(*i*) *an affine open $C' \subset C$ containing $Y$ and $Z$, and*

(*ii*) *a quasi-finite, flat $A$-morphism $C' \to \mathbb{A}^1_A$ that maps $Z$ isomorphically onto a closed subscheme*

$$Z' \subset \mathbb{A}^1_A \quad \text{such that} \quad Z \cong Z' \times_{\mathbb{A}^1_A} C'.$$

We do not include a proof of Lemma 7.3.2, and we refer the reader to the proof of loc. cit.

Using Lemmas 7.3.1-7.3.2, following the argument of [Čes22, Proposition 6.5], we give a proof of the main result of this section below.

**Proposition 7.3.3.** *For a semilocal Prüfer domain $R$ of dimension at most 1, a ring $A$ that is obtained as the semilocalisation of a smooth $R$-domain at finitely many primes, a quasi-split reductive $A$-group scheme $G$ with a maximal torus $T$, a Borel $A$-subgroup $T \subset B \subset G$, and a generically trivial $G$-torsor $E$, there are*

(*i*) *a smooth, affine $A$-scheme $C$ of pure relative dimension 1,*

(*ii*) *a section $s \in C(A)$,*

(iii) *an $A$-finite closed subscheme $Z \subset C$,*

(iv) *a $G_C$-torsor $\mathscr{E}$ whose $s$-pullback is $E$ such that $\mathscr{E}$ reduces to an $\mathscr{R}_u(B)$-torsor over $C \setminus Z$,*

(v) *a quasi-finite, flat $A$-morphism $C \to \mathbb{A}^1_A$ that maps $Z$ isomorphically onto a closed subscheme*

$$Z' \subset \mathbb{A}^1_A \quad \text{such that} \quad Z = Z' \times_{\mathbb{A}^1_A} C.$$

*Proof.* Points (i)-(iv) being supplied by Proposition 7.2.5, we are reduced to show the point (v). By Lemma 7.3.1, possibly by shrinking $C$ and passing to a finite étale cover, we may assume that for all closed points $c \in \operatorname{Spec} A$ with finite residue field $\kappa(c)$ we have that

$$\#\{z \in Z_{\kappa(c)} \mid [\kappa(z)\colon \kappa(c)] = d\} < \#\{z \in \mathbb{A}^1_{\kappa(c)} \mid [\kappa(z)\colon \kappa(c)] = d\}, \quad \text{for every } d \geq 1.$$

This allows us to apply Lemma 7.3.2 with $Y = s$ to arrange (v), possibly by shrinking $C$, and to conclude. □

## 7.4 Descending the torsor $\mathscr{E}$ to $\mathbb{A}^1_A$ by Nisnevich type gluing

In this section, we glue the torsor $\mathscr{E}$ of Proposition 7.3.3 with a trivial torsor on $\mathbb{A}^1_A \setminus Z'$ along the Nisnevich-type Cartesian square (7.3.0.1) (see Proposition 7.4.4). As a consequence, we obtain a torsor on $\mathbb{A}^1_A$ that trivialises away from the $A$-finite closed subscheme $Z'$ such that it pulls-back to $E$ along the zero section. For our purpose, we need Proposition 7.4.1, which is a version of formal gluing, i.e., Beauville–Laszlo gluing, over general bases. We begin with our discussion on Proposition 7.4.1.

Given a scheme $S$ with a closed subscheme $Z \subset S$ and its complement $U \subset X$, a flat morphism $f\colon S' \to S$ of schemes whose base change to $Z$ induces an isomorphism produces a 'Nisnevich type gluing square'[7].

$$\begin{tikzcd} Z' \arrow[r, hook, "i'"] \arrow[d, "f_Z"', "\wr"] \arrow[dr, phantom, "\square"] & S' \arrow[d, "f"] \arrow[dr, phantom, "\square"] & U' \arrow[l, hook', "j'"'] \arrow[d, "f_U"] \\ Z \arrow[r, hook, "i"'] & S & U. \arrow[l, hook', "j"] \end{tikzcd} \tag{7.4.0.1}$$

The following is a consequence of [Ryd11, Theorems A and B], which simultaneously generalises [Mor96, Corollaire 6.5.1(a)] and [FR70, Proposition 4.2].

**Proposition 7.4.1.** *Let $f\colon S' \to S$ be a finitely presented, flat morphism of schemes whose base change to a closed subscheme $Z$ is an isomorphism and let $f_U\colon U' \to U$ be the base change to $U := S \setminus Z$ (as in (7.4.0.1)). There exists an equivalence of categories*

$$\operatorname{QCoh}(S) \xrightarrow{\sim} \operatorname{QCoh}(S') \times_{\operatorname{QCoh}(U')} \operatorname{QCoh}(U), \tag{7.4.1.1}$$

*i.e., the category of quasi-coherent $\mathcal{O}_S$-modules $\mathscr{F}$ is equivalent to the category of triplets formed by a quasi-coherent $\mathcal{O}_{S'}$-module $\mathscr{F}'$, a quasi-coherent $\mathcal{O}_U$-module $\mathscr{F}_U$ and an isomorphism $\varphi_{\mathscr{F}}\colon j'^*\mathscr{F}' \xrightarrow{\sim}$*

[7] Closely related to *a flat Mayer–Vietoris diagram* of [HR16, Definition 1.2, cf. Lemma 3.2(2)], which assumes that $f_{Z^{[n]}}$ is an isomorphism for all $n$-thickenings of $i\colon Z \hookrightarrow X$

*$f_U^*\mathscr{F}_U$. Moreover, for a flat (resp., smooth), finitely presented $S$-group scheme $G$, the natural functor from the groupoid of $G$-torsors on $S$ that trivialise in the flat topology (resp., in the étale topology) to the groupoid of triples consisting of a $G_{S'}$-torsor on $S'$ that trivialise in the flat topology (resp., in the étale topology), a $G_U$-torsor on $U$ that trivialise in the flat topology (resp., in the étale topology) and a $G_{U'}$-torsor isomorphism between the two base changes to $U'$ is an equivalence.*

*Proof.* The equivalence (7.4.1.1) is a consequence of [Ryd11, Theorem A] (although, a priori, loc. cit. requires $f$ to be étale, the paragraph after op. cit. Remark 1.4 clarifies that it is enough to assume that $f$ is flat). The statement about $G$-torsors is a consequence of [Ryd11, Theorem B] (again, the application of loc. cit. is valid because the paragraph after op. cit. Remark 1.4 clarifies that it is enough to assume that $f$ is flat). Indeed, given the hypothesis on $G$, the classifying stack $\mathbf{B}G$ is algebraic (see [Sta22, Tag 06FI]) so that we may apply $\mathrm{Hom}(-, \mathbf{B}G)$ to the co-Cartesian diagram of [Ryd11, Theorem B]. □

To be able to perform the Beauville–Laszlo gluing along the Nisnevich-type gluing square (7.3.0.1) in Proposition 7.4.4, as a part of gluing data, we require to have an isomorphism over $U$ between objects on $S$ restricted to $U$ and objects on $U'$ restricted to $U$ (following notations of (7.3.0.1)). This is guaranteed by the following result, which is essentially [Česnavičius22, Lemma 7.2], whose proof we follow. Strictly speaking, the quasi-compact hypothesis of the open $U \hookrightarrow X$ in Lemma 7.4.2 is not needed but assumed to reduce further complications of the argument.

**Lemma 7.4.2.** *Let $f\colon S' := \operatorname{Spec} A' \to S := \operatorname{Spec} A$ be a finitely presented, flat morphism of affine schemes, let $U \hookrightarrow S$ be a quasi-compact open such that the base change of $f$ to $Z := S \setminus U$ is an isomorphism and let $U' \to U$ be the base change of $f$ to $U$.*

(*a*) *For a quasi-coherent $\mathscr{O}_S$-module $\mathscr{F}$ with pullback $\mathscr{O}_{S'}$-module $\mathscr{F}'$, we have an isomorphism*

$$\mathrm{R}\Gamma_Z(A, \mathscr{F}) \xrightarrow{\sim} \mathrm{R}\Gamma_Z(A', \mathscr{F}').$$

(*b*) *For an affine, smooth $S$-group (resp., $U$-group) $F$ with a filtration*

$$F = F_0 \supset F_1 \supset \ldots \supset F_n = 0 \tag{7.4.2.1}$$

*by normal, affine, smooth $S$-subgroups (resp., $U$-subgroups) such that, for all $i \geq 0$, the quotient $F_i/F_{i+1}$ is a vector group[8] associated to a vector bundle on $S$ (resp., such that the vector group $F_i/F_{i+1}$ is also central in $F/F_{i+1}$), the morphism*

$$H^1(U, F) \to H^1(U', F) \text{ has trivial kernel (resp., is surjective).}$$

(*c*) *For the unipotent radical $\mathscr{R}_u(P)$ of a parabolic $S$-subgroup $P$ of a reductive $S$-group $G$, the morphism*

$$H^1(U, \mathscr{R}_u(P)) \xrightarrow{\sim} H^1(U', \mathscr{R}_u(P)) \text{ is an isomorphism.} \tag{7.4.2.2}$$

[8]For a quasi-coherent sheaf $\mathscr{F}$, the associated *vector group* (see [SGA $3_{\mathrm{I}}$, Exposé I, Définition 4.6.1]) is the étale $\mathscr{O}_S$-module

$$\mathbb{W}(\mathscr{F})(S') = \Gamma(S', \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}).$$

If $\mathscr{F}$ is a vector bundle, $\mathbb{W}(\mathscr{F})$ is representable by an affine $S$-scheme (see op. cit. Notation 4.6.3.1 and Corollaire 4.6.5.1).

*Proof.* Following [Ryd16, Proposition 8.2], we show that there is a finitely presented closed subscheme $Z_{\lambda_0} \subset X$ such that $U = S \setminus Z_{\lambda_0}$. Letting $\mathscr{I} \subset \mathscr{O}_S$ be the defining ideal sheaf of $Z$, we write $\mathscr{I}$ as an increasing union of its finitely generated sub-ideals $\mathscr{I}_\lambda \subset \mathscr{O}_S$, for $\lambda \in \Lambda$. Since $U$ is quasi-compact, the open covering $\{U_\lambda := S \setminus V(\mathscr{I}_\lambda)\}_{\lambda \in \Lambda}$ of $U$ has a finite subcovering, consequently, there is a $\lambda_0 \in \Lambda$ such that $U_{\lambda_0} = U$, in particular, the closed subscheme $Z_{\lambda_0} := V(\mathscr{I}_{\lambda_0})$ works. Let $Z'_{\lambda_0} := Z_{\lambda_0} \times_S S'$.

(a): By the definition of cohomology with supports, since the closed immersion $Z \hookrightarrow Z_{\lambda_0}$ induces an identification of topological spaces,

$$\mathrm{R}\Gamma_Z(A, \mathscr{F}) \xrightarrow{\sim} \mathrm{R}\Gamma_{Z_{\lambda_0}}(A, \mathscr{F}).$$

Similarly, we have

$$\mathrm{R}\Gamma_Z(A', \mathscr{F}') \xrightarrow{\sim} \mathrm{R}\Gamma_{Z'_{\lambda_0}}(A', \mathscr{F}').$$

Taking the above displayed isomorphisms into account, by [Sta22, Tag 0ALZ], there is an isomorphism

$$\mathrm{R}\Gamma_Z(A, \mathscr{F}) \otimes_A A' \xrightarrow{\sim} \mathrm{R}\Gamma_Z(A', \mathscr{F}')$$

(since $A \to A'$ is flat, the derived tensor product coincides with the usual one). As a last step, it suffices to argue that there is an isomorphism

$$H^q_Z(A, \mathscr{F}) \xrightarrow{\sim} H^q_Z(A, \mathscr{F}) \otimes_A A', \quad \text{for all } q \geq 0.$$

The displayed isomorphism is a consequence of the equivalence (7.4.1.1), since as an $A$-module (resp., $A'$-module) $H^q_Z(A, \mathscr{F})$ (resp., $H^q_Z(A', \mathscr{F}')$) is supported in $Z$.

(b): We recall that the quasi-coherent property is preserved by pushforward along quasi-compact open immersions (see [Sta22, Tag 03M9]). The rest of the argument is identical to the proof of [Čes22, Lemma 7.2(b)] (we use Proposition 7.4.1).

(c): By [SGA $3_{\mathrm{III}}$, Exposé XXVI, Proposition 2.1], there is a filtration (7.4.2.1). Therefore, we can apply (b) to conclude. $\square$

Strictly speaking, the torsor $\mathscr{E}$ reduces to an $\mathscr{R}_u(B)$-torsor on $C \setminus Z$, which is, a priori, not an affine scheme. However, in order to formally glue $\mathscr{E}$ with the trivial torsor on $\mathbb{A}^1_A \setminus Z'$ along the Nisnevich-type gluing square (7.3.0.1), Proposition 7.4.4 requires $\mathscr{E}$ to be trivial along $C \setminus Z$. One possible solution is to replace $Z$ by a larger $A$-finite closed subscheme in $\mathbb{A}^1_A$ such that $\mathbb{A}^1_A \setminus Z$ is affine, from which the gluing is guaranteed by Lemma 7.4.2. The following, which shows that such a replacement of $Z$ is possible, is a slight modification of [GLL15, Theorem 5.1], where we have replaced the finitely presented closed subscheme $Z$ of loc. cit. with a closed subscheme whose complement is a quasi-compact open.

**Lemma 7.4.3.** *Given a ring $A$, for*

- *a quasi-projective morphism $\pi\colon X \to \operatorname{Spec} A$ of finite presentation with an $A$-ample line bundle $\mathscr{L}$,*
- *a closed subscheme $Z \subset X$ complementary to a quasi-compact open $U \subseteq X$ such that $Z$ does not contain any positive dimensional component of any fibre of $\pi$ and*
- *a finite set of points $\mathcal{S} \subset X$ disjoint from $Z$,*

*there exists an integer $N \geq 1$ such that for all integers $n \geq N$, there is a section $h \in H^0(X, \mathscr{L}^{\otimes n})$ whose vanishing locus $H$ contains $Z$ as a closed subscheme, does not contain any positive dimensional component of any fibre of $\pi$ and is disjoint from $\mathcal{S}$.*

*Proof.* In the same vein as the beginning of the proof of Lemma 7.4.2, given the defining ideal sheaf $\mathscr{I} \subset \mathcal{O}_X$ of $Z$, we can find a finitely generated sub-ideal $\mathscr{I}_0 \subset \mathcal{O}_X$ whose vanishing set coincides with $|Z|$. Letting $Z_0 := V(\mathscr{I}_0)$, the inclusion of ideals induces a bijective closed immersion $Z \hookrightarrow Z_0$. Therefore, without loss of generality, we may replace $Z$ with the finitely presented closed subscheme $Z_0$, which makes [GLL15, Theorem 5.1] applicable, and we are done. □

We are ready to perform the Beauville–Laszlo gluing of $\mathscr{G}$-torsors along the Nisnevich type square built in Proposition 7.3.3. The following proposition will let us assume that $C = \mathbb{A}^1_A$ in the statement of Proposition 7.1.1.

**Proposition 7.4.4.** *For a semilocal Prüfer domain $R$ of dimension at most 1, a ring $A$ that is obtained as the semilocalisation of a smooth $R$-domain at finitely many primes, a quasi-split, reductive $A$-group scheme $G$ and a generically trivial $G$-torsor $E$, there are*

- *an $A$-finite closed subscheme $Z \subset \mathbb{A}^1_A$, and*
- *a $G_{\mathbb{A}^1_A}$-torsor $\mathscr{E}$ which trivialises over $\mathbb{A}^1_A \setminus Z$ and whose pullback along the zero section is $E$.*

*Proof.* We argue as in the proof of [Čes22, Proposition 7.4]. Proposition 7.3.3 supplies a finitely presented, flat morphism $C \to \mathbb{A}^1_A$ and an $A$-finite closed subscheme $Z \subset \mathbb{A}^1_A$ (called $Z'$ there) that fits into a Nisnevich type gluing square (7.3.0.1), as well as an $s \in C(A)$ and a $G_C$-torsor $\mathscr{E}'$ (called $\mathscr{E}$ there) over $C$ whose $s$-pullback is $E$ such that $\mathscr{E}'$ reduces to an $\mathscr{R}_u(B)_{C \setminus Z}$-torsor over $C \setminus Z$. We note that the topological space $C$ is Noetherian (see Remark 3.9), and in particular, the open subset $U := C \setminus Z \subseteq C$ is quasi-compact. The closed subscheme $Z$ embeds into $\mathbb{P}^1_A$ as a closed subscheme by composing with $\mathbb{A}^1_A \hookrightarrow \mathbb{P}^1_A$ (the composite $Z \subset \mathbb{P}^1_A$ is a proper morphism since $Z$ is $A$-finite and hence, $A$-proper). Applying Lemma 7.4.3 to the data comprising of $X = \mathbb{P}^1_A$, $Z$, and the set of points at infinity over the maximal ideals of $A$ as $\mathcal{S}$, we obtain a hypersurface $Z \subseteq H \subset X$ which misses the infinity section $L$ of $\mathbb{P}^1_A$ (since $H \cap \mathcal{S} = \emptyset \implies H \cap L = \emptyset$). The $A$-scheme $H \subset \mathbb{A}^1_A = \mathbb{P}^1_A \setminus L$ is proper and has finite $A$-fibres (consequence of the fact that $H$ does not contain any positive dimensional component of $\mathbb{P}^1_A$), and hence, by an application of the Zariski Main Theorem [Sta22, Tag 02LS], $H$ is $A$-finite.

The torsor $\mathscr{E}'_{C \setminus Z}$ descends to an $\mathscr{R}_u(B)_{\mathbb{A}^1_A \setminus Z}$-torsor $\mathscr{E}_{\mathbb{A}^1_A \setminus Z}$ thanks to the isomorphism (7.4.2.2), whence by the patching result in Proposition 7.4.1, $\mathscr{E}'$ itself descends to a $G_{\mathbb{A}^1_A}$-torsor $\mathscr{E}$. To ensure that the pullback of $\mathscr{E}$ along the zero section is $E$, we postcompose the section obtained from $s$ with a linear automorphism of $\mathbb{A}^1_A$. Finally, we replace $Z$ by $H$ to assume, in addition, that $\mathbb{A}^1_A \setminus Z$ is affine. In particular, this means that the $\mathscr{R}_u(B)_{\mathbb{A}^1_A \setminus Z}$-torsor $\mathscr{E}_{\mathbb{A}^1_A \setminus Z}$ is trivial (using the filtration (7.4.2.1) and the Serre vanishing of coherent cohomology of quasi-coherent sheaves over affine schemes [Sta22, Tag 01XB], by an induction argument, we can show that torsors under unipotent groups over affine schemes are trivial). □

# Chapter 8

# Quasi-split case of the GS conjecture for Smooth Algebras over Valuation Rings

In this section we prove Theorem A.

**Theorem 8.1.** *Let $R$ be a semilocal Prüfer domain of Krull dimension at most 1 and let $A$ be the semilocalisation at finitely many primes of an $R$-smooth, integral domain. For any quasi-split, reductive $A$-group scheme $G$, the restriction morphism*

$$\theta^1_G \colon H^1(A, G) \to H^1(\mathrm{Frac}(A), G) \text{ has trivial kernel,}$$

*i.e., a $G$-torsor over $A$ has a section (and hence, is trivial) if it has a generic section.*

Starting with a generically trivial torsor $E$ on $\operatorname{Spec} A$, Proposition 7.4.4 produces a torsor $\mathscr{E}$ on $\mathbb{A}^1_A$ that pulls-back to $E$ under the zero section and that trivialises away from an $A$-finite subset $Z \subset \mathbb{A}^1_A$. To conclude the proof of the main theorem, it remains to use the following result.

**Proposition 8.2** ([Čes$_{\text{Preprint}}$, Proposition 5.4], cf. [Čes22, Proposition 8.4] and [Čes22$_{\text{Surv}}$, §3.5] for the 'totally isotropic' case)**.** *For a semilocal, normal ring $A$, a quasi-split, reductive $A$-group $G$, and a $G_{\mathbb{A}^1_A}$-torsor $\mathscr{E}$ on $\mathbb{A}^1_A$ that trivialises away from an $A$-finite subset $Z \subset \mathbb{A}^1_A$, the pulled-back $G$-torsor $s^*\mathscr{E}$ along any section $s \in \mathbb{A}^1_A(A)$ is trivial.*

We formulated only a special case of the results that are proved in the cited literature because this version suffices for our purposes. We present the fundamental steps in proving Proposition 8.2 from [Čes22$_{\text{Surv}}$, §3.5]. Firstly, when $A$ is a field, the discrete valuation ring case of the Grothendieck–Serre conjecture proved in [Nis82] (cf. [Guo20]) shows that $\mathscr{E}$ is a Zariski locally trivial $G_{\mathbb{A}^1_A}$-torsor. Consequently, with the help of [Gil02, Corollaire 3.10(a)], we infer that $\mathscr{E}$ reduces to a torsor under a maximal split $A$-subtorus of $G$. However, since $\mathbb{A}^1_A$ has no nontrivial line bundles, we deduce that $\mathscr{E}$ is already trivial. Therefore, the same applies for its pullback along any section $s \in \mathbb{A}^1_A(A)$. Thus, the proof is complete in the case when $A$ is a field. Secondly, for a local ring $A$ with a maximal ideal $\mathfrak{m}$, the strategy consists of the following two steps:

(i) to glue $\mathscr{E}$ with the trivial $G$-torsor that is defined in a neighbourhood of $\infty \in \mathbb{P}^1_A$ producing a $G_{\mathbb{P}^1_A}$-torsor $\tilde{\mathscr{E}}$ in such a way that $\tilde{\mathscr{E}}_{\kappa(\mathfrak{m})}$ is trivial (see [Čes22$_{\text{Surv}}$, Lemma 3.5.5]), and

(ii) to show that a $G_{\mathbb{P}^1_A}$-torsor on $\mathbb{P}^1_A$ whose reduction to the special fibre is trivial is a pullback of a $G$-torsor on $A$ (see [Čes22, Lemma 8.3]).

This completes the proof in the local case because, by (i), $\tilde{\mathscr{E}}\mid_{\kappa(\mathfrak{m})}$ is trivial and the same is true for $\tilde{\mathscr{E}}$ in a neighbourhood of $\infty$. Indeed, in this case, (ii) demonstrates that $\tilde{\mathscr{E}}$ is a pullback of the trivial $G$-torsor $\infty^*\tilde{\mathscr{E}}$ that is defined on $A$. Lastly, for a general semilocal ring $A$, one can reduce to the local case using Quillen patching [Čes22$_{\text{Surv}}$, Corollary 5.1.9].

**Corollary 8.3.** *For a semilocal Prüfer domain $R$ of Krull dimension at most 1, a ring $A$ that is a semilocalisation at finitely many primes of an $R$-smooth, integral domain and a quasi-split, reductive $A$-group scheme $G$, a Nisnevich locally trivial $G$-torsor over $A$ trivialises Zariski locally, i.e.,*

$$H^1_{\text{Nis}}(A, G) = H^1_{\text{Zar}}(A, G).$$

*Proof.* Define $H^1_{\text{ét,gen}}(A, G)$ to be the set of classes of generically trivial torsors over $A$ that trivialise étale locally on $A$. There are canonical inclusions

$$H^1_{\text{ét,gen}}(A, G) \overset{\alpha}{\hookleftarrow} H^1_{\text{zar}}(A, G) \overset{\beta}{\hookrightarrow} H^1_{\text{Nis}}(A, G).$$

Thanks to Theorem 8.1 and spreading out, the morphism $\alpha$ is an equality. Consequently, it suffices to construct an inverse of $\beta \circ \alpha^{-1}$. Let $E \in H^1_{\text{Nis}}(A, G)$ and let $U_i \to \operatorname{Spec} A$ be étale morphisms such that $\{U_i\}$ forms an affine Nisnevich cover of $\operatorname{Spec} A$ that trivialises $E$. By definition [Nis89, §1.1.1], there exists a $U_i$ which is generically isomorphic to $\operatorname{Spec} A$. This generic isomorphism produces a generic section of the torsor $E$. Therefore, every Nisnevich locally trivial $G$-torsor $E$ has a generic section. Hence, this produces an inverse of $\beta \circ \alpha^{-1}$ and we are done. $\square$

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**ÉCOLE DOCTORALE**
de mathématiques
Hadamard (EDMH)

**Titre:** Torseurs sur les algèbres lisses sur les anneaux de valuation

**Mots clés:** Grothendieck–Serre, torseur, anneau de valuation, domain de Prufer, groupe réductif, lemme de présentation, quasi-déployé

**Résumé:** Nous démontrons une version de la conjecture de Grothendieck–Serre qui affirme que les torseurs génériquement triviaux sous les groupes réductifs quasi-déployés sur les algèbres lisses sur les anneaux de valuation de rang 1 sont triviaux. Ce travail étend celui de Guo et celui de Česnavičius. Tous les résultats sont établis dans leur généralité naturelle sur les domaines de Prüfer semi-locaux. Comme étape intermédiaire, nous établissons pureté pour des torseurs sous tores généralisant le travail de Colliot-Thélène et Sansuc. En particulier, nous déduisons que le groupe cohomologique de Brauer d'une algèbre intègre lisse sur un anneau de valuation s'injecte dans le groupe cohomologique de Brauer de son corps des fractions. L'ingrédient clé est une nouvelle version du "Lemme de Présentation" géométrique sur les anneaux de valuation de rang 1.

**Title:** Torsors on Smooth Algebras over Valuation Rings

**Keywords:** Grothendieck–Serre, torsor, valuation ring, Prufer domain, reductive group, presentation lemma, quasi-split

**Abstract:** We prove a version of the Grothendieck–Serre conjecture to confirm that generically trivial torsors under quasi-split reductive groups over smooth algebras over valuation rings of rank $1$ are trivial, extending the work of Guo and the work of Česnavičius. All the results are stated in their natural generality over semilocal Prüfer domains. As an intermediary step, we establish purity for torsors under tori generalising the work of Colliot-Thélène and Sansuc, in particular, we deduce that the cohomological Brauer group of a smooth domain over a valuation ring injects into the cohomological Brauer group of its fraction field. The key ingredient is a new version of geometric "Presentation Lemma" over valuation rings of rank $1$.

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