Geometric Inequalities and Bounded Mean Oscillation

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ABSTRACT

Geometric Inequalities and Bounded Mean Oscillation

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In this thesis, we study the space of functions of bounded mean oscillation (BMO) on shapes. We prove the boundedness of important nonlinear operators, such as maximal functions and rearrangements, on this space and analyse how the bounds are affected by the underlying geometry of the shapes.

We provide a general definition of BMO on a domain in $\mathbb{R}^n$, where mean oscillation is taken with respect to a basis of shapes, i.e. a collection of open sets covering the domain. We prove many properties inherent to BMO that are valid for any choice of basis; in particular, BMO is shown to be complete. Many shapewise inequalities, which hold for every shape in a given basis, are proven with sharp constants. Moreover, a sharp norm inequality, which holds for the BMO norm that involves taking a supremum over all shapes in a given basis, is obtained for the truncation of a BMO function. When the shapes exhibit some product structure, a product decomposition is obtained for BMO.

We consider the boundedness of maximal functions on BMO on shapes in $\mathbb{R}^n$. We prove that for bases of shapes with an engulfing property, the corresponding maximal function is bounded from BMO to BLO, the collection of functions of bounded lower oscillation. When the basis of shapes does not possess an engulfing property but exhibits a product structure with respect to lower-dimensional shapes coming from bases that do possess an engulfing property, we show that the corresponding maximal function is bounded from BMO to a space we define and call rectangular BLO.

We obtain boundedness and continuity results for rearrangements on BMO. This allows for an improvement of the known bound for the basis of cubes. We show, by example, that the decreasing rearrangement is not continuous on BMO, but that it is both bounded and continuous on VMO, the subspace of functions of vanishing mean oscillation. Boundedness for the symmetric decreasing rearrangement is then established for the basis of balls in $\mathbb{R}^n$.

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Contribution of Authors

This thesis is based on three research articles:

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This joint work was previously published in a Contemporary Mathematics volume. R.G. is responsible for a substantial portion of the analysis, as well as the primary portion of the writing with editing by G. D.

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III : A. Burchard, G. Dafni, R. Gibara. Rearrangement inequalities on spaces defined by mean oscillation.

This joint work is still in preparation and represents a draft. R. G. is responsible for a substantial portion of the analysis, as well as the primary portion of the writing with editing by A. B. and G. D.

Contents

Chapter 1

Introduction

Introduced in [54] by John and Nirenberg, the space BMO of functions of bounded mean oscillation is an important function space in harmonic analysis and PDEs. It has connections to the theory of quasiconformal mappings, Muckenhoupt weights, and Hardy spaces. Most importantly, it has found a role as a remedy for the failure of L^∞ in many situations.

One such situation is the following. A well-known inequality in PDEs, the Gagliardo-Nirenberg-Sobolev inequality implies the Sobolev embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ for $1 \leq p < n$. Here, $p^* = \frac{np}{n-p}$ and $W^{1,p}(\mathbb{R}^n)$ is the Sobolev space of functions in $L^p(\mathbb{R}^n)$ having weak first-order partial derivatives in $L^p(\mathbb{R}^n)$. From this embedding, one might expect that $W^{1,n}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, letting $p \to n^-$. In dimension $n > 1$, however, this is false: there exist unbounded functions in $W^{1,n}(\mathbb{R}^n)$. A correct statement is obtained, thanks to the Poincaré inequality, by enlargening $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$. This is but one of many important examples where replacing $L^\infty(\mathbb{R}^n)$ by $\mathrm{BMO}(\mathbb{R}^n)$ produces a correct statement.

With this said, we come to the definition of $\mathrm{BMO}(\mathbb{R}^n)$. Consider a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For such functions, it makes sense to define its mean oscillation on a cube $Q \subset \mathbb{R}^n$ as

$$\frac{1}{|Q|}\int_Q |f - f_Q|,$$

where $|Q|$ denotes the Lebesgue measure of Q and f_Q denotes the mean of f on Q. We say that f is in $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}} := \sup_Q \fint_Q |f - f_Q| < \infty, \tag{1.1}$$

where the supremum is taken over all cubes Q. Here, a cube is understood to mean having sides parallel to the axes.

As mentioned earlier, $\mathrm{BMO}(\mathbb{R}^n)$ contains $L^\infty(\mathbb{R}^n)$ and, in fact, this inclusion is strict. The quintessential example is given by $f(x) = -\log|x| \in \mathrm{BMO}(\mathbb{R}^n) \setminus L^\infty(\mathbb{R}^n)$. There is, however, a fundamental difference between $\mathrm{BMO}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ that is immediate from the definition. This difference is that $\mathrm{BMO}(\mathbb{R}^n)$ is defined with respect to a fixed, and special, geometry on $\mathbb{R}^n$, namely cubes.

An equivalent characterisation of $\mathrm{BMO}(\mathbb{R}^n)$ is obtained by replacing the cubes in (1.1) by Euclidean balls, an observation that has allowed the study of BMO to transcend Euclidean space into metric measure spaces. The reason for this is that cubes and balls have a similar geometry from the viewpoint of measure. After all, cubes can always be fit inside balls that are not too much bigger and vice versa.

This is not the case, however, for strong $\mathrm{BMO}(\mathbb{R}^n)$, which is defined by replacing the cubes in (1.1) by rectangles (again, with sides parallel to the axes). As cubes are rectangles, every cube can trivially be placed inside a rectangle of the same measure, namely itself. An arbitrary rectangle, however, can be very thin in one direction, and so fitting it inside a cube would necessitate one of much larger measure. From this perspective, cubes and rectangles are somehow incompatible. A consequence of this is that strong $\mathrm{BMO}(\mathbb{R}^n)$ is a strict subset of $\mathrm{BMO}(\mathbb{R}^n)$.

From here, many questions arise. How does the choice of geometry, in this case reflected by the choice of the sets on which to measure mean oscillation, affect BMO? What properties are intrinsic to BMO and which are consequences of our choice of geometry? How does this choice affect the boundedness of various operators defined on BMO, especially if those operators are intimately tied to geometry themselves? These questions are the topic of this thesis.

In Chapter 2, these ideas are made more concrete. The space $\mathrm{BMO}_{\mathscr{S}}(\Omega)$ is defined, where Ω is a domain in $\mathbb{R}^n$, as the space of functions with bounded mean oscillation on sets coming from a fixed collection $\mathscr{S}$, known as a basis of shapes. These shapes are taken to be open sets of positive and finite measure, and are assumed to cover Ω. This point of view of studying

BMO with respect to a basis of shapes is new, and it provides a framework for examining the strongest results that can be obtained about functions in BMO while assuming only the weakest geometric assumptions: if f has bounded mean oscillation with respect to *some* basis, what can be said about f?

A first interesting phenomenon is that a large number of properties with which one classically associates BMO hold for any basis, showing a "geometry-free" side to BMO. These properties are, then, somehow inherent to BMO itself and, in the classical case of cubes, never had anything to do with the cubes, in the first place. An example is the completeness of $\mathrm{BMO}_{\mathscr{S}}(\Omega)$ in the sense of metric spaces (see Theorem 2.3.9).

Even if a certain inequality holds for all $f \in \mathrm{BMO}_{\mathscr{S}}(\Omega)$ for any basis, it is possible that this inequality involves a constant that may depend on the choice of $\mathscr{S}$. Of interest is determining the sharpest (that is, best) constant and to determine its dependence on $\mathscr{S}$.

A distinction must be made between shapewise inequalities and norm inequalities. A shapewise inequality is one that holds for each shape in the basis $\mathscr{S}$, while norm inequalities involve the BMO norm defined in (1.1). An example of the importance of this distinction is the following simple open problem in the area of sharp constants in BMO.

Given a function f that is in $L^1(S)$ for every $S \in \mathscr{S}$, it can easily be shown that $|f|$ satisfies

$$\frac{1}{|S|}\int_S \big||f| - |f|_S\big| \leq \frac{2}{|S|}\int_S |f - f_S|$$

for every $S \in \mathscr{S}$. This is true for any basis $\mathscr{S}$ and an example can show that the 2 is sharp in the sense that for every $S \in \mathscr{S}$ there is a function f for which the constant 2 cannot be decreased (see Example 2.4.11). This shapewise inequality implies the following norm inequality: $\||f|\|_{\mathrm{BMO}_{\mathscr{S}}} \leq 2\|f\|_{\mathrm{BMO}_{\mathscr{S}}}$ for all $f \in \mathrm{BMO}_{\mathscr{S}}(\Omega)$. This constant 2, however, is *not necessarily* sharp! Known results due to Korenovskii are that the sharp constant is, in fact, 1 in one dimension when $\mathscr{S}$ is the basis of open intervals and in n dimensions when $\mathscr{S}$ is the basis of rectangles (see Section 2.5). The problem is open, to my knowledge, for other bases.

A sharp norm inequality that is proven is the following. For $f \in \mathrm{BMO}_{\mathscr{S}}(\Omega)$, define its truncation at height k to be $\mathrm{Tr}(f,k) = \min(\max(f,-k),k)$. In Corollary 2.6.5, we prove that $\|\mathrm{Tr}(f,k)\|_{\mathrm{BMO}_{\mathscr{S}}} \leq \|f\|_{\mathrm{BMO}_{\mathscr{S}}}$ for any basis $\mathscr{S}$. Truncations are a useful tool, as they allow one to approximate the BMO norm of a function by the BMO norm of L^∞ functions.

A situation where the geometry of $\mathscr{S}$ plays a major role is in the product setting. When the shapes exhibit some product structure, namely that the collection of shapes coincides with the collection of Cartesian products of lower-dimensional shapes, we prove that this product nature is passed along to $\mathrm{BMO}_{\mathscr{S}}(\Omega)$ in the sense that $\mathrm{BMO}_{\mathscr{S}}(\Omega)$ has a relationship with spaces defined by uniform lower-dimensional mean oscillations (see Theorem 2.8.3).

In Chapter 3, geometric maximal operators are studied. These are important operators in analysis, as their boundedness very often implies other results. A well-known example of this phenomenon comes from the uncentred Hardy-Littlewood maximal function, Mf. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f|, \tag{1.2}$$

where the supremum is taken over all cubes Q containing the point x. The weak-type $(1,1)$ boundedness of this operator (see Appendix II for more details) implies the Lebesgue differentiation theorem, a fundamental result that will be used throughout this thesis.

The boundedness of M on $\mathrm{BMO}(\mathbb{R}^n)$ was first considered by Bennett-DeVore-Sharpley in [3]. They showed that if $Mf \not\equiv \infty$, then $Mf \in \mathrm{BMO}(\mathbb{R}^n)$ when $f \in \mathrm{BMO}(\mathbb{R}^n)$. This result was improved by Bennett in [2] by showing that if $Mf \not\equiv \infty$, then $Mf \in \mathrm{BLO}(\mathbb{R}^n)$ when $f \in \mathrm{BMO}(\mathbb{R}^n)$. The space $\mathrm{BLO}(\mathbb{R}^n)$ of functions of bounded lower oscillation, introduced by Coifman-Rochberg in [18], is defined by replacing the mean f_Q in (1.1) by the essential infimum of f on Q.

The generalisation to $M_{\mathscr{S}}$, the maximal function with respect to the basis $\mathscr{S}$, is done by replacing the cubes in (1.2) by shapes from $\mathscr{S}$. This definition is not new; there is an entire area of analysis devoted to the study of such maximal functions (see the monograph [44]). Having developed a theory of BMO with respect to a basis $\mathscr{S}$, a natural question is the following: for what bases $\mathscr{S}$ is $M_{\mathscr{S}}$ bounded from $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ to $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$, whenever $M_{\mathscr{S}}f$ is finite almost everywhere?

A partial answer is provided by considering a class of bases. We define what it means for a basis to be engulfing. This is most easily described for balls: given two intersecting balls with one much larger than the other, the larger of the two balls can be dilated in such a way as to engulf both balls without having to grow too much. This is in sharp contrast to the case of rectangles, where either of two intersecting rectangles that are long and narrow in orthogonal directions would need to grow a lot, in terms of measure, to engulf the other. In Theorem 3.3.2, the geometric property of being engulfing is exploited to show that for such bases, $M_{\mathscr{S}}$ is bounded from $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ to $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$, where $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ is defined analogously to $\mathrm{BLO}(\mathbb{R}^n)$ by replacing cubes with shapes in $\mathscr{S}$.

A product decomposition is shown for the class $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ when the shapes exhibit some product structure (see Theorem 3.4.6). This is done much in the same way as the corresponding result for $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$. In this case, however, a sharp constant is obtained.

As both $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ and $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ inherit some product structure from the shapes in $\mathscr{S}$, perhaps $M_{\mathscr{S}}$ inherits some boundedness properties when the shapes $\mathscr{S}$ are products of lower-dimensional shapes that are engulfing. Only relying on this product structure and the engulfing property of lower-dimensional shapes, we prove that $M_{\mathscr{S}}$ is bounded from $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ to a space we define, rectangular $\mathrm{BLO}_{\mathscr{S}}$ (see Theorem 3.6.1).

The model case of a basis $\mathscr{S}$ for which the boundedness on $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ is unknown is the basis of rectangles. In this case, the maximal function $M_{\mathscr{S}}$ is known as the strong maximal function, and has been studied going back to the work of Jessen-Marcinkiewicz-Zygmund ([52]). This basis satisfies the hypotheses of Theorem 3.6.1, and so we have shown, in particular, that the strong maximal function is bounded from strong $\mathrm{BMO}(\mathbb{R}^n)$ to rectangular BLO.

In Chapter 4, we consider equimeasurable rearrangement operators. Given a measurable function f on $\mathbb{R}^n$, one such rearrangement is its decreasing rearrangement, the unique decreasing function f^* on $\mathbb{R}_+ = (0, \infty)$ that is right-continuous and equimeasurable with $|f|$. This rearrangement is important in areas such as interpolation theory, and there is interest in studying function spaces that are invariant under equimeasurable rearrangements (see [4]).

The work of Bennett-DeVore-Sharpley in [3] implies that the decreasing rearrangement $f \mapsto f^*$ is bounded from $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}_+)$ with $\|f^*\|_{\mathrm{BMO}} \leq 2^{n+5}\|f\|_{\mathrm{BMO}}$. We ask: for what other bases $\mathscr{S}$ is the decreasing rearrangement bounded on BMO?

Our main result in this direction is Lemma 4.3.2, which determines a class of bases for which the decreasing rearrangement is bounded from $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ to $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}_+)$, even in the generality of a metric measure space. In particular, this is used to improve the Bennett-DeVore-Sharpley bound from 2^{n+5} to $2^{\frac{n+3}{2}}$ (see Theorem 4.3.10).

The decreasing rearrangement is a nonlinear operator. As such, we no longer have access to the equivalence of boundedness and continuity that is familiar from working with linear operators. It turns out that there is a simple example showing that the decreasing rearrangement can fail to be continuous on BMO (see Theorem 4.4.1).

When looking for a subspace of BMO on which the decreasing rearrangement might be continuous, a natural candidate is VMO, the space of functions of vanishing mean oscillation. If Q_0 is a cube in $\mathbb{R}^n$, we say that a function $f \in \mathrm{BMO}(Q_0)$ is in $\mathrm{VMO}(Q_0)$ if

$$\lim_{t \to 0^+} \sup_{\delta(Q) \leq t} \Omega(f, Q) = 0,$$

where the supremum is taken over all cubes Q with diameter at most t. This function space, introduced by Sarason ([77]), often plays the role of the continuous functions in $\mathrm{BMO}(Q_0)$.

This candidate turns out to be a good choice: it is proven that the decreasing rearrangement is bounded and continuous from $\mathrm{VMO}(Q_0)$ to $\mathrm{VMO}(0, |Q_0|)$ (see Theorems 4.4.6 and 4.4.9), assuming that we normalize functions to have mean zero.

Another important rearrangement is the symmetric decreasing rearrangement, which associates a measurable function f on $\mathbb{R}^n$ to a measurable function Sf on $\mathbb{R}^n$ that is radially decreasing and symmetric in such a way that $|f|$ and Sf are equimeasurable. This rearrangement is important in the study of geometric functional analysis and PDEs. The symmetric decreasing rearrangement may be defined by means of the formula $Sf(x) = f^*(\omega_n|x|^n)$ for $x \in \mathbb{R}^n$, where ω_n denotes the measure of the unit ball in $\mathbb{R}^n$. As such, the symmetric decreasing rearrangement is intimately connected to the decreasing rearrangement and one may ask whether BMO-boundedness results for f^* can be transferred to Sf.

The theory of BMO on shapes provides the proper point of view to achieve this result. By the definition of the symmetric decreasing rearrangement, mean oscillations of f^* on intervals can be compared with mean oscillations of Sf on balls and annuli centred at the origin. Such shapes, along with sectors thereof, are then comparable with general balls (in the sense of Definition 2.2.2). As such, we obtain that the symmetric decreasing rearrangement is bounded from $\mathrm{BMO}_{\mathcal{B}}(\mathbb{R}^n)$ to $\mathrm{BMO}_{\mathcal{B}}(\mathbb{R}^n)$ (see Theorem 4.5.1).

As this thesis is a compilation of three separate works, there is bound to be quite a bit of repetition. I am sure that BMO will be defined at least three times more throughout this text. Hopefully, all occurrences of a given definition will be the same in each instance. Even worse, notation may vary from chapter to chapter, sometimes in a significant way. Hopefully, everything is written in such a way that meanings remain clear. In general, the following is maintained: $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$, possibly with $\mathbb{R}^n$ replaced by some other domain of definition, denotes the space of locally integrable functions on $\mathbb{R}^n$ of bounded mean oscillation with respect to some fixed basis $\mathscr{S}$; and, $\mathrm{BMO}(\mathbb{R}^n)$ is reserved for the classical case $\mathscr{S} = \mathcal{Q}$ or $\mathscr{S} = \mathcal{B}$.

At the end of the thesis, there is an appendix containing auxiliary results and some additional details. Throughout the text, footnotes are used to signify a moment when the reader may wish to visit the appendix, along with the specific location of the relevant material therein.

Chapter 2

BMO on shapes and sharp constants

2.1 Introduction

First defined by John and Nirenberg in [54], the space BMO of functions of bounded mean oscillation has served as the replacement for L^∞ in situations where considering bounded functions is too restrictive. BMO has proven to be important in areas such as harmonic analysis, partly due to the duality with the Hardy space established by Fefferman in [32], and partial differential equations, where its connection to elasticity motivated John to first consider the mean oscillation of functions in [53]. Additionally, one may regard BMO as a function space that is interesting to study in its own right. As such, there exist many complete references to the classical theory and its connection to various areas; for instance, see [37, 42, 56, 79].

The mean oscillation of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ was initially defined over a cube Q with sides parallel to the axes as

$$\fint_Q |f - f_Q|, \tag{2.1}$$

where $f_Q = \fint_Q f$ and $\fint_Q = \frac{1}{|Q|}\int_Q$. A function f was then said to be in BMO if the quantity (2.1) is bounded independently of Q. Equivalently, as will be shown, the same space can be obtained by considering the mean oscillation with respect to balls; that is, replacing the cube

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Q by a ball B in (2.1). Using either characterization, BMO has since been defined in more general settings such as on domains, manifolds, and metric measure spaces ([8, 19, 55]).

There has also been some attention given to the space defined by a mean oscillation condition over rectangles with sides parallel to the axes, either in $\mathbb{R}^n$ or on a domain in $\mathbb{R}^n$, appearing in the literature under various names. For instance, in [60], the space is called "anisotropic BMO" to highlight the contrast with cubes, while in papers such as [22, 30, 31, 35], it goes by "little BMO" and is denoted by bmo. The notation bmo, however, had already been used for the "local BMO" space of Goldberg ([40]), a space that has been established as an independent topic of study (see, for instance, [11, 28, 87]). Yet another name for the space defined by mean oscillations on rectangles - the one we prefer - is the name "strong BMO". This name has been used in at least one paper ([64]), and it is analogous to the terminology of strong differentiation of the integral and the strong maximal function ([20, 44, 51, 52, 76]), as well as strong Muckenhoupt weights ([5, 68]).

In this paper we consider BMO on domains of $\mathbb{R}^n$ with respect to a geometry (what will be called a basis of shapes) more general than cubes, balls, or rectangles. The purpose of this is to provide a framework for examining the strongest results that can be obtained about functions in BMO by assuming only the weakest assumptions. To illustrate this, we provide the proofs of many basic properties of BMO functions that are known in the literature for the specialised bases of cubes, balls, or rectangles but that hold with more general bases of shapes. In some cases, the known proofs are elementary themselves and so our generalisation serves to emphasize the extent to which they are elementary and to which these properties are intrinsic to the definition of BMO. In other cases, the known results follow from deeper theory and we are able to provide elementary proofs. We also prove many properties of BMO functions that may be well known, and may even be referred to in the literature, but for which we could not find a proof written down. An example of such a result is the completeness of BMO, which is often deduced as a consequence of duality, or proven only for cubes in $\mathbb{R}^n$. We prove this result (Theorem 2.3.9) for a general basis of shapes on a domain.

The paper has two primary focuses, the first being constants in inequalities related to BMO. Considerable attention will be given to their dependence on an integrability parameter p, the basis of shapes used to define BMO, and the dimension of the ambient Euclidean space.

References to known results concerning sharp constants are given and connections between the sharp constants of various inequalities are established. We distinguish between shapewise inequalities, that is, inequalities that hold on any given shape, and norm inequalities. We provide some elementary proofs of several shapewise inequalities and obtain sharp constants in the distinguished cases $p = 1$ and $p = 2$. An example of such a result is the bound on truncations of a BMO function (Proposition 2.6.3). Although sharp shapewise inequalities are available for estimating the mean oscillation of the absolute value of a function in BMO, the constant 2 in the implied norm inequality - a statement of the boundedness of the map $f \mapsto |f|$ - is not sharp. Rearrangements are a valuable tool that compensate for this, and we survey some known deep results giving norm bounds for decreasing rearrangements.

A second focus of this paper is on the product nature that BMO spaces may inherit from the shapes that define them. In the case where the shapes defining BMO have a certain product structure, namely that the collection of shapes coincides with the collection of Cartesian products of lower-dimensional shapes, a product structure is shown to be inherited by BMO under a mild hypothesis related to the theory of differentiation (Theorem 2.8.3). This is particularly applicable to the case of strong BMO. It is important to note that the product nature studied here is different from that considered in the study of the space known as product BMO (see [13, 14]).

Following the preliminaries, Section 2.3 presents the basic theory of BMO on shapes. Section 2.4 concerns shapewise inequalities and the corresponding sharp constants. In Section 2.5, two rearrangement operators are defined and their boundedness on various function spaces is examined, with emphasis on BMO. Section 2.6 discusses truncations of BMO functions and the cases where sharp inequalities can be obtained without the need to appeal to rearrangements. Section 2.7 gives a short survey of the John-Nirenberg inequality. Finally, in Section 2.8 we state and prove the product decomposition of certain BMO spaces.

This introduction is not meant as a review of the literature since that is part of the content of the paper, and references are given throughout the different sections. The bibliography is by no means exhaustive, containing only a selection of the available literature, but it is collected with the hope of providing the reader with some standard or important references to the different topics touched upon here.

2.2 Preliminaries

Consider $\mathbb{R}^n$ with the Euclidean topology and Lebesgue measure, denoted by $|\cdot|$. By a domain we mean an open and connected set.

Definition 2.2.1. We call a shape in $\mathbb{R}^n$ any open set S such that $0 < |S| < \infty$. For a given domain $\Omega \subset \mathbb{R}^n$, we call a basis of shapes in Ω a collection $\mathscr{S}$ of shapes S such that $S \subset \Omega$ for all $S \in \mathscr{S}$ and $\mathscr{S}$ forms a cover of Ω.

Common examples of bases are the collections of all Euclidean balls, $\mathcal{B}$, all cubes with sides parallel to the axes, $\mathcal{Q}$, and all rectangles with sides parallel to the axes, $\mathcal{R}$. In one dimension, these three choices degenerate to the collection of all (finite) open intervals, $\mathcal{I}$. A variant of $\mathcal{B}$ is $\mathcal{C}$, the basis of all balls centred around some central point (usually the origin). Another commonly used collection is $\mathcal{Q}_d$, the collection of all dyadic cubes, but the open dyadic cubes cannot cover Ω unless Ω itself is a dyadic cube, so the proofs of some of the results below which rely on $\mathscr{S}$ being an open cover (e.g. Proposition 2.3.8 and Theorem 2.3.9) may not apply.

One may speak about shapes that are balls with respect to a (quasi-)norm on $\mathbb{R}^n$, such as the p-"norms" $\|\cdot\|_p$ for $0 < p \leq \infty$ when $n \geq 2$. The case $p = 2$ coincides with the basis $\mathcal{B}$ and the case $p = \infty$ coincides with the basis $\mathcal{Q}$, but other values of p yield other interesting shapes. On the other hand, $\mathcal{R}$ is not generated from a p-norm.

Further examples of interesting bases have been studied in relation to the theory of differentiation of the integral, such as the collection of all rectangles with j of the sidelengths being equal and the other $n-j$ being arbitrary ([89]), as well as the basis of all rectangles with sides parallel to the axes and sidelengths of the form $\big(\ell_1, \ell_2, \ldots, \phi(\ell_1, \ell_2, \ldots, \ell_{n-1})\big)$, where ϕ is a positive function that is monotone increasing in each variable separately ([21]).

Definition 2.2.2. Given two bases of shapes, $\mathscr{S}$ and $\widetilde{\mathscr{S}}$, we say that $\mathscr{S}$ is comparable to $\widetilde{\mathscr{S}}$, written $\mathscr{S} \trianglelefteq \widetilde{\mathscr{S}}$, if there exist lower and upper comparability constants $c > 0$ and $C > 0$, depending only on n, such that for all $S \in \mathscr{S}$ there exist $S_1, S_2 \in \widetilde{\mathscr{S}}$ for which $S_1 \subset S \subset S_2$ and $c|S_2| \leq |S| \leq C|S_1|$. If $\mathscr{S} \trianglelefteq \widetilde{\mathscr{S}}$ and $\widetilde{\mathscr{S}} \trianglelefteq \mathscr{S}$, then we say that $\mathscr{S}$ and $\widetilde{\mathscr{S}}$ are equivalent, and write $\mathscr{S} \approx \widetilde{\mathscr{S}}$.

An example of equivalent bases are $\mathcal{B}$ and $\mathcal{Q}$: one finds that $\mathcal{B} \trianglelefteq \mathcal{Q}$ with $c = \frac{\omega_n}{2^n}$ and $C = \omega_n \left(\frac{\sqrt{n}}{2}\right)^n$, and $\mathcal{Q} \trianglelefteq \mathcal{B}$ with $c = \frac{1}{\omega_n}\left(\frac{2}{\sqrt{n}}\right)^n$ and $C = \frac{2^n}{\omega_n}$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$, and so $\mathcal{B} \approx \mathcal{Q}$. The bases of shapes given by the balls in the other p-norms $\|\cdot\|_p$ for $1 \leq p \leq \infty$ are also equivalent to these.

If $\mathscr{S} \subset \widetilde{\mathscr{S}}$ then $\mathscr{S} \trianglelefteq \widetilde{\mathscr{S}}$ with $c = C = 1$. In particular, $\mathcal{Q} \subset \mathcal{R}$ and so $\mathcal{Q} \trianglelefteq \mathcal{R}$, but $\mathcal{R} \ntrianglelefteq \mathcal{Q}$ and so $\mathcal{Q} \not\approx \mathcal{R}$.

Unless otherwise specified, we maintain the convention that $1 \leq p < \infty$. Moreover, many of the results implicitly assume that the functions are real-valued, but others may hold also for complex-valued functions. This should be understood from the context.

2.3 BMO spaces with respect to shapes

Consider a basis of shapes $\mathscr{S}$. Given a shape $S \in \mathscr{S}$, for a function $f \in L^1(S)$, denote by f_S its mean over S.

Definition 2.3.1. We say that a function satisfying $f \in L^1(S)$ for all shapes $S \in \mathscr{S}$ is in the space $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ if there exists a constant $K \geq 0$ such that

$$\left(\fint_S |f - f_S|^p\right)^{1/p} \leq K, \tag{2.2}$$

holds for all $S \in \mathscr{S}$.

The quantity on the left-hand side of (2.2) is called the p-mean oscillation of f on S. For $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$, we define $\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}$ as the infimum of all K for which (2.2) holds for all $S \in \mathscr{S}$. Note that the p-mean oscillation does not change if a constant is added to f; as such, it is sometimes useful to assume that a function has mean zero on a given shape.

In the case where $p = 1$, we will write $\mathrm{BMO}^1_{\mathscr{S}}(\Omega) = \mathrm{BMO}_{\mathscr{S}}(\Omega)$. For the classical BMO spaces we reserve the notation $\mathrm{BMO}^p(\Omega)$ without explicit reference to the underlying basis of shapes ($\mathcal{Q}$ or $\mathcal{B}$).

We mention a partial answer to how $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ relate for different values of p. This question will be taken up again in a later section when some more machinery has been developed.

Proposition 2.3.2. *For any basis of shapes,* $\mathrm{BMO}^{p_2}_{\mathscr{S}}(\Omega) \subset \mathrm{BMO}^{p_1}_{\mathscr{S}}(\Omega)$ *with* $\|f\|_{\mathrm{BMO}^{p_1}_{\mathscr{S}}} \leq \|f\|_{\mathrm{BMO}^{p_2}_{\mathscr{S}}}$ *for* $1 \leq p_1 \leq p_2 < \infty$*. In particular, this implies that* $\mathrm{BMO}^{p}_{\mathscr{S}}(\Omega) \subset \mathrm{BMO}_{\mathscr{S}}(\Omega)$ *for all* $1 \leq p < \infty$*.*

Proof. This follows from Jensen's inequality with $p = \frac{p_2}{p_1} \geq 1$. □

Next we show a lemma that implies, in particular, the local integrability of functions in $\mathrm{BMO}^{p}_{\mathscr{S}}(\Omega)$.

Lemma 2.3.3. *For any basis of shapes,* $\mathrm{BMO}^{p}_{\mathscr{S}}(\Omega) \subset L^p_{loc}(\Omega)$*.*

Proof. Fix a shape $S \in \mathscr{S}$ and a function $f \in \mathrm{BMO}^{p}_{\mathscr{S}}(\Omega)$. By Minkowski's inequality on $L^p(S, \frac{dx}{|S|})$,

$$\left(\fint_S |f|^p \right)^{1/p} \leq \left(\fint_S |f - f_S|^p \right)^{1/p} + |f_S|$$

and so

$$\left(\int_S |f|^p \right)^{1/p} \leq |S|^{1/p} \left(\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + |f_S| \right).$$

As $\mathscr{S}$ covers all of Ω, for any compact set $K \subset \Omega$ there exists a collection $\{S_i\}_{i=1}^N \subset \mathscr{S}$ for some finite N such that $K \subset \bigcup_{i=1}^N S_i$. Hence, using the previous calculation,

$$\left(\int_K |f|^p \right)^{1/p} \leq \sum_{i=1}^N \left(\int_{S_i} |f|^p \right)^{1/p} \leq \sum_{i=1}^N |S_i|^{1/p} \left(\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + |f_{S_i}| \right) < \infty.$$

□

In spite of this, a function in $\mathrm{BMO}^{p}_{\mathscr{S}}(\Omega)$ need not be locally bounded. If Ω contains the origin or is unbounded, $f(x) = \log |x|$ is the standard example of a function in $\mathrm{BMO}(\Omega) \setminus L^\infty(\Omega)$. The reverse inclusion, however, does hold:

Proposition 2.3.4. *For any basis of shapes,* $L^\infty(\Omega) \subset \mathrm{BMO}^{p}_{\mathscr{S}}(\Omega)$ *with*

$$\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} \leq \begin{cases} \|f\|_{L^\infty}, & 1 \leq p \leq 2; \\ 2\|f\|_{L^\infty}, & 2 < p < \infty. \end{cases}$$

Proof. Fix $f \in L^\infty(\Omega)$ and a shape $S \in \mathscr{S}$. For any $1 \leq p < \infty$, Minkowski's and Jensen's inequalities give

$$\left(\fint_S |f - f_S|^p \right)^{1/p} \leq 2 \left(\fint_S |f|^p \right)^{1/p} \leq 2\|f\|_\infty.$$

Restricting to $1 \leq p \leq 2$, one may use Proposition 2.3.2 with $p_2 = 2$ to arrive at

$$\left(\fint_S |f - f_S|^p\right)^{1/p} \leq \left(\fint_S |f - f_S|^2\right)^{1/2}.$$

Making use of the Hilbert space structure on $L^2(S, \frac{dx}{|S|})$, observe that $f - f_S$ is orthogonal to constants and so it follows that[1]

$$\fint_S |f - f_S|^2 = \fint_S |f|^2 - |f_S|^2 \leq \fint_S |f|^2 \leq \|f\|_\infty^2.$$

$\square$

A simple example shows that the constant 1 obtained for $1 \leq p \leq 2$ is, in fact, sharp:

Example 2.3.5. Let S be a shape on Ω and consider a function $f = \chi_E - \chi_{E^c}$, where E is a measurable subset of S such that $|E| = \frac{1}{2}|S|$ and $E^c = S \setminus E$. Then $f_S = 0$, $|f - f_S| = |f| \equiv 1$ on S and so

$$\fint_S |f - f_S|^p = 1.$$

Thus, $\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} \geq 1 = \|f\|_{L^\infty}$.

There is no reason to believe that the constant 2 for $2 < p < \infty$ is sharp, however, and so we pose the following question:

Problem 2.3.6. What is the smallest constant $c_\infty(p, \mathscr{S})$ such that the inequality $\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} \leq c_\infty(p, \mathscr{S})\|f\|_{L^\infty}$ holds for all $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$?

The solution to this problem was obtained by Leonchik in the case when $\Omega \subset \mathbb{R}$ and $\mathscr{S} = \mathcal{I}$.

Theorem 2.3.7 ([63, 60])**.**

$$c_\infty(p, \mathcal{I}) = 2 \sup_{0<h<1} \{h(1-h)^p + h^p(1-h)\}^{1/p}.$$

An analysis of this expression ([60]) shows that $c_\infty(p, \mathcal{I}) = 1$ for $1 \leq p \leq 3$, improving on Proposition 2.3.4 for $2 < p \leq 3$. Moreover, $c_\infty(p, \mathcal{I})$ is monotone in p with $1 < c_\infty(p, \mathcal{I}) < 2$ for $p > 3$ and $c_\infty(p, \mathcal{I}) \to 2$ as $p \to \infty$.

[1]See Proposition 5.1.5 in Appendix I for more details.

It is easy to see that $\|\cdot\|_{\mathrm{BMO}^p_{\mathscr{S}}}$ defines a seminorm. It cannot be a norm, however, as a function f that is almost everywhere equal to a constant will satisfy $\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} = 0$. What we can show is that the quantity $\|\cdot\|_{\mathrm{BMO}^p_{\mathscr{S}}}$ defines a norm modulo constants.

In the classical case of $\mathrm{BMO}(\mathbb{R}^n)$, the proof (see [42]) relies on the fact that $\mathcal{B}$ contains $\mathcal{C}$ and so $\mathbb{R}^n$ may be written as the union of countably-many concentric shapes. When on a domain that is also a shape, the proof is immediate. In our general setting, however, we may not be in a situation where Ω is a shape or $\mathscr{S}$ contains a distinguished subcollection of nested shapes that exhausts all of Ω; for an example, consider the case where Ω is a rectangle that is not a cube with $\mathscr{S} = \mathcal{Q}$. As such, the proof must be adapted. We do so in a way that relies on shapes being open sets that cover Ω, and on Ω being connected and Lindelöf.

Proposition 2.3.8. *For any basis of shapes, $\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} = 0$ if and only if f is almost everywhere equal to some constant.*

Proof. One direction is immediate. For the other direction, fix $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ such that $\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} = 0$. It follows that $f = f_S$ almost everywhere on S for each $S \in \mathscr{S}$.

Fix $S_0 \in \mathscr{S}$ and let $C_0 = f_{S_0}$. Set

$$U = \bigcup \{S \in \mathscr{S} : f_S = C_0\}.$$

Using the Lindelöf property of Ω, we may assume that U is defined by a countable union. It follows that $f = C_0$ almost everywhere on U since $f = f_S = C_0$ almost everywhere for each S comprising U. The goal now is to show that $\Omega = U$.

Now, let

$$V = \bigcup \{S \in \mathscr{S} : f_S \neq C_0\}.$$

Since $\mathscr{S}$ covers Ω, we have $\Omega = U \cup V$. Thus, in order to show that $\Omega = U$, we need to show that V is empty. To do this, we note that both U and V are open sets and so, since Ω is connected, it suffices to show that U and V are disjoint.

Suppose that $x \in U \cap V$. Then there is an S_1 containing x which is in U and an S_2 containing x which is in V. In particular, $f = C_0$ almost everywhere on S_1 and $f \neq C_0$ almost everywhere on S_2. However, this is impossible as S_1 and S_2 are open sets of positive measure with non-empty intersection and so $S_1 \cap S_2$ must have positive measure. Therefore, $U \cap V = \emptyset$ and the result follows. $\square$

As Proposition 2.3.8 implies that $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)/\mathbb{C}$ is a normed linear space, a natural question is whether it is complete. For $\mathrm{BMO}(\mathbb{R}^n)/\mathbb{C}$ this is a corollary of Fefferman's theorem that identifies it as the dual of the real Hardy space ([32, 33]).

A proof that $\mathrm{BMO}(\mathbb{R}^n)/\mathbb{C}$ is a Banach space that does not pass through duality came a few years later and is due to Neri ([70]). This is another example of a proof that relies on the fact that $\mathcal{B}$ contains $\mathcal{C}$ (or, equivalently, that $\mathcal{Q}$ contains an analogue of $\mathcal{C}$ but for cubes). The core idea, however, may be adapted to our more general setting. The proof below makes use of the fact that shapes are open and that Ω is path connected since it is both open and connected.

Theorem 2.3.9. *For any basis of shapes,* $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ *is complete.*

Proof. Let $\{f_i\}$ be Cauchy in $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$. Then, for any shape $S \in \mathscr{S}$, the sequence $\{f_i - (f_i)_S\}$ is Cauchy in $L^p(S)$. Since $L^p(S)$ is complete, there exists a function $f^S \in L^p(S)$ such that $f_i - (f_i)_S \to f^S$ in $L^p(S)$. The function f^S can be seen to have mean zero on S: since $f_i - (f_i)_S$ converges to f^S in $L^1(S)$, it follows that

$$\fint_S f^S = \lim_{i\to\infty} \fint_S f_i - (f_i)_S = 0. \tag{2.3}$$

If we have two shapes $S_1, S_2 \in \mathscr{S}$ such that $S_1 \cap S_2 \neq \emptyset$, by the above there is a function $f^{S_1} \in L^p(S_1)$ such that $f_i - (f_i)_{S_1} \to f^{S_1}$ in $L^p(S_1)$ and a function $f^{S_2} \in L^p(S_2)$ such that $f_i - (f_i)_{S_2} \to f^{S_2}$ in $L^p(S_2)$. Since both of these hold in $L^p(S_1 \cap S_2)$, we have

$$(f_i)_{S_2} - (f_i)_{S_1} = [f_i - (f_i)_{S_1}] - [f_i - (f_i)_{S_2}] \to f^{S_1} - f^{S_2} \quad \text{in } L^p(S_1 \cap S_2).$$

This implies that the sequence $C_i(S_1, S_2) = (f_i)_{S_2} - (f_i)_{S_1}$ converges as constants to a limit that we denote by $C(S_1, S_2)$, with

$$f^{S_1} - f^{S_2} \equiv C(S_1, S_2) \quad \text{on } S_1 \cap S_2.$$

From their definition, these constants are antisymmetric:

$$C(S_1, S_2) = -C(S_2, S_1).$$

Moreover, they possess an additive property that will be useful in later computations. By a finite chain of shapes we mean a finite sequence $\{S_j\}_{j=1}^k \subset \mathscr{S}$ such that $S_j \cap S_{j+1} \neq \emptyset$ for all

$1 \leq j \leq k-1$. Furthermore, by a loop of shapes we mean a finite chain $\{S_j\}_{j=1}^k$ such that $S_1 \cap S_k \neq \emptyset$. If $\{S_j\}_{j=1}^k$ is a loop of shapes, then

$$C(S_1, S_k) = \sum_{j=1}^{k-1} C(S_j, S_{j+1}). \tag{2.4}$$

To see this, consider the telescoping sum

$$(f_i)_{S_k} - (f_i)_{S_1} = \sum_{j=1}^{k-1} (f_i)_{S_{j+1}} - (f_i)_{S_j},$$

for a fixed i. The formula (2.4) follows from this as each $(f_i)_{S_{j+1}} - (f_i)_{S_j}$ converges to $C(S_j, S_{j+1})$ since $S_j \cap S_{j+1} \neq \emptyset$ and $(f_i)_{S_k} - (f_i)_{S_1}$ converges to $C(S_1, S_k)$ since $S_1 \cap S_k \neq \emptyset$.

Let us now fix a shape $S_0 \in \mathscr{S}$ and consider another shape $S \in \mathscr{S}$ such that $S_0 \cap S = \emptyset$. Since Ω is a path-connected set, for any pair of points $(x, y) \in S_0 \times S$ there exists a path $\gamma_{x,y} : [0,1] \to \Omega$ such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$. Since $\mathscr{S}$ covers Ω and the image of $\gamma_{x,y}$ is a compact set, we may cover $\gamma_{x,y}$ by a finite number of shapes. From this we may extract a finite chain connecting S to S_0.

We now come to building the limit function f. If $x \in S_0$, then set $f(x) = f^{S_0}(x)$, where f^{S_0} is as defined earlier. If $x \notin S_0$, then there is some shape S containing x and, by the preceding argument, a finite chain of shapes $\{S_j\}_{j=1}^k$ where $S_k = S$. In this case, set

$$f(x) = f^{S_k}(x) + \sum_{j=0}^{k-1} C(S_j, S_{j+1}). \tag{2.5}$$

The first goal is to show that this is well defined. Let $\{\widetilde{S}_j\}_{j=1}^{\ell}$ be another finite chain connecting some $\widetilde{S}_\ell$ with $x \in \widetilde{S}_\ell$ to $S_0 = \widetilde{S}_0$. Then we need to show that

$$f^{S_k}(x) + \sum_{j=0}^{k-1} C(S_j, S_{j+1}) = f^{\widetilde{S}_\ell}(x) + \sum_{j=0}^{\ell-1} C(\widetilde{S}_j, \widetilde{S}_{j+1}). \tag{2.6}$$

First, we use the fact that $x \in S_k \cap \widetilde{S}_\ell$ to write $f^{S_k}(x) - f^{\widetilde{S}_\ell}(x) = C(S_k, \widetilde{S}_\ell)$. Then, from the antisymmetry property of the constants, (2.6) is equivalent to

$$C(S_k, \widetilde{S}_\ell) = C(S_k, S_{k-1}) + \cdots + C(S_1, S_0) + C(S_0, \widetilde{S}_1) + \cdots + C(\widetilde{S}_{\ell-1}, \widetilde{S}_\ell),$$

which follows from (2.4) as $\{S_k, S_{k-1}, \ldots, S_1, S_0, \widetilde{S}_1, \widetilde{S}_2, \ldots, \widetilde{S}_\ell\}$ forms a loop.

Finally, we show that $f_i \to f$ in $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$. Fixing a shape $S \in \mathscr{S}$, choose a finite chain $\{S_j\}_{j=1}^k$ such that $S_k = S$. By (2.5), on S we have that $f = f^S$ modulo constants, and so, using (2.3) and the definition of f^S,

$$\begin{aligned}
⨍_S |(f_i(x) - f(x)) - (f_i - f)_S|^p \, dx &= ⨍_S |(f_i(x) - f^S(x)) - (f_i - f^S)_S|^p \, dx \\
&= ⨍_S |f_i(x) - (f_i)_S - f^S(x)|^p \, dx \\
&\to 0 \qquad \text{as } i \to \infty.
\end{aligned}$$

□

2.4 Shapewise inequalities on BMO

A "shapewise" inequality is an inequality that holds for each shape S in a given basis. In this section, considerable attention will be given to highlighting those situations where the constants in these inequalities are known to be sharp. A recurring theme is the following: sharp results are mainly known for $p = 1$ and for $p = 2$ and, in fact, the situation for $p = 2$ is usually simple. The examples given demonstrating sharpness are straightforward generalisations of some of those found in [60].

For this section, we assume that $\mathscr{S}$ is an arbitrary basis of shapes and that $f \in L^1(S)$ for every $S \in \mathscr{S}$.

We begin by considering inequalities that provide equivalent characterizations of the space $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$. As with the classical BMO space, one can estimate the mean oscillation of a function on a shape by a double integral that is often easier to use for calculations but that comes at the loss of a constant.

Proposition 2.4.1. *For any shape $S \in \mathscr{S}$,*

$$\frac{1}{2}\left(⨍_S ⨍_S |f(x) - f(y)|^p \, dy \, dx\right)^{\frac{1}{p}} \le \left(⨍_S |f - f_S|^p\right)^{\frac{1}{p}} \le \left(⨍_S ⨍_S |f(x) - f(y)|^p \, dy \, dx\right)^{\frac{1}{p}}.$$

Proof. Fix a shape $S \in \mathscr{S}$. By Jensen's inequality we have that

$$⨍_S |f(x) - f_S|^p \, dx = ⨍_S \left| ⨍_S \big(f(x) - f(y)\big) \, dy \right|^p dx \le ⨍_S ⨍_S |f(x) - f(y)|^p \, dy \, dx$$

and by Minkowski's inequality on $L^p(S \times S, \frac{dxdy}{|S|^2})$ we have that

$$\left(\fint_S \fint_S |f(x) - f(y)|^p \, dy \, dx\right)^{1/p} = \left(\fint_S \fint_S \big|\big(f(x) - f_S\big) - \big(f(y) - f_S\big)\big|^p \, dy \, dx\right)^{1/p}$$
$$\leq 2\left(\fint_S |f - f_S|^p\right)^{1/p}.$$

□

When $p = 1$, the following examples show that the constants in this inequality are sharp.

Example 2.4.2. Let S be a shape in Ω and consider a function $f = \chi_E$, where E is a measurable subset of S such that $|E| = \frac{1}{2}|S|$. Then $f_S = \frac{1}{2}$ and so $|f - f_S| = \frac{1}{2}\chi_S$, yielding

$$\fint_S |f - f_S| = \frac{1}{2}.$$

Writing $E^c = S \setminus E$, we have that $|f(x) - f(y)| = 0$ for $(x, y) \in E \times E$ or $(x, y) \in E^c \times E^c$, and that $|f(x) - f(y)| = 1$ for $(x, y) \in E^c \times E$; hence,

$$\fint_S \fint_S |f(x) - f(y)| \, dy \, dx = \frac{2}{|S|^2} \int_{E^c} \int_E |f(x) - f(y)| \, dy \, dx = \frac{2|E^c||E|}{|S|^2} = \frac{1}{2}.$$

Therefore, the right-hand side constant 1 is sharp.

Example 2.4.3. Now consider a function $f = \chi_{E_1} - \chi_{E_3}$, where E_1, E_2, E_3 are measurable subsets of S such that $S = E_1 \cup E_2 \cup E_3$ is a disjoint union (up to a set of measure zero) and $|E_1| = |E_3| = \beta|S|$ for some $0 < \beta < \frac{1}{2}$. Then, $f_S = 0$ and so

$$\fint_S |f - f_S| = \fint_S |f| = 2\beta.$$

Since $|f(x) - f(y)| = 0$ for $(x, y) \in E_j \times E_j$, $j = 1, 2, 3$, $|f(x) - f(y)| = 1$ for $(x, y) \in E_i \times E_j$ when $|i - j| = 1$, and $|f(x) - f(y)| = 2$ for $(x, y) \in E_i \times E_j$ when $|i - j| = 2$, we have that

$$\fint_S \fint_S |f(x) - f(y)| \, dy \, dx = \frac{4\beta|S|(|S| - 2\beta|S|) + 4\beta^2|S|^2}{|S|^2} = 4\beta(1 - 2\beta) + 4\beta^2.$$

As

$$\frac{2\beta}{4\beta(1 - 2\beta) + 4\beta^2} = \frac{1}{2 - 2\beta} \to \frac{1}{2}$$

as $\beta \to 0^+$, the left-hand side constant $\frac{1}{2}$ is sharp.

For other values of p, however, these examples tell us nothing about the sharpness of the constants in Proposition 2.4.1. In fact, the constants are not sharp for $p = 2$. In the following proposition, as in many to follow, the additional Hilbert space structure afforded to us yields a sharp statement (in this case, an equality) for little work.

Proposition 2.4.4. *For any shape $S \in \mathscr{S}$,*

$$\left(\fint_S \fint_S |f(x) - f(y)|^2 \, dy \, dx \right)^{1/2} = \sqrt{2} \left(\fint_S |f - f_S|^2 \right)^{1/2}.$$

Proof. Observe that as elements of $L^2(S \times S, \frac{dx\,dy}{|S|^2})$, $f(x) - f_S$ is orthogonal to $f(y) - f_S$. Thus,

$$\fint_S \fint_S |f(x) - f(y)|^2 \, dy \, dx = \fint_S \fint_S |f(x) - f_S|^2 \, dy \, dx + \fint_S \fint_S |f(y) - f_S|^2 \, dy \, dx$$

and so

$$\left(\fint_S \fint_S |f(x) - f(y)|^2 \, dy \, dx \right)^{1/2} = \sqrt{2} \left(\fint_S |f - f_S|^2 \right)^{1/2}.$$

□

In a different direction, it is sometimes easier to consider not the oscillation of a function from its mean, but its oscillation from a different constant. Again, this can be done at the loss of a constant.

Proposition 2.4.5. *For any shape $S \in \mathscr{S}$,*

$$\inf_c \left(\fint_S |f - c|^p \right)^{1/p} \leq \left(\fint_S |f - f_S|^p \right)^{1/p} \leq 2 \inf_c \left(\fint_S |f - c|^p \right)^{1/p},$$

where the infimum is taken over all constants c.

Proof. The first inequality is trivial. To show the second inequality, fix a shape $S \in \mathscr{S}$. By Minkowski's inequality on $L^p(S, \frac{dx}{|S|})$ and Jensen's inequality,

$$\left(\fint_S |f - f_S|^p \right)^{1/p} \leq \left(\fint_S |f - c|^p \right)^{1/p} + (|f_S - c|^p)^{1/p} \leq 2 \left(\fint_S |f - c|^p \right)^{1/p}.$$

□

As with Proposition 2.4.1, simple examples show that the constants are sharp for $p = 1$.

Example 2.4.6. Let S be a shape on Ω and consider the function $f = \chi_E - \chi_{E^c}$ as in Example 2.3.5, with $|E| = |S|/2$, $E^c = S \setminus E$. Then

$$\fint_S |f - f_S| = 1,$$

and for any constant c we have that

$$\fint_S |f - c| = \frac{|1-c||E| + |1+c||E^c|}{|S|} = \frac{|1-c| + |1+c|}{2} \geq 1$$

with equality if $c \in [-1, 1]$. Thus

$$\inf_c \fint_S |f - c| = \fint_S |f - f_S|,$$

showing the left-hand side constant 1 in Proposition 2.4.5 is sharp when $p = 1$.

Example 2.4.7. Consider, now, the function $f = \chi_E$ where E is a measurable subset of S such that $|E| = \alpha|S|$ for some $0 < \alpha < \frac{1}{2}$. Then, $f_S = \alpha$ and

$$\fint_S |f - f_S| = \frac{(1-\alpha)\alpha|S|}{|S|} + \frac{(|S| - \alpha|S|)\alpha}{|S|} = 2\alpha(1-\alpha).$$

For any constant c, we have that

$$\fint_S |f - c| = \frac{\big|1 - |c|\big||E|}{|S|} + \frac{|c|(|S| - |E|)}{|S|} = \alpha\big|1 - |c|\big| + |c|(1-\alpha).$$

The right-hand side is at least $\alpha(1 - |c|) + |c|(1-\alpha) = \alpha + |c|(1 - 2\alpha)$, which is at least α with equality for $c = 0$, and so

$$\inf_c \fint_S |f - c| = \alpha.$$

As $\alpha \to 0^+$,

$$\frac{2\alpha(1-\alpha)}{\alpha} \to 2,$$

showing the right-hand side constant 2 in Proposition 2.4.5 is sharp when $p = 1$.

When $p = 1$, it turns out that we know for which constant the infimum in Proposition 2.4.5 is achieved.

Proposition 2.4.8. *Let f be real-valued. For any shape $S \in \mathscr{S}$,*

$$\inf_c \fint_S |f - c| = \fint_S |f - m|,$$

where m is a median of f on S: that is, a (possibly non-unique) number such that $|\{x \in S : f(x) > m\}| \leq \frac{1}{2}|S|$ and $|\{x \in S : f(x) < m\}| \leq \frac{1}{2}|S|$.

Note that the definition of a median makes sense for real-valued measurable functions. A proof of this proposition can be found in the appendix of [24], along with the fact that such functions always have a median on a measurable set of positive and finite measure (in particular, on a shape). Also, note that from the definition of a median, it follows that

$$|\{x \in S : f(x) \geq m\}| = |S| - |\{x \in S : f(x) > m\}| \geq |S| - \frac{1}{2}|S| = \frac{1}{2}|S|$$

and, likewise,

$$|\{x \in S : f(x) \leq m\}| \geq \frac{1}{2}|S|.$$

Proof. Fix a shape $S \in \mathscr{S}$ and a median m of f on S. For any constant c,

$$\begin{aligned}
\int_S |f(x) - m| \, dx &= \int_{\{x \in S : f(x) \geq m\}} \big(f(x) - m\big) \, dx + \int_{\{x \in S : f(x) < m\}} \big(m - f(x)\big) \, dx \\
&= \int_{\{x \in S : f(x) \geq m\}} \big(f(x) - c\big) \, dx + (c - m)|\{x \in S : f(x) \geq m\}| \\
&\qquad + \int_{\{x \in S : f(x) < m\}} \big(c - f(x)\big) \, dx + (m - c)|\{x \in S : f(x) < m\}|.
\end{aligned}$$

Assuming that $m > c$, we have that

$$\int_{\{x \in S : f(x) \geq m\}} \big(f(x) - c\big) \, dx \leq \int_{\{x \in S : f(x) \geq c\}} \big(f(x) - c\big) \, dx$$

and

$$\int_{\{x \in S : c \leq f(x) < m\}} \big(c - f(x)\big) \, dx \leq 0,$$

and so we can write

$$\begin{aligned}
\int_S |f(x) - m| \, dx &\leq \int_{\{x \in S : f(x) \geq c\}} \big(f(x) - c\big) \, dx + (c - m)|\{x \in S : f(x) \geq m\}| \\
&\qquad + \int_{\{x \in S : f(x) < c\}} \big(c - f(x)\big) \, dx + (m - c)|\{x \in S : f(x) < m\}| \\
&= \int_S |f(x) - c| \, dx \\
&\qquad + (m - c)\big[|\{x \in S : f(x) < m\}| - |\{x \in S : f(x) \geq m\}|\big] \\
&\leq \int_S |f(x) - c| \, dx + (m - c)\left[\frac{1}{2}|S| - \frac{1}{2}|S|\right] \\
&= \int_S |f(x) - c| \, dx.
\end{aligned}$$

In the case where $m < c$, we may apply the previous calculation to $-f$ and use the fact that $-m$ is a median of $-f$:

$$\int_S |f(x) - m|\, dx = \int_S |-f(x) - (-m)|\, dx \leq \int_S |-f(x) - (-c)|\, dx = \int_S |f(x) - c|\, dx.$$

□

For $p = 2$, we are able to do a few things at once. We are very simply able to obtain an equality that automatically determines the sharp constants for Proposition 2.4.5 and the constant c for which the infimum is achieved.

Proposition 2.4.9. *For any shape $S \in \mathscr{S}$,*

$$\inf_c \left(⨍_S |f - c|^2 \right)^{1/2} = \left(⨍_S |f - f_S|^2 \right)^{1/2},$$

where the infimum is taken over all constants c.

Proof. Fix a shape $S \in \mathscr{S}$ and a constant c. As previously observed, $f - f_S$ is orthogonal to any constant in the sense of $L^2(S, \frac{dx}{|S|})$; in particular, $f - f_S$ is orthogonal to $f_S - c$. Thus,

$$⨍_S |f - c|^2 = ⨍_S |f - f_S|^2 + ⨍_S |f_S - c|^2 \geq ⨍_S |f - f_S|^2$$

with the minimum achieved when $c = f_S$. □

The following proposition shows that the action of Hölder continuous maps preserves the bound on the p-mean oscillation, up to a constant.

Proposition 2.4.10. *Let $F : \mathbb{R} \to \mathbb{R}$ be α-Hölder continuous for $0 < \alpha \leq 1$ with Hölder coefficient L. Fix any shape $S \in \mathscr{S}$ and suppose $f \in L^1(S)$ is real-valued. Then, for $1 \leq p < \infty$,*

$$\left(⨍_S \left|F \circ f - (F \circ f)_S\right|^p \right)^{\frac{1}{p}} \leq 2L \left(⨍_S |f - f_S|^p \right)^{\frac{\alpha}{p}}. \tag{2.7}$$

When $p = 2$,

$$\left(⨍_S \left|F \circ f - (F \circ f)_S\right|^2 \right)^{\frac{1}{2}} \leq L \left(⨍_S |f - f_S|^2 \right)^{\frac{\alpha}{2}}. \tag{2.8}$$

Proof. Fix a shape $S \in \mathscr{S}$. By Proposition 2.4.5 and Jensen's inequality, we have that

$$\begin{aligned}
⨍_S \left|F \circ f - (F \circ f)_S\right|^p &\le 2^p ⨍_S \left|F(f(x)) - F(f_S)\right|^p dx \\
&\le 2L^p ⨍_S |f(x) - f_S|^{\alpha p}\, dx \\
&\le 2^p L^p \left(⨍_S |f - f_S|^p\right)^{\alpha}.
\end{aligned}$$

When $p = 2$, Proposition 2.4.9 shows that the factor of 2^p in the first inequality can be dropped. □

As has been pointed out in [24], if one uses the equivalent norm defined by Proposition 2.4.5, the result of Proposition 2.4.10 holds with constant L for any $p \ge 1$, since the factor of 2 comes from $(F \circ f)_S \ne F(f_S)$.

The following example demonstrates that the constants are sharp when $p = 1$ and $p = 2$.

Example 2.4.11. Consider the function $F(x) = |x|$, so that $\alpha = 1 = L$, and fix a shape S in Ω. Taking the function $f = \chi_{E_1} - \chi_{E_3}$, as in Example 2.4.3, where S is a disjoint union $E_1 \cup E_2 \cup E_3$ and $|E_1| = |E_3| = \beta|S|$ for some $0 < \beta < \frac{1}{2}$, we have $|f|_S = 2\beta$ and so

$$⨍_S \left||f| - |f|_S\right| = 4(1 - 2\beta)\beta.$$

Since $⨍_S |f - f_S| = 2\beta$ and $\frac{4(1-2\beta)\beta}{2\beta} \to 2$ as $\beta \to 0^+$, the constant 2 is sharp for $p = 1$.

For $p = 2$, when $f \ge 0$ we have $F(f) = f$ and $F(f)_S = F(f_S)$, so equality holds in (2.8).

While we have shown that the shapewise inequalities (2.7), for $p = 1$, and (2.8) are sharp for $F(x) = |x|$, in the next section it will be shown that better constants can be obtained for norm inequalities.

Now we address how the $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ spaces relate for different bases.

Proposition 2.4.12. *For any shape $S \in \mathscr{S}$, if $\widetilde{S}$ is another shape (from possibly another basis) such that $\widetilde{S} \subset S$ and $|\widetilde{S}| \ge c|S|$ for some constant c, then*

$$\left(⨍_{\widetilde{S}} |f - f_{\widetilde{S}}|^p\right)^{\frac{1}{p}} \le 2c^{-1/p} \left(⨍_S |f - f_S|^p\right)^{\frac{1}{p}}.$$

Proof. From Proposition 2.4.5,

$$\fint_{\widetilde{S}} |f - f_{\widetilde{S}}|^p \leq 2^p \fint_{\widetilde{S}} |f - f_S|^p \leq 2^p c^{-1} \fint_S |f - f_S|^p.$$

$\square$

An immediate consequence of this is that $\mathrm{BMO}^p_{\mathscr{S}}(\Omega) \subset \mathrm{BMO}^p_{\widetilde{\mathscr{S}}}(\Omega)$ if $\widetilde{\mathscr{S}} \trianglelefteq \mathscr{S}$. Moreover, if $\mathscr{S} \approx \widetilde{\mathscr{S}}$ then $\mathrm{BMO}^p_{\mathscr{S}}(\Omega) \cong \mathrm{BMO}^p_{\widetilde{\mathscr{S}}}(\Omega)$. In particular, it follows that $\mathrm{BMO}^p_{\mathcal{B}}(\Omega) \cong \mathrm{BMO}^p_{\mathcal{Q}}(\Omega)$. Since $\mathcal{Q} \subset \mathcal{R}$, it is automatic without passing through Proposition 2.4.12 that $\mathrm{BMO}^p_{\mathcal{R}}(\Omega) \subset \mathrm{BMO}^p(\Omega)$ with $\|f\|_{\mathrm{BMO}^p} \leq \|f\|_{\mathrm{BMO}^p_{\mathcal{R}}}$. The reverse inclusion is false. The following example of a function in $\mathrm{BMO}(\Omega)$ that is not in $\mathrm{BMO}_{\mathcal{R}}(\Omega)$ is taken from [60], where the calculations proving the claim can be found:

Example 2.4.13. Consider $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$. The function

$$f = \sum_{k=1}^{\infty} \chi_{\left(0, 2^{-k+1}\right) \times \left(0, \frac{1}{k}\right)}$$

belongs to $\mathrm{BMO}(\Omega) \setminus \mathrm{BMO}_{\mathcal{R}}(\Omega)$.

2.5 Rearrangements and the absolute value

Consider two measure spaces (M, μ) and (N, ν) such that $\mu(M) = \nu(N)$.

Definition 2.5.1. We say that measurable functions $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ are equimeasurable if for all $s \in \mathbb{R}$ the quantities $\mu_f(s) = \mu\big(\{x \in M : f(x) > s\}\big)$ and $\nu_g(s) = \nu\big(\{y \in N : g(y) > s\}\big)$ coincide.

It is important to note that this is not the standard definition of equimeasurability. Typically (see, for example, [4]) equimeasurability means $\mu_{|f|}(s) = \mu_{|g|}(s)$ for all $s \geq 0$; however, for our purposes, it will be useful to distinguish between two functions being equimeasurable and the absolute value of two functions being equimeasurable. That said, it is true that

Lemma 2.5.2. *Let f and g be measurable functions such that $\mu_f(s) = \nu_g(s) < \infty$ for all s. Then, $\mu_{|f|}(s) = \nu_{|g|}(s)$ for all s.*

Proof. Fix $s \in \mathbb{R}$. Writing

$$\{x \in M : f(x) < -s\} = \bigcup_{n \in \mathbb{N}} \left\{x \in M : f(x) \leq -s - \frac{1}{n}\right\},$$

we have that

$$\mu\big(\{x \in M : f(x) < -s\}\big) = \lim_{n \to \infty} \mu(M) - \mu_f\left(-s - \frac{1}{n}\right).$$

Here we use the convention that infinity minus a finite number is infinity and use the fact that $\mu_f < \infty$. Since $\mu_f = \nu_g$, by assumption, it follows that

$$\mu\big(\{x \in M : f(x) < -s\}\big) = \nu\big(\{x \in N : g(x) < -s\}\big).$$

If $s \geq 0$, then

$$\begin{aligned}\mu\big(\{x \in M : |f(x)| > s\}\big) &= \mu\big(\{x \in M : f(x) > s\}\big) + \mu\big(\{x \in M : f(x) < -s\}\big)\\ &= \nu\big(\{y \in N : g(y) > s\}\big) + \nu\big(\{y \in N : g(y) < -s\}\big)\\ &= \nu\big(\{y \in N : |g(x)| > s\}\big).\end{aligned}$$

If $s < 0$, then

$$\mu\big(\{x \in M : |f(x)| > s\}\big) = \mu(M) = \nu(N) = \nu\big(\{y \in N : |g(x)| > s\}\big).$$

□

A useful tool is the following lemma. It is a consequence of Cavalieri's principle, also called the layer cake representation, which provides a way of expressing the integral of $\varphi(|f|)$ for a suitable transformation φ in terms of a weighted integral of $\mu_{|f|}$. The simplest incarnation of this principal states that for any measurable set A,

$$\int_A |f|^p = \int_0^\infty p\alpha^{p-1} |\{x \in A : |f(x)| > \alpha\}| \, d\alpha,$$

where $0 < p < \infty$. A more general statement can be found in [66], Theorem 1.13 and its remarks.

Lemma 2.5.3. *Let $M \subset \mathbb{R}^m$, $N \subset \mathbb{R}^n$ be Lebesgue measurable sets of equal measure, and $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ be measurable functions such that $|f|$ and $|g|$ are equimeasurable. Then, for $0 < p < \infty$,*

$$\int_M |f|^p = \int_N |g|^p \qquad \textit{and} \qquad \operatorname*{ess\,sup}_M |f| = \operatorname*{ess\,sup}_N |g|.$$

Furthermore, under the hypothesis of Lemma 2.5.2, for any constant c,

$$\int_M \big||f| - c\big| = \int_N \big||g| - c\big|.$$

Moving back to the setting of this paper, for this section we assume that f is a measurable function on Ω that satisfies the condition

$$|\{x \in \Omega : |f(x)| > s\}| \to 0 \qquad \text{as} \qquad s \to \infty. \tag{2.9}$$

This guarantees that the rearrangements defined below are finite on their domains (see [81], V.3).

Definition 2.5.4. Let $I_\Omega = (0, |\Omega|)$. The decreasing rearrangement of f is the function

$$f^*(t) = \inf\{s \geq 0 : |\{x \in \Omega : |f(x)| > s\}| \leq t\}, \quad t \in I_\Omega.$$

This rearrangement is studied in the theory of interpolation and rearrangement-invariant function spaces. In particular, it can be used to define the Lorentz spaces, $L^{p,q}$, which are a refinement of the scale of Lebesgue spaces and can be used to strengthen certain inequalities such as those of Hardy-Littlewood-Sobolev and Hausdorff-Young. For standard references on these topics, see [4] or [81].

A related rearrangement is the following.

Definition 2.5.5. The signed decreasing rearrangement of f is defined as

$$f^\circ(t) = \inf\{s \in \mathbb{R} : |\{x \in \Omega : f(x) > s\}| \leq t\}, \quad t \in I_\Omega.$$

Clearly, f° coincides with f^* when $f \geq 0$ and, more generally, $|f|^\circ = f^*$. Further information on this rearrangement can be found in [17, 60].

Here we collect some of the basic properties of these rearrangements, the proofs for which are adapted from [81].

Lemma 2.5.6. *Let $f : \Omega \to \mathbb{R}$ be measurable and satisfying* (2.9)*. Then*

a) its signed decreasing rearrangement $f^\circ : I_\Omega \to (-\infty, \infty)$ is decreasing and equimeasurable with f;

b) its decreasing rearrangement $f^* : I_\Omega \to [0, \infty)$ is decreasing and equimeasurable with $|f|$.

Proof. If $t_1 \geq t_2$, it follows that

$$\{s : |\{x \in \Omega : f(x) > s\}| \leq t_2\} \subset \{s : |\{x \in \Omega : f(x) > s\}| \leq t_1\}.$$

Since this is equally true for $|f|$ in place of f, it shows that both f^* and f° are decreasing functions.

Fix s. For $t \in I_\Omega$, $f^\circ(t) > s$ if and only if $t < |\{x \in \Omega : f(x) > s\}|$, from where it follows that

$$|\{t \in I_\Omega : f^\circ(t) > s\}| = |\{x \in \Omega : f(x) > s\}|.$$

Again, applying this to $|f|$ in place of f yields the corresponding statement for f^*.

□

One may ask how the rearrangement f^* behaves when additional conditions are imposed on f. In particular, is the map $f \mapsto f^*$ a bounded operator on various function spaces? A well-known result in this direction is that this map is an isometry on L^p, which follows immediately from Lemmas 2.5.3 and 2.5.6.

Proposition 2.5.7. *For all $1 \leq p \leq \infty$, if $f \in L^p(\Omega)$ then $f^* \in L^p(I_\Omega)$ with $\|f^*\|_{L^p(I_\Omega)} = \|f\|_{L^p(\Omega)}$.*

Another well-known result is the Pólya-Szegő inequality, which asserts that the Sobolev norm decreases under the symmetric decreasing rearrangement ([10]), yet another kind of rearrangement. From this one can deduce the following (see, for instance, [17]).

Theorem 2.5.8. *If $f \in W^{1,p}(\mathbb{R}^n)$ then*

$$n\omega_n^{1/n} \left(\int_0^\infty \left| \frac{d}{dt} f^*(t) \right|^p t^{p/n'} \, dt \right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}^n} |\nabla f|^p \right)^{\frac{1}{p}},$$

where n' is the Hölder dual exponent of n and ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

Despite these positive results, there are some closely related spaces on which the operator $f \mapsto f^*$ is not bounded. One such example is the John-Nirenberg space $\mathrm{JN}_p(\Omega)$. We say that

$f \in L^1_{\text{loc}}(\Omega)$ is in $\text{JN}_p(\Omega)$ if there exists a constant $K \geq 0$ such that

$$\sup \sum |Q_i| \left(\fint_{Q_i} |f - f_{Q_i}| \right)^p \leq K^p, \tag{2.10}$$

where the supremum is taken over all collections of pairwise disjoint cubes Q_i in Ω. We define the quantity $\|f\|_{\text{JN}_p}$ as the smallest K for which (2.10) holds. One can show that this is a norm on $\text{JN}_p(\Omega)$ modulo constants. These spaces have been considered in the case where Ω is a cube in [27, 54] and a general Euclidean domain in [49], and generalised to a metric measure space in [1, 69].

While it is well known that $L^p(\Omega) \subset \text{JN}_p(\Omega) \subset L^{p,\infty}(\Omega)$, the strictness of these inclusions has only recently been addressed ([1, 27]).

In the case where $\Omega = I$, a (possibly unbounded) interval, the following is obtained:

Theorem 2.5.9 ([27]). *Let $f : I \to \mathbb{R}$ be a monotone function with $f \in L^1(I)$. Then there exists $c = c(p) > 0$ such that*

$$\|f\|_{\text{JN}_p} \geq c\|f - C\|_{L^p}$$

for some $C \in \mathbb{R}$.

In other words, monotone functions are in $\text{JN}_p(I)$ if and only if they are also in $L^p(I)$. In [27], an explicit example of a function $f \in \text{JN}_p(I) \setminus L^p(I)$ is constructed when I is a finite interval. This leads to the observation that the decreasing rearrangement is not bounded on $\text{JN}_p(I)$.

Corollary 2.5.10. *If I is a finite interval, there exists an $f \in \text{JN}_p(I)$ such that $f^* \notin \text{JN}_p(I_I)$.*

Proof. Since $f \notin L^p(I)$, it follows from Proposition 2.5.7 that $f^* \notin L^p(I_I)$. As f^* is monotone, it follows from the previous theorem that $f^* \notin \text{JN}_p(I_I)$. □

We consider now the question of boundedness of rearrangements on $\text{BMO}^p_{\mathscr{S}}(\Omega)$ spaces.

Problem 2.5.11. Does there exist a constant c such that $\|f^*\|_{\text{BMO}^p(I_\Omega)} \leq c\|f\|_{\text{BMO}^p_{\mathscr{S}}(\Omega)}$ holds for all $f \in \text{BMO}^p_{\mathscr{S}}(\Omega)$? If so, what is the smallest constant, written $c_*(p, \mathscr{S})$, for which this holds?

Problem 2.5.12. Does there exist a constant c such that $\|f^\circ\|_{\mathrm{BMO}^p(I_\Omega)} \le c\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)}$ holds for all $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$? If so, what is the smallest constant, written $c_\circ(p, \mathscr{S})$, for which this holds?

Clearly, if such constants exist, then they are at least equal to one. The work of Garsia-Rodemich and Bennett-DeVore-Sharpley implies an answer to the first problem and that $c_*(1, \mathcal{Q}) \le 2^{n+5}$ when $\Omega = \mathbb{R}^n$:

Theorem 2.5.13 ([3, 39])**.** *If $f \in \mathrm{BMO}(\mathbb{R}^n)$, then $f^* \in \mathrm{BMO}\big((0,\infty)\big)$ and*

$$\|f^*\|_{\mathrm{BMO}} \le 2^{n+5}\|f\|_{\mathrm{BMO}}.$$

These results were obtained by a variant of the Calderón-Zygmund decomposition ([78]). Riesz' rising sun lemma, an analogous one-dimensional result that can often be used to obtain better constants, was then used by Klemes to obtain the sharp estimate that for $\Omega = I$, a finite interval, $c_\circ(1, \mathcal{I}) = 1$.

Theorem 2.5.14 ([57])**.** *If $f \in \mathrm{BMO}(I)$, then $f^\circ \in \mathrm{BMO}(I_I)$ and*

$$\|f^\circ\|_{\mathrm{BMO}} \le \|f\|_{\mathrm{BMO}}.$$

An elementary but key element of Klemes' proof that can be generalised to our context of general shapes is the following shapewise identity.

Lemma 2.5.15. *For any shape S, if $f \in L^1(S)$ then*

$$\fint_S |f - f_S| = \frac{2}{|S|}\int_{\{f > f_S\}} (f - f_S) = \frac{2}{|S|}\int_{\{f < f_S\}} (f_S - f).$$

Proof. Write

$$\int_S |f(x) - f_S|\,dx = \int_{\{x \in S : f(x) > f_S\}} (f(x) - f_S)\,dx + \int_{\{x \in S : f(x) < f_S\}} (f_S - f(x))\,dx.$$

Since

$$\int_{\{x \in S : f(x) > f_S\}} (f(x) - f_S)\,dx + \int_{\{x \in S : f(x) < f_S\}} (f(x) - f_S)\,dx = \int_S (f(x) - f_S)\,dx = 0,$$

it follows that

$$\int_{\{x \in S : f(x) > f_S\}} (f(x) - f_S)\,dx = \int_{\{x \in S : f(x) < f_S\}} (f_S - f(x))\,dx,$$

which gives the identity. $\square$

The next sharp result concerning rearrangements is due to Korenovskii, showing that for $\Omega = I$, a finite interval, $c_*(1,\mathcal{I}) = 1$. The proof of this result makes direct use of Klemes' theorem.

Theorem 2.5.16 ([58]). *If $f \in \mathrm{BMO}(I)$, then $f^* \in \mathrm{BMO}(I_I)$ and*

$$\|f^*\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}}.$$

Important in Korenovskii's transition from a sharp estimate for $c_\circ(1,\mathscr{S})$ to one for $c_*(1,\mathscr{S})$ is the fact that $|f|^\circ = f^*$, bringing us to consider the boundedness of the absolute value operator. Recall from Example 2.4.11 that $F(x) = |x|$ gives us the sharp shapewise inequality in Proposition 2.4.10 with $p = 1$, which implies that $\||f|\|_{\mathrm{BMO}_{\mathscr{S}}} \leq 2\|f\|_{\mathrm{BMO}_{\mathscr{S}}}$. However, this need not be sharp as a norm inequality, and so it is natural to ask

Problem 2.5.17. What is the smallest constant $c_{|\cdot|}(p,\mathscr{S})$ such that for all $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$, $\||f|\|_{\mathrm{BMO}^p_{\mathscr{S}}} \leq c_{|\cdot|}(p,\mathscr{S})\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}$ holds?

It is clear that $c_{|\cdot|}(p,\mathscr{S}) \geq 1$ and Proposition 2.4.10 implies that $c_{|\cdot|}(p,\mathscr{S}) \leq 2$. Applying this estimate along with Klemes' theorem yields the non-sharp bound $c_*(1,\mathcal{I}) \leq 2$.

In order for Korenovskii to obtain a sharp result for $c_*(1,\mathcal{I})$, a more subtle argument was needed that allowed him to conclude that $c_{|\cdot|}(1,\mathcal{I}) = 1$ when $\Omega = I$:

Theorem 2.5.18 ([58]). *If $f \in \mathrm{BMO}(I)$, then $\||f|\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}}$.*

The following is one of the essential parts of this argument. It demonstrates that the behaviour of the absolute value operator is more easily analyzed for decreasing functions.

Theorem 2.5.19 ([58]). *Let I be a finite interval and $f \in L^1(I)$ be a decreasing function. Then,*

$$⨍_I \big||f| - |f|_I\big| \leq \sup_{J \subset I} ⨍_J |f - f_J|$$

where the supremum is taken over all subintervals J of I.

Further sharp results were obtained by Korenovskii in the case where $\Omega = R$, a rectangle, and $\mathscr{S} = \mathcal{R}$: it was shown that, similar to the one-dimensional case just discussed, $c_{|\cdot|}(1,\mathcal{R}) = c_\circ(1,\mathcal{R}) = c_*(1,\mathcal{R}) = 1$.

Theorem 2.5.20 ([59]). *If $f \in \mathrm{BMO}_{\mathcal{R}}(R)$, then $|f| \in \mathrm{BMO}_{\mathcal{R}}(R)$ and $f^\circ, f^* \in \mathrm{BMO}(I_R)$ with*

$$\||f|\|_{\mathrm{BMO}_{\mathcal{R}}} \leq \|f\|_{\mathrm{BMO}_{\mathcal{R}}}, \qquad \|f^\circ\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}_{\mathcal{R}}}, \qquad \|f^*\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}_{\mathcal{R}}}.$$

This demonstrates the paradigm that rectangles behave more similarly to one-dimensional intervals than cubes do. In particular, the generalization of Klemes' theorem to the higher-dimensional case of rectangles (the result that $c_\circ(1, \mathcal{R}) = 1$) employs a multidimensional analogue of Riesz' rising sun lemma using rectangles ([61]) when such a theorem could not exist for arbitrary cubes.

Following the techniques of [58], general relationships can be found between the constants $c_{|\cdot|}(1, \mathscr{S})$, $c_\circ(1, \mathscr{S})$, $c_*(1, \mathscr{S})$ for an arbitrary basis of shapes. First, we show that $c_{|\cdot|}(1, \mathscr{S}) \leq c_\circ(1, \mathscr{S})$.

Proposition 2.5.21. *For any collection of shapes $\mathscr{S}$, if $f \in \mathrm{BMO}_{\mathscr{S}}(\Omega)$ and $\|f^\circ\|_{\mathrm{BMO}(I_\Omega)} \leq c\|f\|_{\mathrm{BMO}_{\mathscr{S}}(\Omega)}$, then $\||f|\|_{\mathrm{BMO}_{\mathscr{S}}(\Omega)} \leq c\|f\|_{\mathrm{BMO}_{\mathscr{S}}(\Omega)}$ for the same constant c.*

Proof. Fix a shape $S \in \mathscr{S}$ and assume that $f \in \mathrm{BMO}_{\mathscr{S}}(\Omega)$ is supported on S.

Since f is equimeasurable with f° by Lemma 2.5.6, it follows from Lemma 2.5.2 that $|f|$ is equimeasurable with $|f^\circ|$ (recall that $|S| < \infty$). Writing $E = (0, |S|)$, by Lemma 2.5.3 we have that $|f^\circ|_E = |f|_S$ and also, then, that

$$\int_0^{|S|} \big||f^\circ| - |f^\circ|_E\big| = \int_0^{|S|} \big||f^\circ| - |f|_S\big| = \int_S \big||f| - |f|_S\big|.$$

Thus, by Theorem 2.5.19,

$$⨍_S \big||f| - |f|_S\big| = ⨍_{(0,|S|)} \big||f^\circ| - |f^\circ|_E\big| \leq \sup_{J \subset (0,|S|)} ⨍_J |f^\circ - (f^\circ)_J|.$$

For all $J \subset (0, |S|)$, we have that

$$⨍_J |f^\circ - (f^\circ)_J| \leq \|f^\circ\|_{\mathrm{BMO}\left((0,|S|)\right)} \leq \|f^\circ\|_{\mathrm{BMO}(I_\Omega)} \leq c\|f\|_{\mathrm{BMO}_{\mathscr{S}}(\Omega)},$$

and, therefore,

$$⨍_S \big||f| - |f|_S\big| \leq c\|f\|_{\mathrm{BMO}_{\mathscr{S}}(\Omega)}.$$

Taking a supremum over all shapes $S \in \mathscr{S}$ yields the result.

□

This result in turn allows us to prove the following relationship:

$$c_*(1,\mathscr{S}) \le c_{|\cdot|}(1,\mathscr{S})c_\circ(1,\mathscr{S}) \le c_\circ(1,\mathscr{S})^2.$$

Proposition 2.5.22. *For any collection of shapes $\mathscr{S}$, if $f \in \mathrm{BMO}_{\mathscr{S}}(\Omega)$ and $\|f^\circ\|_{\mathrm{BMO}(I_\Omega)} \le c\|f\|_{\mathrm{BMO}_{\mathscr{S}}(\Omega)}$, then $\|f^*\|_{\mathrm{BMO}(I_\Omega)} \le c^2\|f\|_{\mathrm{BMO}_{\mathscr{S}}(\Omega)}$.*

Proof. By Proposition 2.5.21, it follows that

$$\||f|\|_{\mathrm{BMO}_{\mathscr{S}}(\Omega)} \le c\|f\|_{\mathrm{BMO}_{\mathscr{S}}(\Omega)}.$$

Writing $f^* = |f|^\circ$, we have that

$$\|f^*\|_{\mathrm{BMO}(I_\Omega)} = \||f|^\circ\|_{\mathrm{BMO}(I_\Omega)} \le c\||f|\|_{\mathrm{BMO}_{\mathscr{S}}(\Omega)} \le c^2\|f\|_{\mathrm{BMO}_{\mathscr{S}}(\Omega)}.$$

□

From these results, we see that a sharp result of the form $c_\circ(1,\mathscr{S}) = 1$ would immediately imply two more sharp results, $c_{|\cdot|}(1,\mathscr{S}) = 1$ and $c_*(1,\mathscr{S}) = 1$.

Although the dyadic cubes do not, in general, cover a domain, the space dyadic BMO has been extensively studied in the literature (see [38] for an early work illustrating its connection to martingales). In fact, many of the results in this section hold for that space; as such, extending our notation to include $\mathscr{S} = \mathcal{Q}_d$ even though it does not form a basis, we provide here a sample of the known sharp results.

Klemes' theorem was extended to the higher-dimensional dyadic case by Nikolidakis, who shows, for $\Omega = Q$, that $c_\circ(1,\mathcal{Q}_d) \le 2^n$:

Theorem 2.5.23 ([71])**.** *If $f \in \mathrm{BMO}_{\mathcal{Q}_d}(Q)$, then $f^\circ \in \mathrm{BMO}(I_Q)$ and*

$$\|f^\circ\|_{\mathrm{BMO}} \le 2^n\|f\|_{\mathrm{BMO}_{\mathcal{Q}_d}}.$$

As a corollary of Proposition 2.5.22 and Theorem 2.5.23 we have the following, which shows that $c_*(1,\mathcal{Q}_d) \le 2^{n+1}$, an improvement on Theorem 2.5.13.

Corollary 2.5.24. *If $f \in \mathrm{BMO}_{\mathcal{Q}_d}(Q)$, then $f^* \in \mathrm{BMO}(I_Q)$ and*

$$\|f^*\|_{\mathrm{BMO}} \le 2^{n+1}\|f\|_{\mathrm{BMO}_{\mathcal{Q}_d}}.$$

The previous discussion emphasized the situation when $p = 1$. For $p = 2$, even more powerful tools are available: using probabilistic methods, Stolyarov, Vasyunin and Zatitskiy prove the following sharp result.

Theorem 2.5.25 ([84]). $c_*(2, \mathcal{Q}_d) = \frac{1+2^n}{2^{1+n/2}}$.

2.6 Truncations

An immediate consequence of the bounds for the absolute value is the following result demonstrating that BMO is a lattice.

Proposition 2.6.1. *For any basis of shapes, if f_1, f_2 are real-valued functions in $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$, then $\max(f_1, f_2)$ and $\min(f_1, f_2)$ are in $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$, with*

$$\|\max(f_1, f_2)\|_{\mathrm{BMO}^p_{\mathscr{S}}} \leq \frac{1 + c_{|\cdot|}(p, \mathscr{S})}{2} \left(\|f_1\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \|f_2\|_{\mathrm{BMO}^p_{\mathscr{S}}}\right).$$

and

$$\|\min(f_1, f_2)\|_{\mathrm{BMO}^p_{\mathscr{S}}} \leq \frac{1 + c_{|\cdot|}(p, \mathscr{S})}{2} \left(\|f_1\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \|f_2\|_{\mathrm{BMO}^p_{\mathscr{S}}}\right).$$

Proof. This follows from writing

$$\max(f_1, f_2) = \frac{(f_1 + f_2) + |f_1 - f_2|}{2} \quad \text{and} \quad \min(f_1, f_2) = \frac{(f_1 + f_2) - |f_1 - f_2|}{2}$$

and using the estimate for the absolute value:

$$\||f_1 - f_2|\|_{\mathrm{BMO}^p_{\mathscr{S}}} \leq c_{|\cdot|}(p, \mathscr{S})(\|f_1\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \|f_2\|_{\mathrm{BMO}^p_{\mathscr{S}}}).$$

□

In particular, applying Theorems 2.5.18 and 2.5.20, this yields the sharp constant 1 for $p = 1$ when $\mathscr{S} = \mathcal{I}$ or $\mathcal{R}$.

We can also obtain the sharp constant 1 via a sharp shapewise inequality for the cases $p = 1$ and $p = 2$, regardless of the basis. The proof of the case $p = 2$ in the following result is given by Reimann and Rychener [73] for the basis $\mathscr{S} = \mathcal{Q}$.

Proposition 2.6.2. *Let $p = 1$ or $p = 2$. Let $\mathscr{S}$ be any basis of shapes and fix a shape $S \in \mathscr{S}$. If $f_1, f_2 \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ and $f = \max(f_1, f_2)$ or $f = \min(f_1, f_2)$, we have*

$$\fint_S |f - f_S|^p \leq \fint_S |f_1 - (f_1)_S|^p + \fint_S |f_2 - (f_2)_S|^p. \tag{2.11}$$

Consequently

$$\|\max(f_1, f_2)\|_{\mathrm{BMO}^p_{\mathscr{S}}} \leq \|f_1\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \|f_2\|_{\mathrm{BMO}^p_{\mathscr{S}}}$$

and

$$\|\min(f_1, f_2)\|_{\mathrm{BMO}^p_{\mathscr{S}}} \leq \|f_1\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \|f_2\|_{\mathrm{BMO}^p_{\mathscr{S}}}.$$

Proof. First, for $p = 1$, fix a shape $S \in \mathscr{S}$ and let $f = \min(f_1, f_2)$. Note that $f_S \leq (f_i)_S$ as $f \leq f_i$ on S for $i = 1, 2$. Let

$$E_1 = \{x \in S : f_1(x) \leq f_2(x)\} = \{x \in S : f(x) = f_1(x)\}, \quad E_2 = S \setminus E_1.$$

By Lemma 2.5.15,

$$\begin{aligned}
\fint_S |f(x) - f_S|\,dx &= \frac{2}{|S|} \int_{\{x \in S: f(x) < f_S\}} (f_S - f(x))\,dx \\
&= \frac{2}{|S|} \sum_{i=1}^{2} \int_{\{x \in E_i: f_i(x) < f_S\}} (f_S - f_i(x))\,dx \\
&\leq \frac{2}{|S|} \sum_{i=1}^{2} \int_{\{x \in S: f_i(x) < (f_i)_S\}} ((f_i)_S - f_i(x))\,dx \\
&= \fint_S |f_1 - (f_1)_S| + \fint_S |f_2 - (f_2)_S|.
\end{aligned}$$

For $f = \max(f_1, f_2)$, the previous arguments follow in a similar way, except that we apply Lemma 2.5.15 to write the mean oscillation in terms of an integral over the set $\{x \in S : f(x) > f_S\}$.

For $p = 2$, we include, for the benefit of the reader, the proof from [73], with cubes replaced by shapes. Let $f = \max(f_1, f_2)$. We may assume without loss of generality that $(f_1)_S \geq (f_2)_S$. Consider the sets

$$S_1 = \{x \in S : f_2(x) < f_1(x)\}, \quad S_2 = \{x \in S : f_2(x) \geq f_1(x) \text{ and } f_2(x) \geq (f_1)_S\}$$

and

$$S_3 = S \setminus (S_1 \cup S_2) = \{x \in S : f_1(x) \leq f_2(x) < (f_1)_S\}.$$

Then

$$\begin{aligned}\int_S |f-(f_1)_S|^2 &= \int_{S_1} |f_1(x)-(f_1)_S|^2\,dx + \int_{S_2} |f_2(x)-(f_1)_S|^2\,dx + \int_{S_3} |f_2(x)-(f_1)_S|^2\,dx \\ &\leq \int_{S_1} |f_1(x)-(f_1)_S|^2\,dx + \int_{S_2} |f_2(x)-(f_2)_S|^2\,dx + \int_{S_3} |f_1(x)-(f_1)_S|^2\,dx \\ &\leq \int_S |f_1(x)-(f_1)_S|^2\,dx + \int_S |f_2(x)-(f_2)_S|^2\,dx\end{aligned}$$

Using Proposition 2.4.9 and dividing by $|S|$ gives (2.11). Similarly, the result can be shown for $\min(f_1, f_2)$. □

For a real-valued measurable function f on Ω, define its truncation from above at level $k \in \mathbb{R}$ by

$$f^k = \min(f, k)$$

and its truncation from below at level $j \in \mathbb{R}$ as

$$f_j = \max(f, j).$$

We use the preceding propositions to prove boundedness of the upper and lower truncations on $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$.

Proposition 2.6.3. *Let $\mathscr{S}$ be any basis of shapes and fix a shape $S \in \mathscr{S}$. If $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ then for all $k, j \in \mathbb{R}$,*

$$\max\left(\fint_S |f^k(x)-(f^k)_S|^p\,dx, \fint_S |f_j(x)-(f_j)_S|^p\,dx\right) \leq c^p \fint_S |f(x)-f_S|^p\,dx \tag{2.12}$$

where

$$c = \begin{cases} 1, & p = 1 \text{ or } p = 2 \\ \min\left(2, \frac{1+c_{|\cdot|}(p,\mathscr{S})}{2}\right), & \text{otherwise.} \end{cases}$$

Consequently

$$\|f^k\|_{\mathrm{BMO}^p_{\mathscr{S}}} \leq c\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} \text{ and } \|f_j\|_{\mathrm{BMO}^p_{\mathscr{S}}} \leq c\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}.$$

Proof. For the truncation from above, observing that $|f^k(x) - f^k(y)| \leq |f(x) - f(y)|$ almost everywhere and for any k, Proposition 2.4.1 gives (2.12) with $c = 2$. On the other hand, applying Proposition 2.6.1 to $f^k = \min(f, k)$ and using the fact that constant functions have zero mean oscillation, we get the constant $c = \frac{1+c_{|\cdot|}(p,\mathscr{S})}{2}$. These calculations hold for any $p \geq 1$.

For $p = 1$ and $p = 2$, we are able to strengthen this by deriving the sharp shapewise inequality, namely (2.12) with $c = 1$, from (2.11) in Proposition 2.6.2 and the fact that constants have zero mean oscillation. Alternatively, for $p = 2$, fix a shape $S \in \mathscr{S}$ and assume, without loss of generality, that $f_S = 0$. Then, by Proposition 2.4.9,

$$\fint_S |f^{(k)} - (f^{(k)})_S|^2 \leq \fint_S |f^{(k)} - f_S|^2 = \fint_S |f^{(k)}|^2 \leq \fint_S |f|^2 = \fint_S |f - f_S|^2.$$

The calculations for the truncation from below are analogous or can be derived by writing $f_j = -\min(-f, -j)$. □

For $p = 1$ and $p = 2$, this is clearly a sharp result: take any bounded function and either $k > \sup f$ or $j < \inf f$.

Also note that for $p = 1$, the sharp shapewise inequalities (2.11) and (2.12) give the sharp shapewise inequality (2.7) (with constant 2) for the absolute value by writing

$$|f| = f_+ + f_- = \max(f_+, f_-), \quad f_+ = f_0,\ f_- = -(f^0).$$

For a measurable function f on Ω, define its (full) truncation at level k as

$$\mathrm{Tr}(f,k)(x) = \begin{cases} k, & f(x) > k \\ f(x), & -k \leq f(x) \leq k \\ -k, & f(x) < -k. \end{cases}$$

Note that $\mathrm{Tr}(f,k) = (f^k)_{-k} = (f_{-k})^k$ and that $\mathrm{Tr}(f,k) \in L^\infty(\Omega)$ for each k. Moreover, $\mathrm{Tr}(f,k) \to f$ pointwise and in $L^1_{\mathrm{loc}}(\Omega)$ as $k \to \infty$ ([56]).

For a general function $f \in \mathrm{BMO}(\Omega)$, it is not true that $\mathrm{Tr}(f,k) \to f$ in $\mathrm{BMO}(\Omega)$, unless Ω is bounded and $f \in \mathrm{VMO}(\Omega)$, the space of functions of vanishing mean oscillation ([9]). Nonetheless, as shown in Corollary 2.6.5 below, $\|\mathrm{Tr}(f,k)\|_{\mathrm{BMO}} \to \|f\|_{\mathrm{BMO}}$ (as mentioned without proof in [79, 80]), and in fact for any basis $\mathscr{S}$.

Problem 2.6.4. Does there exist a constant c such that, for all k and for all $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$, $\|\mathrm{Tr}(f,k)\|_{\mathrm{BMO}^p_{\mathscr{S}}} \leq c\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}$, and if so, what is the smallest constant $c_T(p,\mathscr{S})$ for which this holds?

If $c_T(p, \mathscr{S})$ exists, then clearly $c_T(p, \mathscr{S}) \geq 1$. The results above provide a positive answer to the first question and some information about $c_T(p, \mathscr{S})$.

Corollary 2.6.5. *For any choice of shapes, if* $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ *then* $\mathrm{Tr}(f,k) \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ *for all* k *and* $\|\mathrm{Tr}(f,k)\|_{\mathrm{BMO}^p_{\mathscr{S}}} \leq c\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}$*, where*

$$c = \begin{cases} 1, & p = 1 \text{ or } p = 2 \\ \min\left(2, \left(\frac{1+c_{|\cdot|}(p,\mathscr{S})}{2}\right)^2\right), & \text{otherwise} \end{cases}.$$

Moreover, for $p = 1$ *or* $p = 2$*,* $\|\mathrm{Tr}(f,k)\|_{\mathrm{BMO}^p_{\mathscr{S}}} \to \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}$ *as* $k \to \infty$*.*

Proof. Writing $\mathrm{Tr}(f,k) = (f^k)_{-k}$ and applying inequality (2.12) in Proposition 2.6.3 gives us the shapewise inequality

$$\fint_S |\mathrm{Tr}(f,k) - (\mathrm{Tr}(f,k))_S|^p \leq c \fint_S |f - f_S|^p \tag{2.13}$$

with constant $c = 1$ for $p = 1$ and $p = 2$, and $c \leq \left(\min\left(2, \frac{1+c_{|\cdot|}(p,\mathscr{S})}{2}\right)\right)^2$. We get $c \leq 2$ (as opposed to 2^2) by applying Proposition 2.4.1 directly with the estimate $|\mathrm{Tr}(f,k)(x) - \mathrm{Tr}(f,k)(y)| \leq |f(x) - f(y)|$.

For the convergence of the norms in the case $p = 1$ and $p = 2$, we have that $|\mathrm{Tr}(f,k)| \leq |f| \in L^1(S)$ implies $(\mathrm{Tr}(f,k))_S \to f_S$, and since $\mathrm{Tr}(f,k) \to f$ pointwise a.e. on S, we can apply Fatou's lemma and (2.13) with $c = 1$ to get

$$\begin{aligned} \fint_S |f - f_S|^p &\leq \liminf_{k\to\infty} \fint_S |\mathrm{Tr}(f,k) - (\mathrm{Tr}(f,k))_S|^p \\ &\leq \limsup_{k\to\infty} \fint_S |\mathrm{Tr}(f,k) - (\mathrm{Tr}(f,k))_S|^p \leq \fint_S |f - f_S|^p. \end{aligned}$$

$\square$

This result gives the sharp constant for $p = 1, 2$ and an upper bound for $c_T(p, \mathscr{S})$. Of course, the known upper bound $c_{|\cdot|}(p, \mathscr{S}) \leq 2$ implies

$$\left(\frac{1 + c_{|\cdot|}(p,\mathscr{S})}{2}\right)^2 \leq \frac{9}{4}$$

(which appears, for example, in Exercise 3.1.4 in [42]), but this is worse than the truncation bound $c_T(p, \mathscr{S}) \leq 2$. On the other hand, if $c_{|\cdot|}(p, \mathscr{S}) \leq 2\sqrt{2} - 1$, then the bound depends on $c_{|\cdot|}(p, \mathscr{S})$. In particular, a result of $c_{|\cdot|}(p, \mathscr{S}) = 1$ would imply that $c_T(p, \mathscr{S}) = 1$.

2.7 The John-Nirenberg inequality

We now come to the most important inequality in the theory of BMO, originating in the paper of John and Nirenberg [54].

Definition 2.7.1. Let $X \subset \mathbb{R}^n$ be a set of finite Lebesgue measure. We say that $f \in L^1(X)$ satisfies the John-Nirenberg inequality on X if there exist constants $c_1, c_2 > 0$ such that

$$|\{x \in X : |f(x) - f_X| > \alpha\}| \le c_1 |X| e^{-c_2 \alpha}, \qquad \alpha > 0. \tag{2.14}$$

The following is sometimes referred to as the John-Nirenberg lemma.

Theorem 2.7.2 ([54])**.** *If $X = Q$, a cube in $\mathbb{R}^n$, then there exist constants c and C such that for all $f \in \mathrm{BMO}(Q)$,* (2.14) *holds with $c_1 = c$, $c_2 = C/\|f\|_{\mathrm{BMO}}$.*

More generally, given a basis of shapes $\mathscr{S}$ on a domain $\Omega \subset \mathbb{R}^n$, $|\Omega| < \infty$, one can pose the following problem.

Problem 2.7.3. Does $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ imply that f satisfies the John-Nirenberg inequality on Ω? That is, do there exist constants $c, C > 0$ such that

$$|\{x \in \Omega : |f(x) - f_\Omega| > \alpha\}| \le c|\Omega| \exp\left(-\frac{C}{\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}}\alpha\right), \qquad \alpha > 0$$

holds for all $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$? If so, what is the smallest constant $c_{\Omega,\mathrm{JN}}(p, \mathscr{S})$ and the largest constant $C_{\Omega,\mathrm{JN}}(p, \mathscr{S})$ for which this inequality holds?

When $n = 1$, $\Omega = I$, a finite interval, and $\mathscr{S} = \mathcal{I}$, the positive answer is a special case of Theorem 2.7.2. Sharp constants are known for the cases $p = 1$ and $p = 2$. For $p = 1$, it is shown in [65] that $c_{I,\mathrm{JN}}(1, \mathcal{I}) = \frac{1}{2}e^{4/e}$ and in [58] that $C_{I,\mathrm{JN}}(1, \mathcal{I}) = 2/e$. For $p = 2$, Bellman function techniques are used in [86] to give $c_{I,\mathrm{JN}}(2, \mathcal{I}) = 4/e^2$ and $C_{I,\mathrm{JN}}(2, \mathcal{I}) = 1$.

When $n \ge 2$, $\Omega = R$, a rectangle, and $\mathscr{S} = \mathcal{R}$, a positive answer is provided by a less well-known result due to Korenovskii in [59], where he also shows the sharp constant $C_{R,\mathrm{JN}}(1, \mathcal{R}) = 2/e$.

Dimension-free bounds on these constants are also of interest. In [25], Cwikel, Sagher, and Shvartsman conjecture a geometric condition on cubes and prove dimension-free bounds for $c_{\Omega,\mathrm{JN}}(1, \mathcal{Q})$ and $C_{\Omega,\mathrm{JN}}(1, \mathcal{Q})$ conditional on this hypothesis being true.

Rather than just looking at Ω, we can also consider whether the John-Nirenberg inequality holds for all shapes S.

Definition 2.7.4. We say that a function $f \in L^1_{\text{loc}}(\Omega)$ has the John-Nirenberg property with respect to a basis $\mathscr{S}$ of shapes on Ω if there exist constants $c_1, c_2 > 0$ such that for all $S \in \mathscr{S}$, (2.14) holds for $X = S$.

We can now formulate a modified problem.

Problem 2.7.5. For which bases $\mathscr{S}$ and $p \in [1, \infty)$ does $f \in \text{BMO}^p_{\mathscr{S}}(\Omega)$ imply that f has the John-Nirenberg property with respect to $\mathscr{S}$? If this is the case, what is the smallest constant $c = c_{\text{JN}}(p, \mathscr{S})$ and the largest constant $C = C_{\text{JN}}(p, \mathscr{S})$ for which (2.14) holds for all $f \in \text{BMO}_{\mathscr{S}}(\Omega)$ and $S \in \mathscr{S}$ with $c_1 = c$, $c_2 = C/\|f\|_{\text{BMO}^p_{\mathscr{S}}}$?

Since Theorem 2.7.2 holds for any cube Q in $\mathbb{R}^n$ with constants independent of Q, it follows that for a domain $\Omega \subset \mathbb{R}^n$, any $f \in \text{BMO}(\Omega)$ has the John-Nirenberg property with respect to $\mathcal{Q}$, and equivalently $\mathcal{B}$. Similarly, every $f \in \text{BMO}_{\mathcal{R}}(\Omega)$ has the John-Nirenberg property with respect to $\mathcal{R}$.

In the negative direction, $f \in \text{BMO}^p_{\mathcal{C}}(\mathbb{R}^n)$ does not necessarily have the John-Nirenberg property with respect to $\mathcal{C}$ ([62, 67]). This space, known in the literature as CMO for central mean oscillation or CBMO for central bounded mean oscillation, was originally defined with the additional constraint that the balls have radius at least 1 ([15, 36]).

We now state the converse to Theorem 2.7.2, namely that the John-Nirenberg property is sufficient for BMO, in more generality.

Theorem 2.7.6. *If $f \in L^1_{loc}(\Omega)$ and f has the John-Nirenberg property with respect to $\mathscr{S}$, then $f \in \text{BMO}^p_{\mathscr{S}}(\Omega)$ for all $p \in [1, \infty)$, with*

$$\|f\|_{\text{BMO}^p_{\mathscr{S}}} \leq \frac{(c_1 p \Gamma(p))^{1/p}}{c_2}.$$

Proof. Take $S \in \mathscr{S}$. By Cavalieri's principle and (2.14),

$$\begin{aligned} \fint_S |f - f_S|^p &= \frac{p}{|S|} \int_0^\infty \alpha^{p-1} |\{x \in S : |f(x) - f_S| > \alpha\}| \, d\alpha \\ &\leq p c_1 \int_0^\infty \alpha^{p-1} \exp\left(-c_2 \alpha\right) \, d\alpha \\ &= \frac{c_1 p \Gamma(p)}{c_2{}^p}, \end{aligned}$$

from where the result follows. $\square$

By Lemma 2.3.3, this theorem shows that every f with the John-Nirenberg property is in $L^p_{\mathrm{loc}}(\Omega)$.

A consequence of the John-Nirenberg lemma, Theorem 2.7.2, is that

$$\mathrm{BMO}^{p_1}(\mathbb{R}^n) \cong \mathrm{BMO}^{p_2}(\mathbb{R}^n)$$

for all $1 \le p_1, p_2 < \infty$. This can be stated in more generality as a corollary of the preceding theorem.

Corollary 2.7.7. *If there exists $p_0 \in [1, \infty)$ such that every $f \in \mathrm{BMO}^{p_0}_{\mathscr{S}}(\Omega)$ has the John-Nirenberg property with respect to $\mathscr{S}$, then*

$$\mathrm{BMO}^{p_1}_{\mathscr{S}}(\Omega) \cong \mathrm{BMO}^{p_2}_{\mathscr{S}}(\Omega), \quad p_0 \le p_1, p_2 < \infty.$$

Proof. The hypothesis means that there are constants $c_{\mathrm{JN}}(p_0, \mathscr{S})$, $C_{\mathrm{JN}}(p_0, \mathscr{S})$ such that if $f \in \mathrm{BMO}^{p_0}_{\mathscr{S}}(\Omega)$ then f satisfies (2.14) for all $S \in \mathscr{S}$, with $c_1 = c_{\mathrm{JN}}(p_0, \mathscr{S})$, $c_2 = C_{\mathrm{JN}}(p_0, \mathscr{S})/\|f\|_{\mathrm{BMO}^{p_0}_{\mathscr{S}}}$.

From the preceding theorem, this implies that $\mathrm{BMO}^{p_0}_{\mathscr{S}}(\Omega) \subset \mathrm{BMO}^{p}_{\mathscr{S}}(\Omega)$ for all $p \in [1, \infty)$, with

$$\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} \le \frac{(c_{\mathrm{JN}}(p_0, \mathscr{S}) p\Gamma(p))^{1/p}}{C_{\mathrm{JN}}(p_0, \mathscr{S})} \|f\|_{\mathrm{BMO}^{p_0}_{\mathscr{S}}}.$$

Conversely, Proposition 2.3.2 gives us that $\mathrm{BMO}^{p}_{\mathscr{S}}(\Omega) \subset \mathrm{BMO}^{p_0}_{\mathscr{S}}(\Omega)$ whenever $p_0 \le p < \infty$, with $\|f\|_{\mathrm{BMO}^{p_0}_{\mathscr{S}}} \le \|f\|_{\mathrm{BMO}^{p}_{\mathscr{S}}}$.

Thus all the spaces $\mathrm{BMO}^{p}_{\mathscr{S}}(\Omega)$, $p_0 \le p < \infty$, are congruent to $\mathrm{BMO}^{p_0}_{\mathscr{S}}(\Omega)$. $\square$

The John-Nirenberg lemma gives the hypothesis of Corollary 2.7.7 for the bases $\mathcal{Q}$ and $\mathcal{B}$ on $\mathbb{R}^n$ with $p_0 = 1$. As pointed out, by results of [59] this also applies to the basis $\mathcal{R}$, showing that $\mathrm{BMO}^{p_1}_{\mathcal{R}}(\Omega) \cong \mathrm{BMO}^{p_2}_{\mathcal{R}}(\Omega)$ for all $1 \le p_1, p_2 < \infty$.

Problem 2.7.8. If the hypothesis of Corollary 2.7.7 is satisfied with $p_0 = 1$, what is the smallest constant $c(p, \mathscr{S})$ such that $\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} \le c(p, \mathscr{S})\|f\|_{\mathrm{BMO}_{\mathscr{S}}}$ holds for all $f \in \mathrm{BMO}^{p}_{\mathscr{S}}(\Omega)$?

The proof of Corollary 2.7.7 gives a well-known upper bound on $c(p, \mathscr{S})$:

$$c(p, \mathscr{S}) \le \frac{\left(c_{\mathrm{JN}}(1, \mathscr{S}) p\Gamma(p)\right)^{1/p}}{C_{\mathrm{JN}}(1, \mathscr{S})}.$$

2.8 Product decomposition

In this section, we assume that the domain Ω can be decomposed as

$$\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_k \tag{2.15}$$

for $2 \le k \le n$, where each Ω_i is a domain in $\mathbb{R}^{m_i}$ for $1 \le m_i \le n-1$, having its own basis of shapes $\mathscr{S}_i$.

We will require some compatibility between the basis $\mathscr{S}$ on all of Ω and these individual bases.

Definition 2.8.1. We say that $\mathscr{S}$ satisfies the weak decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$ if for every $S \in \mathscr{S}$ there exist $S_i \in \mathscr{S}_i$ for each $i = 1, \ldots, k$ such that $S = S_1 \times S_2 \times \ldots \times S_k$. If, in addition, for every $\{S_i\}_{i=1}^k$, $S_i \in \mathscr{S}_i$, the set $S_1 \times S_2 \times \ldots \times S_k \in \mathscr{S}$, then we say that the basis $\mathscr{S}$ satisfies the strong decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$.

Using $\mathcal{R}_i$ to denote the basis of rectangles in Ω_i (interpreted as $\mathcal{I}_i$ if $m_i = 1$), note that the basis $\mathcal{R}$ on Ω satisfies the strong decomposition property with respect to $\{\mathcal{R}_i\}_{i=1}^k$ and for any k. Meanwhile, the basis $\mathcal{Q}$ on Ω satisfies the weak decomposition property with respect to $\{\mathcal{R}_i\}_{i=1}^k$ for any k, but not the strong decomposition property.

We now turn to the study of the spaces $\mathrm{BMO}^p_{\mathscr{S}_i}(\Omega)$, first defined using different notation in the context of the bidisc $\mathbb{T} \times \mathbb{T}$ in [22]. For simplicity, we only define $\mathrm{BMO}^p_{\mathscr{S}_1}(\Omega)$, as the other $\mathrm{BMO}^p_{\mathscr{S}_i}(\Omega)$ for $i = 2, \ldots, k$ are defined analogously. We write a point in Ω as (x, y), where $x \in \Omega_1$ and $y \in \widetilde{\Omega} = \Omega_2 \times \cdots \times \Omega_k$. Writing $f_y(x) = f(x, y)$, functions in $\mathrm{BMO}^p_{\mathscr{S}_1}(\Omega)$ are those for which f_y is in $\mathrm{BMO}^p_{\mathscr{S}_1}(\Omega_1)$, uniformly in y:

Definition 2.8.2. A function $f \in L^1_{\mathrm{loc}}(\Omega)$ is said to be in $\mathrm{BMO}^p_{\mathscr{S}_1}(\Omega)$ if

$$\|f\|_{\mathrm{BMO}^p_{\mathscr{S}_1}(\Omega)} := \sup_{y \in \widetilde{\Omega}} \|f_y\|_{\mathrm{BMO}^p_{\mathscr{S}_1}(\Omega_1)} < \infty,$$

where $f_y(x) = f(x, y)$.

Although this norm combines a supremum with a BMO norm, we are justified in calling $\mathrm{BMO}^p_{\mathscr{S}_1}(\Omega)$ a BMO space as it inherits many properties from $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$. In particular, for each $i = 1, 2, \dots, k$, $L^\infty(\Omega) \subset \mathrm{BMO}^p_{\mathscr{S}_i}(\Omega)$ and $\mathrm{BMO}^{p_1}_{\mathscr{S}_i}(\Omega) \cong \mathrm{BMO}^{p_2}_{\mathscr{S}_i}(\Omega)$ for all $1 \le p_1, p_2 < \infty$ if the hypothesis of Corollary 2.7.7 is satisfied with $p_0 = 1$ for $\mathscr{S}_i$ on Ω_i. Moreover, under certain conditions, the spaces $\mathrm{BMO}^p_{\mathscr{S}_i}(\Omega)$ can be quite directly related to $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$, revealing the product nature of $\mathrm{BMO}^p_{\mathscr{S}}(\Omega)$. This depends on the decomposition property of the basis $\mathscr{S}$, as well as some differentiation properties of the $\mathscr{S}_i$.

Before stating the theorem, we briefly recall the main definitions related to the theory of differentiation of the integral; see the survey [44] for a standard reference. For a basis of shapes $\mathscr{S}$, denote by $\mathscr{S}(x)$ the subcollection of shapes that contain $x \in \Omega$. We say that $\mathscr{S}$ is a differentiation basis if for each $x \in \Omega$ there exists a sequence of shapes $\{S_k\} \subset \mathscr{S}(x)$ such that $\delta(S_k) \to 0$ as $k \to \infty$. Here, $\delta(\cdot)$ is the Euclidean diameter. For $f \in L^1_{\mathrm{loc}}(\Omega)$, we define the upper derivative of $\int f$ with respect to $\mathscr{S}$ at $x \in \Omega$ by

$$\overline{D}\Big(\int f, x\Big) = \sup\left\{\limsup_{k\to\infty} \fint_{S_k} f : \{S_k\} \subset \mathscr{S}(x),\ \delta(S_k) \to 0 \text{ as } k \to \infty\right\}$$

and the lower derivative of $\int f$ with respect to $\mathscr{S}$ at $x \in \Omega$ by

$$\underline{D}\Big(\int f, x\Big) = \inf\left\{\liminf_{k\to\infty} \fint_{S_k} f : \{S_k\} \subset \mathscr{S}(x),\ \delta(S_k) \to 0 \text{ as } k \to \infty\right\}.$$

We say, then, that a differentiation basis $\mathscr{S}$ differentiates $L^1_{\mathrm{loc}}(\Omega)$ if for every $f \in L^1_{\mathrm{loc}}(\Omega)$ and for almost every $x \in \Omega$, $\overline{D}(\int f, x) = \underline{D}(\int f, x) = f(x)$. The classical Lebesgue differentiation theorem is a statement that the basis $\mathcal{B}$ (equivalently, $\mathcal{Q}$) differentiates $L^1_{\mathrm{loc}}(\Omega)$. It is known, however, that the basis $\mathcal{R}$ does not differentiate $L^1_{\mathrm{loc}}(\Omega)$, but does differentiate the Orlicz space $L(\log L)^{n-1}(\Omega)$ ([52]).

Theorem 2.8.3. *Let Ω be a domain satisfying* (2.15)*, $\mathscr{S}$ be a basis of shapes for Ω and $\mathscr{S}_i$ be a basis of shapes for Ω_i, $1 \le i \le k$.*

a) Let $f \in \bigcap_{i=1}^k \mathrm{BMO}^p_{\mathscr{S}_i}(\Omega)$. If $\mathscr{S}$ satisfies the weak decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$, then $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ with

$$\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)} \le \sum_{i=1}^k \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_i}(\Omega)}.$$

b) Let $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$. If $\mathscr{S}$ satisfies the strong decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$ and each $\mathscr{S}_i$ contains a differentiation basis that differentiates $L^1_{loc}(\Omega_i)$, then

$f \in \bigcap_{i=1}^{k} \mathrm{BMO}^p_{\mathscr{S}_i}(\Omega)$ *with*

$$\max_{i=1,\ldots,k}\{\|f\|_{\mathrm{BMO}^p_{\mathscr{S}_i}(\Omega)}\} \leq 2^{k-1}\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)}.$$

When $p = 2$, the constant 2^{k-1} can be replaced by 1.

This theorem was first pointed out in [22] in the case of $\mathcal{R}$ in $\mathbb{T}\times\mathbb{T}$. Here, we prove it in the setting of more general shapes and domains, clarifying the role played by the theory of differentiation and keeping track of constants.

Proof. We first present the proof in the case of $k = 2$. We write $\Omega = X \times Y$, denoting by (x, y) an element in Ω with $x \in X$ and $y \in Y$. The notations $\mathscr{S}_X$ and $\mathscr{S}_Y$ will be used for the basis in X and Y, respectively. Similarly, the measures dx and dy will be used for Lebesgue measure on X and Y, respectively, while dA will be used for the Lebesgue measure on Ω.

To prove (a), assume that $\mathscr{S}$ satisfies the weak decomposition property with respect to $\{\mathscr{S}_X, \mathscr{S}_Y\}$ and let $f \in \mathrm{BMO}^p_{\mathscr{S}_X}(\Omega) \cap \mathrm{BMO}^p_{\mathscr{S}_Y}(\Omega)$. Fixing a shape $R \in \mathscr{S}$, write $R = S \times T$ for $S \in \mathscr{S}_X$ and $T \in \mathscr{S}_Y$. Then, by Minkowski's inequality,

$$\left(\fint_R |f(x,y) - f_R|^p dA\right)^{\frac{1}{p}} \leq \left(\fint_R |f(x,y) - (f_y)_S|^p dA\right)^{\frac{1}{p}} + \left(\fint_T |(f_y)_S - f_R|^p dy\right)^{\frac{1}{p}}.$$

For the first integral, we estimate

$$\fint_R |f(x,y) - (f_y)_S|^p \, dA \leq \fint_T \|f_y\|^p_{\mathrm{BMO}^p_{\mathscr{S}_X}(X)} \, dy \leq \|f\|^p_{\mathrm{BMO}^p_{\mathscr{S}_X}(\Omega)}. \tag{2.16}$$

For the second integral, Jensen's inequality gives

$$\begin{aligned}
\left(\fint_T |(f_y)_S - f_R|^p \, dy\right)^{1/p} &= \left(\fint_T \left|\fint_S f_y(x)\, dx - \fint_S \fint_T f(x,y)\, dy\, dx\right|^p dy\right)^{1/p} \\
&= \left(\fint_T \left|\fint_S \big(f(x,y) - (f_x)_T\big)\, dx\right|^p dy\right)^{1/p} \\
&\leq \left(\fint_R |f(x,y) - (f_x)_T|^p \, dA\right)^{1/p} \\
&\leq \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_Y}(\Omega)},
\end{aligned}$$

where the last inequality follows in the same way as (2.16). Therefore, we may conclude that $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ with $\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)} \leq \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_X}(\Omega)} + \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_Y}(\Omega)}$.

We now come to the proof of (b). To simplify the notation, we use $\mathcal{O}_p(f,S)$ for the p-mean oscillation of the function f on the shape S, i.e.

$$\mathcal{O}_p(f,S) := \fint_S |f - f_S|^p.$$

Assume that $\mathscr{S}$ satisfies the strong decomposition property with respect to $\{\mathscr{S}_X, \mathscr{S}_Y\}$, and that $\mathscr{S}_X$ and $\mathscr{S}_Y$ each contain a differentiation basis that differentiates $L^1_{\text{loc}}(X)$ and $L^1_{\text{loc}}(Y)$, respectively.

Let $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$. Fix a shape $S_0 \in \mathscr{S}_X$ and consider the p-mean oscillation of f_y on S_0, $\mathcal{O}_p(f_y, S_0)$, as a function of y. For any $T \in \mathscr{S}_Y$, writing $R = S_0 \times T \in \mathscr{S}$, we have that $R \in \mathscr{S}$ by the strong decomposition property of $\mathscr{S}$, so $f \in \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ implies $f \in L^1(R)$ and therefore

$$\int_T \mathcal{O}_p(f_y, S_0)\, dy = \frac{1}{|S_0|}\int_T\int_{S_0} |f_y(x) - (f_y)_{S_0}|^p \, dx\, dy \le \frac{2^p}{|S_0|}\int_R |f(x,y)|^p \, dx\, dy < \infty.$$

By Lemma 2.3.3, this is enough to guarantee that $\mathcal{O}_p(f_y, S_0) \in L^1_{\text{loc}}(Y)$.

Let $\varepsilon > 0$. Since $\mathscr{S}_Y$ contains a differentiation basis, for almost every $y_0 \in Y$ there exists a shape $T_0 \in \mathscr{S}_Y$ containing y_0 such that

$$\left| \fint_{T_0} \mathcal{O}_p(f_y, S_0)\, dy - \mathcal{O}_p(f_{y_0}, S_0) \right| < \varepsilon. \tag{2.17}$$

Fix such an x_0 and a T_0 and let $R_0 = S_0 \times T_0$. Then the strong decomposition property implies that $R_0 \in \mathscr{S}$, and by Proposition 2.4.5 applied to the mean oscillation of f_y on S_0, we have

$$\begin{aligned}
\fint_{T_0} \mathcal{O}_p(f_y, S_0)\, dy &= \fint_{T_0}\fint_{S_0} |f_y(x) - (f_y)_{S_0}|^p \, dx\, dy \\
&\le 2^p \fint_{T_0}\fint_{S_0} |f(x,y) - f_{R_0}|^p \, dx\, dy \\
&= 2^p \fint_{R_0} |f(x,y) - f_{R_0}|^p \, dA \\
&\le 2^p \|f\|^p_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)}.
\end{aligned}$$

Note that when $p = 2$, Proposition 2.4.9 implies that the factor of 2^p can be dropped.

Combining this with (2.17), it follows that

$$\mathcal{O}_p(f_{y_0}, S_0) < \varepsilon + 2^p \|f\|^p_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)}.$$

Taking $\varepsilon \to 0^+$, since S_0 is arbitrary, this implies that $f_{y_0} \in \mathrm{BMO}^p_{\mathscr{S}_X}(X)$ with

$$\|f_{y_0}\|_{\mathrm{BMO}^p_{\mathscr{S}_X}(X)} \le 2\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)}. \tag{2.18}$$

The fact that this is true for almost every $y_0 \in Y$ implies that $\|f\|_{\mathrm{BMO}^p_{\mathscr{S}_X}(\Omega)} \le 2\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)}$.

Similarly, one can show that $\|f\|_{\mathrm{BMO}^p_{\mathscr{S}_Y}(\Omega)} \le 2\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)}$. Thus we have shown $f \in \mathrm{BMO}^p_{\mathscr{S}_X}(\Omega) \cap \mathrm{BMO}^p_{\mathscr{S}_Y}(\Omega)$ with

$$\max\{\|f\|_{\mathrm{BMO}^p_{\mathscr{S}_X}(\Omega)}, \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_Y}(\Omega)}\} \le 2\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)}.$$

Again, when $p = 2$ the factor of 2 disappears.

For $k > 2$, let us assume the result holds for $k-1$. Write $X = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_{k-1}$ and $Y = \Omega_k$. Set $\mathscr{S}_Y = \mathscr{S}_k$. By the weak decomposition property of $\mathscr{S}$, we can define the projection of the basis $\mathscr{S}$ onto X, namely

$$\mathscr{S}_X = \{S_1 \times S_2 \times \ldots \times S_{k-1} : S_i \in \mathscr{S}_i, \exists S_k \in \mathscr{S}_k, \prod_{i=1}^{k} S_i \in \mathscr{S}\}, \tag{2.19}$$

and this is a basis of shapes on X which by definition also has the weak decomposition property. Moreover, $\mathscr{S}$ has the weak decomposition property with respect to $\mathscr{S}_X$ and $\mathscr{S}_Y$.

To prove part (a) for k factors, we first apply the result of part (a) proved above for $k = 2$, followed by the definitions and part (a) applied again to X, since we are assuming it is valid with $k-1$ factors. This gives us the inclusion $\bigcap_{i=1}^{k} \mathrm{BMO}^p_{\mathscr{S}_i}(\Omega) \subset \mathrm{BMO}^p_{\mathscr{S}}(\Omega)$ with the following estimates on the norms (we use the notation $\widehat{x_i}$ for the $k-2$ tuple of variables obtained from $(x_1, \ldots, x_{k-1})$ by removing x_i):

$$\begin{aligned}
\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)} &\le \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_X}(\Omega)} + \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_Y}(\Omega)} \\
&= \sup_{y \in Y} \|f_y\|_{\mathrm{BMO}^p_{\mathscr{S}_X}(X)} + \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_Y}(\Omega)} \\
&\le \sup_{y \in Y} \sum_{i=1}^{k-1} \|f_y\|_{\mathrm{BMO}^p_{\mathscr{S}_i}(X)} + \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_Y}(\Omega)} \\
&= \sup_{y \in Y} \sum_{i=1}^{k-1} \sup_{\widehat{x_i}} \|(f_y)_{\widehat{x_i}}\|_{\mathrm{BMO}^p_{\mathscr{S}_i}(\Omega_i)} + \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_Y}(\Omega)} \\
&\le \sum_{i=1}^{k-1} \sup_{(\widehat{x_i}, y)} \|f_{(\widehat{x_i}, y)}\|_{\mathrm{BMO}^p_{\mathscr{S}_i}(\Omega_i)} + \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_Y}(\Omega)}
\end{aligned}$$

$$= \sum_{i=1}^{k-1} \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_i}(\Omega)} + \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_k}(\Omega)}$$
$$= \sum_{i=1}^{k} \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_i}(\Omega)}.$$

To prove part (b) for $k > 2$, we have to be more careful. First note that if $\mathscr{S}$ has the strong decomposition property, then so does $\mathscr{S}_X$ defined by (2.19). We repeat the first part of the proof of (b) for the case $k = 2$ above, with $X = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_{k-1}$ and $Y = \Omega_k$, leading up to the estimate (2.18) for the function f_{y_0} for some $y_0 \in Y$. Note that in this part we only used the differentiation properties of Y, which hold by hypothesis in this case since $Y = \Omega_k$. Now we repeat the process for the function f_{y_0} instead of f, with $X_1 = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_{k-2}$ and $Y_1 = \Omega_{k-1}$. This gives

$$\|(f_{y_0})_{y_1}\|_{\mathrm{BMO}^p_{\mathscr{S}_{X_1}}(X_1)} \leq 2\|f_{y_0}\|_{\mathrm{BMO}^p_{\mathscr{S}_X}(X)} \leq 4\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)} \quad \forall y_1 \in \Omega_{k-1}, y_0 \in \Omega_k.$$

We continue until we get to $X_{k-1} = \Omega_1$, for which $\mathscr{S}_{X_k} = \mathscr{S}_1$, yielding the estimate

$$\|f_{(y_{k-2},\ldots,y_0)}\|_{\mathrm{BMO}^p_{\mathscr{S}_1}(\Omega_1)} \leq \ldots \leq 2^{k-2}\|f_{y_0}\|_{\mathrm{BMO}^p_{\mathscr{S}_X}(X)} \leq 2^{k-1}\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)}$$

for all $k-1$-tuples $y = (y_{k-2}, \ldots, y_0) \in \widetilde{\Omega} = \Omega_2 \times \ldots \times \Omega_k$. Taking the supremum over all such y, we have, by Definition 2.8.2, that $f \in \mathrm{BMO}^p_{\mathscr{S}_1}(\Omega)$ with

$$\|f\|_{\mathrm{BMO}^p_{\mathscr{S}_1}(\Omega)} = \sup_{y \in \widetilde{\Omega}} \|f_y\|_{\mathrm{BMO}^p_{\mathscr{S}_1}(\Omega_1)} \leq 2^{k-1}\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)}.$$

A similar process for $i = 2, \ldots k$ shows that $f \in \mathrm{BMO}^p_{\mathscr{S}_i}(\Omega)$ with

$$\|f\|_{\mathrm{BMO}^p_{\mathscr{S}_i}(\Omega)} \leq 2^{k-1}\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\Omega)}.$$

As the factor of 2 appears in the proof for $k = 2$ only when $p \neq 2$, the same will happen here. $\square$

Since $\mathcal{Q}$ satisfies the weak decomposition property, the claim of part (a) holds for BMO, a fact pointed out in [79] without proof. Also, it is notable that there was no differentiation assumption required for this direction.

In the proof of part (b), differentiation is key and the strong decomposition property of the basis cannot be eliminated as there would be no guarantee that arbitrary S and T

would yield a shape R in Ω. In fact, if the claim were true for bases with merely the weak decomposition property, this would imply that BMO and $\mathrm{BMO}_{\mathcal{R}}$ are congruent, which is not true (see Example 2.4.13).

Chapter 3

Geometric Maximal Operators and BMO on Product Basess

3.1 Introduction

The uncentred Hardy-Littlewood maximal function, Mf, of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined as

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f| = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f|, \tag{3.1}$$

where the supremum is taken over all cubes Q containing the point x and $|Q|$ is the measure of the cube. Note that, unless otherwise stated, cubes in this paper will mean cubes with sides parallel to the axes. The well-known Hardy-Littlewood-Wiener theorem states that the operator M is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$ and from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ (see Stein [79]).

This maximal function is a classical object of study in real analysis due to its connection with differentiation of the integral. When the cubes in (3.1) are replaced by rectangles (the Cartesian product of intervals), we have the strong maximal function, M_s, which is also bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$ but is not bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. Its connection to what is known as strong differentiation of the integral is also quite classical (see Jessen-Marcinkiewicz-Zygmund [52]).

[0]This is a post-peer-review, pre-copyedit version of an article that is to be appear in *The Journal of Geometric Analysis*.

When the cubes in (3.1) are replaced by more general sets taken from a basis $\mathscr{S}$, we obtain a geometric maximal operator, $M_{\mathscr{S}}$ (we follow the nomenclature of [47]). Here the subscript $\mathscr{S}$ emphasizes that the behaviour of this operator depends on the geometry of the sets in $\mathscr{S}$, which we call shapes. Such maximal operators have been extensively studied; see, for instance, the monograph of de Guzmán ([44]). A key theme in this area is the identification of the weakest assumptions needed on $\mathscr{S}$ to guarantee certain properties of $M_{\mathscr{S}}$. For examples of the kind of research currently being done in this area, including its connection to the theory of A_p weights, see [43, 46, 47, 68, 82, 83].

Introduced by John and Nirenberg in [54] for functions supported on a cube, the space of functions of bounded mean oscillation, $\mathrm{BMO}(\mathbb{R}^n)$, is the set of all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\sup_Q \fint_Q |f - f_Q| < \infty, \tag{3.2}$$

where $f_Q = \fint_Q f$ is the mean of f over the cube Q and the supremum is taken over all cubes Q.

An important subset of $\mathrm{BMO}(\mathbb{R}^n)$, introduced by Coifman and Rochberg in [18], is the class of functions of bounded lower oscillation, $\mathrm{BLO}(\mathbb{R}^n)$. The definition of this class is obtained by replacing the mean f_Q in (3.2) by $\operatorname*{ess\,inf}_Q f$, the essential infimum of f on the cube Q.

Just as cubes can be replaced by rectangles in the definition of the maximal function, the same can be done with the definition of $\mathrm{BMO}(\mathbb{R}^n)$ (and, likewise, with $\mathrm{BLO}(\mathbb{R}^n)$). The resulting space, strong BMO, has appeared in the literature under different names (see [22, 30, 60]).

Pushing the analogy with maximal functions even further, one may replace the cubes in (3.2) by more general shapes, coming from a basis $\mathscr{S}$. This space, $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$, was introduced in previous work of two of the authors in [26]. In this work, a product characterisation of $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ was shown when the shapes in $\mathscr{S}$ exhibit some product structure.

In the two-parameter setting of $\mathbb{R}^n \times \mathbb{R}^m$, there is a related space, rectangular BMO, that is larger than strong BMO. The unacquainted reader is invited to see [12, 14, 34, 35] for surveys connecting rectangular BMO to the topic of the product Hardy space and its dual, known as product BMO, which will not be considered in this paper.

Considering shapes in a basis $\mathscr{S}$ that exhibit a product structure like what was investigated in [26] naturally leads to a definition of rectangular BMO with respect to $\mathscr{S}$. As will be shown, this product structure can also be exploited to define a rectangular BLO space, which can easily be defined in even a multiparameter setting. The relationship between rectangular BLO and rectangular BMO will be shown to mirror, in some ways, the relationship between BLO and BMO.

The boundedness of M on $\mathrm{BMO}(\mathbb{R}^n)$ was first considered by Bennett-DeVore-Sharpley ([3]). They showed that if $Mf \not\equiv \infty$, then $Mf \in \mathrm{BMO}(\mathbb{R}^n)$ when $f \in \mathrm{BMO}(\mathbb{R}^n)$. In [2], Bennett refined this result, showing that if $Mf \not\equiv \infty$, then M is bounded from $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{BLO}(\mathbb{R}^n)$. In fact, he showed the stronger result with M defined by averages of f as opposed to $|f|$. Further work in this direction can be found in [16, 23, 41, 64, 72, 75, 88].

As the geometric maximal operator $M_{\mathscr{S}}$ generalises the Hardy-Littlewood maximal operator M and the space $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ generalises $\mathrm{BMO}(\mathbb{R}^n)$, it makes sense to consider the following problem:

Open Problem. *For what bases $\mathscr{S}$ is the geometric maximal operator $M_{\mathscr{S}}$ bounded on $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$?*

Although the result of Bennett-DeVore-Sharpley implies that the basis of cubes is one such basis, it is currently unknown whether this holds for the basis of rectangles.

This problem is the topic of the present paper. The first purpose of the paper is to establish a class of bases for which $M_{\mathscr{S}}$ is bounded on $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$. A basis is said to be engulfing if, roughly speaking, one of two intersecting shapes can be expanded to engulf the other without having to grow too large. This class includes the basis of cubes but excludes the basis of rectangles. It is shown, under an assumption on the basis $\mathscr{S}$, that (see Theorem 3.3.2):

Theorem I (Engulfing bases)**.** *If $\mathscr{S}$ is an engulfing basis, then $M_{\mathscr{S}}$ is bounded from $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ to $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$.*

As an intermediary step to defining and studying rectangular BLO spaces, the product nature of $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ is studied in more detail. When the shapes exhibit a certain product structure, it is shown that a product decomposition for $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ holds (see Theorem 3.4.6). This is analogous to what was done for $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ in [26].

The third purpose of this paper is to address the situation when $\mathscr{S}$ does not possess an engulfing property but is instead a product basis. By this we mean that the shapes in $\mathscr{S}$ exhibit some product structure with respect to lower-dimensional shapes coming from bases that *do* have engulfing. Purely using this product structure, the following theorem is shown in Section 3.6, under certain assumptions on the basis $\mathscr{S}$ (see Theorem 3.6.1 for the exact statement):

Theorem II (Product bases)**.** *If $\mathscr{S}$ is a strong product basis, then $M_{\mathscr{S}}$ is bounded from* $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ *to rectangular* $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k})$*, where $n_1 + n_2 + \ldots + n_k = n$.*

In particular, this theorem applies to the basis of rectangles, and so it follows that the strong maximal operator M_s takes functions from strong BMO to rectangular BLO.

3.2 Preliminaries

Consider $\mathbb{R}^n$ with the Euclidean topology and Lebesgue measure. We call a shape in $\mathbb{R}^n$ any open set S such that $0 < |S| < \infty$. By a basis of shapes we mean a collection $\mathscr{S}$ of shapes S that forms a cover of $\mathbb{R}^n$. Unless otherwise stated, $1 \leq p < \infty$.

Common examples of bases are the collections of all Euclidean balls, $\mathcal{B}$, all cubes, $\mathcal{Q}$, and all rectangles, $\mathcal{R}$. In one dimension, these three choices degenerate to the collection of all (finite) open intervals, $\mathcal{I}$. Other examples of bases are the collection of all ellipses and balls coming from p-norms on $\mathbb{R}^n$.

Fix a basis of shapes $\mathscr{S}$. We assume here and throughout the paper that f is a measurable function satisfying $f \in L^1(S)$ for all shapes $S \in \mathscr{S}$. This implies that f is locally integrable.

Definition 3.2.1. The maximal function of f with respect to the basis $\mathscr{S}$ is defined as

$$M_{\mathscr{S}} f(x) = \sup_{\mathscr{S} \ni S \ni x} \fint_S |f|.$$

Since shapes are open, it follows that $M_{\mathscr{S}} f$ is lower semicontinuous, hence measurable[1].

[1]See Section II.1 in the appendix for the proof of this and a basic discussion of the Hardy-Littlewood maximal function.

One shows this in much the same way as one shows the lower semicontinuity of the Hardy-Littlewood maximal function.

An important feature of a basis is the question of the boundedness of the corresponding maximal operator on L^p for $1 < p < \infty$. Indeed, there exist bases for which no such p exists: the basis of all rectangles, not necessarily having sides parallel to the coordinate axes ([29]).

In [26], the space of functions of bounded mean oscillation with respect to a general basis $\mathscr{S}$ was introduced:

Definition 3.2.2. We say that f belongs to $\mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} := \sup_{S \in \mathscr{S}} \left(\fint_S |f - f_S|^p \right)^{1/p} < \infty.$$

The notation $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ will be reserved for the case where $p = 1$. By Jensen's inequality, $\mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n) \subset \mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ for any $1 < p < \infty$ with $\|f\|_{\mathrm{BMO}_{\mathscr{S}}} \le \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}$. If the opposite inclusion holds, that is $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n) \subset \mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)$ for some $1 < p < \infty$ with $\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} \le c\|f\|_{\mathrm{BMO}_{\mathscr{S}}}$ for some constant $c > 0$, then we write $\mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n) \cong \mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$. This holds, in fact for all $1 < p < \infty$, if the John-Nirenberg inequality is valid for every $f \in \mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ with uniform constants (see [26]). This is the case for the basis $\mathcal{Q}$, for instance, as well as the basis $\mathcal{R}$ ([60]).

There do exist bases that fail to satisfy $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n) \subset \mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)$ for any p. An example is the basis $\mathcal{Q}_c$ of cubes centred at the origin with sides parallel to the axes ([67]).

Note that the maximal function of an f in $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ need not be finite almost everywhere. For example, $M_{\mathcal{Q}} f \equiv \infty$ if $f(z) = -\log|z| \in \mathrm{BMO}_{\mathcal{Q}}(\mathbb{R}^n)$.

Many familiar BMO properties were shown in [26] to hold at this level of generality, even when working with functions defined on a domain in $\mathbb{R}^n$. In particular, $\mathrm{BMO}^p_{\mathscr{S}}$ is a Banach space modulo constants. Moreover, $\mathrm{BMO}^p_{\mathscr{S}}$ is a lattice: if $f, g \in \mathrm{BMO}^p_{\mathscr{S}}$, then $h \in \mathrm{BMO}^p_{\mathscr{S}}$, where h is either $\max(f,g)$ or $\min(f,g)$. This follows readily from writing $\max(f,g) = \frac{1}{2}(f + g + |f - g|)$ and $\min(f,g) = \frac{1}{2}(f + g - |f - g|)$, because the operator $f \mapsto |f|$ is bounded on $\mathrm{BMO}^p_{\mathscr{S}}$ and $\mathrm{BMO}^p_{\mathscr{S}}$ is a linear space.

An important subset of BMO that often arises is the class of functions of bounded lower oscillation. Analogously to what was done in [26] for BMO, we define this set with respect to a general basis:

Definition 3.2.3. We say that f belongs to $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BLO}_{\mathscr{S}}} := \sup_{S\in\mathscr{S}} \fint_S [f - \operatorname*{ess\,inf}_S f] < \infty.$$

Note that $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n) \subset \mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ because, for any shape $S \in \mathscr{S}$,

$$\fint_S |f - f_S| \leq 2\fint_S |f - \alpha|$$

holds for any constant α and so, in particular, for $\alpha = \operatorname*{ess\,inf}_S f$. Moreover, the inclusion can be strict: the function $f(z) = \log|z|$ is an element of $\mathrm{BMO}_{\mathcal{Q}}(\mathbb{R}^n) \setminus \mathrm{BLO}_{\mathcal{Q}}(\mathbb{R}^n)$. The function $f(z) = -\log|z|$, however, *is* in $\mathrm{BLO}_{\mathcal{Q}}(\mathbb{R}^n)$. This example shows that, in general, $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ fails to be a linear space.

As such, the approach used above to argue that $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ is a lattice is not immediately applicable to $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$. The following establishes that $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ is an upper semilattice; that is, $\max(f,g) \in \mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ whenever $f, g \in \mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$.

Proposition 3.2.4. *For any basis $\mathscr{S}$, $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ is an upper semilattice with*

$$\|\max(f,g)\|_{\mathrm{BLO}_{\mathscr{S}}} \leq \|f\|_{\mathrm{BLO}_{\mathscr{S}}} + \|g\|_{\mathrm{BLO}_{\mathscr{S}}}.$$

Proof. Let $f, g \in \mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ and fix a shape $S \in \mathscr{S}$. Writing $h = \max(f,g)$ and considering the set $E = \{z \in S : f(z) \geq g(z)\}$, we have that

$$\begin{aligned}
\int_S [h - \operatorname*{ess\,inf}_S h] &= \int_E [f - \operatorname*{ess\,inf}_S h] + \int_{S\setminus E} [g - \operatorname*{ess\,inf}_S h] \\
&\leq \int_E [f - \operatorname*{ess\,inf}_S f] + \int_{S\setminus E} [g - \operatorname*{ess\,inf}_S g] \\
&\leq \int_S [f - \operatorname*{ess\,inf}_S f] + \int_S [g - \operatorname*{ess\,inf}_S g] \\
&\leq |S| \left[\|f\|_{\mathrm{BLO}_{\mathscr{S}}} + \|g\|_{\mathrm{BLO}_{\mathscr{S}}}\right].
\end{aligned}$$

Dividing by $|S|$ and taking a supremum over $S \in \mathscr{S}$ yields the result. □

3.3 Engulfing bases

In this section, we provide a generalisation of Bennett's theorem that the maximal function is bounded from BMO to BLO. What is essentially the same proof as that of Bennett holds for a class of bases. The key property is that $\mathscr{S}$ is an engulfing basis.

Definition 3.3.1. We say that $\mathscr{S}$ is an engulfing basis if there exist constants $c_d, c_e > 1$, that may depend on the dimension n, such that to each $S \in \mathscr{S}$ we can associate a shape $\widetilde{S} \in \mathscr{S}$ satisfying the following:

(i) $\widetilde{S} \supset S$ with $|\widetilde{S}| \leq c_d|S|$;

(ii) if $T \in \mathscr{S}$ is such that $S \cap T \neq \emptyset$ and $\widetilde{S}^c \cap T \neq \emptyset$, where $\widetilde{S}^c$ denotes the complement of $\widetilde{S}$, then there exists a $\overline{T} \in \mathscr{S}$ such that $\overline{T} \supset \widetilde{S} \cup T$ with $|\overline{T}| \leq c_e|T|$.

Note that the choice of engulfing shape $\overline{T}$ depends on S, T, and the choice of shape $\widetilde{S}$ to associate to S.

An example of an engulfing basis is the family of open balls in $\mathbb{R}^n$ with respect to a p-metric, $1 \leq p \leq \infty$. The bases $\mathcal{B}$ and $\mathcal{Q}$ are special cases, with $p = 2$ and $p = \infty$, respectively.

More generally, the basis of open balls in any doubling metric measure space is an engulfing basis. Denote by $B(z, r)$ a ball with centre z and radius $r > 0$. Every ball $B_1 = B(z, r)$ has a natural double $\widetilde{B}_1 = B(z, 2r)$ satisfying $\widetilde{B}_1 \supset B_1$ and $|\widetilde{B}_1| \leq c_d|B_1|$ for some $c_d > 1$. In $\mathbb{R}^n$, we have $c_d = 2^n$. Furthermore, if $B_2 = B(w, R)$ satisfies $B_1 \cap B_2 \neq \emptyset$ and $\widetilde{B}_1^c \cap B_2 \neq \emptyset$, then $R > r/2$ and there is a ball $\overline{B}_2$ centred at a point in $B_1 \cap B_2$ of radius $\max(2R, 3r) \leq 6r$. This ball satisfies $\overline{B}_2 \supset \widetilde{B}_1 \cup B_2$ and $|\overline{B}_2| \leq c_e|B_2|$ for some $c_e > 1$. In $\mathbb{R}^n$, we have $c_e = 6^n$.

An example of a basis which does not satisfy an engulfing property is $\mathcal{R}$. No matter what choice of $\widetilde{R}$ is made that satisfies (i), there is no c_e for which condition (ii) holds. To see this, consider the case $n = 2$, as well as the intersecting rectangles $R_1 = (0, w) \times (0, H)$ and $R_2 = (0, W) \times (0, h)$ for $H > h$ and $W > w$. Any engulfing rectangle $\overline{R}_2$ would have to contain $(0, W) \times (0, H)$. Thus,

$$\frac{|\overline{R}_2|}{|R_2|} \geq \frac{HW}{hW} = \frac{H}{h} \to \infty$$

as either $H \to \infty$ or $h \to 0^+$, and so there can be no $c_e < \infty$ satisfying condition (ii) uniformly for all rectangles.

Now we come to the statement of the theorem.

Theorem 3.3.2. *Let $\mathscr{S}$ be an engulfing basis such that there exists a $p \in (1,\infty)$ for which $M_{\mathscr{S}}$ is bounded on $L^p(\mathbb{R}^n)$ with norm A_p. If $f \in \mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)$, then*

$$\fint_S M_{\mathscr{S}} f \leq c\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \operatorname*{ess\,inf}_S M_{\mathscr{S}} f \tag{3.3}$$

for all $S \in \mathscr{S}$, where c is a constant depending on p, n, c_d, c_e, and A_p. Assuming the right-hand side of (3.3) *is finite for every shape $S \in \mathscr{S}$, it follows that $M_{\mathscr{S}} f$ is finite almost everywhere and $M_{\mathscr{S}} f \in \mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ with*

$$\|M_{\mathscr{S}} f\|_{\mathrm{BLO}_{\mathscr{S}}} \leq c\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}.$$

Moreover, if $\mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n) \cong \mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$, then $\|M_{\mathscr{S}} f\|_{\mathrm{BLO}_{\mathscr{S}}} \leq C\|f\|_{\mathrm{BMO}_{\mathscr{S}}}$ holds for all $f \in \mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ for which $M_{\mathscr{S}} f$ is finite almost everywhere.

Remark 3.3.3. This theorem contains not only that of Bennett, but also the corresponding result of Guzmán-Partida ([45]) for the basis $\mathcal{Q}_c$. This is an engulfing basis and the boundedness of $M_{\mathcal{Q}_c}$ on L^p follows from the fact that $\mathcal{Q}_c \subset \mathcal{Q}$ and the boundedness of $M_{\mathcal{Q}}$ on L^p.

Proof. Fix $f \in \mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)$ and $S \in \mathscr{S}$. Write $f = g + h$, where $g = (f - f_{\widetilde{S}})\chi_{\widetilde{S}}$ and $h = f_{\widetilde{S}}\chi_{\widetilde{S}} + f\chi_{\widetilde{S}^c}$. Then, by the boundedness of $M_{\mathscr{S}}$ on $L^p(\mathbb{R}^n)$,

$$\fint_S M_{\mathscr{S}} g \leq \frac{1}{|S|^{1/p}}\|M_{\mathscr{S}} g\|_{L^p} \leq \frac{A_p}{|S|^{1/p}}\|g\|_{L^p} \leq A_p c_d^{1/p}\left(\fint_{\widetilde{S}} |f - f_{\widetilde{S}}|^p\right)^{1/p}.$$

Thus,

$$\fint_S M_{\mathscr{S}} g \leq A_p c_d^{1/p}\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}. \tag{3.4}$$

Fix a point $z_0 \in S$ and a shape $T \in \mathscr{S}$ such that $T \ni z_0$. If $T \subset \widetilde{S}$, then

$$\fint_T |h| = |f_{\widetilde{S}}| \leq \fint_{\widetilde{S}} |f| \leq M_{\mathscr{S}} f(z)$$

for every $z \in \widetilde{S}$. In particular, this is true for every $z \in S$, and so

$$\fint_T |h| \leq \operatorname*{ess\,inf}_S M_{\mathscr{S}} f. \tag{3.5}$$

If $T \cap \widetilde{S}^c \neq \emptyset$, then by the engulfing property there exists a shape $\overline{T}$ containing T and $\widetilde{S}$ such that $|\overline{T}| \leq c_e|T|$. Hence,

$$\begin{aligned}
\fint_T |h - f_{\overline{T}}| \leq c_e \fint_{\overline{T}} |h - f_{\overline{T}}| &= \frac{c_e}{|\overline{T}|}\left[|\widetilde{S}||f_{\widetilde{S}} - f_{\overline{T}}| + \int_{\overline{T}\cap\widetilde{S}^c} |f - f_{\overline{T}}|\right] \\
&\leq \frac{c_e}{|\overline{T}|}\left[\int_{\widetilde{S}} |f - f_{\overline{T}}| + \int_{\overline{T}\cap\widetilde{S}^c} |f - f_{\overline{T}}|\right] \\
&= c_e \fint_{\overline{T}} |f - f_{\overline{T}}| \leq c_e \left(\fint_{\overline{T}} |f - f_{\overline{T}}|^p\right)^{1/p} \leq c_e\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}.
\end{aligned}$$

Thus,

$$\fint_T |h| \leq \fint_T |h - f_{\overline{T}}| + \fint_{\overline{T}} |f| \leq c_e\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + M_{\mathscr{S}}f(z)$$

for every $z \in \overline{T}$. In particular, this is true for every $z \in S$, and so

$$\fint_T |h| \leq c_e\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \operatorname*{ess\,inf}_S M_{\mathscr{S}}f. \tag{3.6}$$

Combining (3.5) and (3.6), we have the pointwise bound

$$M_{\mathscr{S}}h(z_0) \leq c_e\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \operatorname*{ess\,inf}_S M_{\mathscr{S}}f. \tag{3.7}$$

Therefore, combining (3.4) and (3.7), we arrive at

$$\fint_S M_{\mathscr{S}}f \leq \fint_S M_{\mathscr{S}}g + \fint_S M_{\mathscr{S}}h \leq c\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \operatorname*{ess\,inf}_S M_{\mathscr{S}}f.$$

□

3.4 Product structure

In this section, we follow Section 8 of [26]. Write $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_k}$, where $n_1 + n_2 + \ldots + n_k = n$ and $2 \leq k \leq n$. Let $\mathscr{S}$ be a basis of shapes in $\mathbb{R}^n$ and, for each $1 \leq i \leq k$, let $\mathscr{S}_i$ be a basis of shapes in $\mathbb{R}^{n_i}$. For $z \in \mathbb{R}^n$, write $\hat{z}_i$ when the ith component (coming from $\mathbb{R}^{n_i}$) has been deleted and define $f_{\hat{z}_i}$ to be the function on $\mathbb{R}^{n_i}$ obtained from f by fixing the other components equal to $\hat{z}_i$.

We can define a BMO space on $\mathbb{R}^n$ that measures uniform "lower-dimensional" bounded mean oscillation with respect to $\mathscr{S}_i$ in the following way.

Definition 3.4.1. A function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be in $\text{BMO}^p_{\mathscr{S}_i}(\mathbb{R}^n)$ if $f_{\hat{z}_i} \in \text{BMO}^p_{\mathscr{S}_i}(\mathbb{R}^{n_i})$ uniformly in $\hat{z}_i$; i.e.

$$\|f\|_{\text{BMO}^p_{\mathscr{S}_i}(\mathbb{R}^n)} := \sup_{\hat{z}_i} \|f_{\hat{z}_i}\|_{\text{BMO}^p_{\mathscr{S}_i}(\mathbb{R}^{n_i})} < \infty.$$

It turns out that under certain conditions on the relationship between the bases $\{\mathscr{S}_i\}_{i=1}^k$ and the overall basis $\mathscr{S}$, there is a relationship between $\text{BMO}_{\mathscr{S}_i}(\mathbb{R}^n)$ and $\text{BMO}_{\mathscr{S}}(\mathbb{R}^n)$. We present the theorem, after a definition, below.

Definition 3.4.2. Let $\mathscr{S}$ be a basis of shapes in $\mathbb{R}^n$ and $\mathscr{S}_i$ be a basis of shapes for $\mathbb{R}^{n_i}$, $1 \leq i \leq k$, where $n_1 + n_2 + \ldots + n_k = n$.

1. We say that $\mathscr{S}$ satisfies the weak decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$ if for every $S \in \mathscr{S}$, there exist $S_i \in \mathscr{S}_i$, $1 \leq i \leq k$, such that $S = S_1 \times S_2 \times \ldots \times S_k$.

2. If, in addition, for every $\{S_i\}_{i=1}^k$, $S_i \in \mathscr{S}_i$, the set $S_1 \times S_2 \times \ldots \times S_k \in \mathscr{S}$, then we say that the basis $\mathscr{S}$ satisfies the strong decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$.

Starting with bases $\mathscr{S}_i$ in $\mathbb{R}^{n_i}$, $1 \leq i \leq k$, the Cartesian product $\mathscr{S}_1 \times \mathscr{S}_2 \times \ldots \times \mathscr{S}_k$ is a basis of shapes in $\mathbb{R}^n$ with the strong decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$.

When $\mathscr{S}_i = \mathcal{R}_i$, where $\mathcal{R}_i$ denotes the basis of rectangles in $\mathbb{R}^{n_i}$, the Cartesian product above coincides with the basis $\mathcal{R}$ in $\mathbb{R}^n$. As such, $\mathcal{R}$ satisfies the strong decomposition property with respect to $\{\mathcal{R}_i\}_{i=1}^k$. In particular, when $k = n$ and so $n_i = 1$ for every $1 \leq i \leq n$, $\mathcal{R}$ satisfies the strong decomposition property with respect to $\{\mathcal{I}_i\}_{i=1}^n$.

The basis $\mathcal{Q}$ does not satisfy the strong decomposition property, however, with respect to $\{\mathcal{Q}_i\}_{i=1}^k$ for any $2 \leq k \leq n$, where $\mathcal{Q}_i$ denotes the basis of cubes in $\mathbb{R}^{n_i}$, as the product of arbitrary cubes (or intervals) may not necessarily be a cube. Nevertheless, $\mathcal{Q}$ does satisfy the weak decomposition property with respect to $\{\mathcal{Q}_i\}_{i=1}^k$.

Theorem 3.4.3 ([26])**.** *Let $\mathscr{S}$ be a basis of shapes in $\mathbb{R}^n$ and $\mathscr{S}_i$ be a basis of shapes for $\mathbb{R}^{n_i}$, $1 \leq i \leq k$, where $n_1 + n_2 + \ldots + n_k = n$.*

a) Let $f \in \bigcap_{i=1}^k \text{BMO}^p_{\mathscr{S}_i}(\mathbb{R}^n)$. If $\mathscr{S}$ satisfies the weak decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$, then $f \in \text{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)$ with

$$\|f\|_{\text{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)} \leq \sum_{i=1}^k \|f\|_{\text{BMO}^p_{\mathscr{S}_i}(\mathbb{R}^n)}.$$

b) Let $f \in \mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)$. If $\mathscr{S}$ satisfies the strong decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$ and each $\mathscr{S}_i$ contains a differentiation basis that differentiates $L^1_{loc}(\mathbb{R}^{n_i})$, then $f \in \bigcap_{i=1}^k \mathrm{BMO}^p_{\mathscr{S}_i}(\mathbb{R}^n)$ with

$$\max_{1\le i\le k}\{\|f\|_{\mathrm{BMO}^p_{\mathscr{S}_i}(\mathbb{R}^n)}\} \le 2^{k-1}\|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)}.$$

When $p = 2$, the constant 2^{k-1} can be replaced by 1.

Remark 3.4.4. The condition that a basis $\mathscr{S}$ contains a differentiation basis that differentiates $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ implies that for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\varepsilon > 0$, for almost every z there exists a shape $S \in \mathscr{S}$ such that $S \ni z$ and

$$\left| \fint_S f - f(z) \right| < \varepsilon.$$

The bases of $\mathcal{B}$ and $\mathcal{Q}$ are examples of differentiation bases that differentiate $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The basis $\mathcal{R}$ does not differentiate $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, but it contains $\mathcal{Q}$ and so $\mathcal{R}$ also satisfies the assumptions of this theorem.

Just as there are "lower-dimensional" BMO spaces, one may define "lower-dimensional" BLO spaces in an analogous manner.

Definition 3.4.5. A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be in $\mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n)} := \sup_{\hat{z}_i} \|f_{\hat{z}_i}\|_{\mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^{n_i})} < \infty.$$

It turns out that a BLO-version of Theorem 3.4.3 is true. The proof follows the same lines as that of Theorem 3.4.3 given in [26], but we include it here to illustrate how the nature of BLO allows us to attain a better constant in part (b).

Theorem 3.4.6. *Let $\mathscr{S}$ be a basis of shapes in $\mathbb{R}^n$ and $\mathscr{S}_i$ be a basis of shapes for $\mathbb{R}^{n_i}$, $1 \le i \le k$, where $n_1 + n_2 + \ldots + n_k = n$.*

a) Let $f \in \bigcap_{i=1}^k \mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n)$. If $\mathscr{S}$ satisfies the weak decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$, then $f \in \mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ with

$$\|f\|_{\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)} \le \sum_{i=1}^k \|f\|_{\mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n)}.$$

b) Let $f \in \mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$. If $\mathscr{S}$ satisfies the strong decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$ and each $\mathscr{S}_i$ contains a differentiation basis that differentiates $L^1_{loc}(\mathbb{R}^{n_i})$, then $f \in \bigcap_{i=1}^k \mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n)$ with

$$\max_{1\le i\le k}\{\|f\|_{\mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n)}\} \le \|f\|_{\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)}.$$

Proof. We begin by proving the case $k=2$, where $\mathbb{R}^n = \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ for $n_1+n_2=n$. Write $\mathscr{S}_x$ for the basis in $\mathbb{R}^{n_1}$ and x for points in $\mathbb{R}^{n_1}$; write $\mathscr{S}_y$ for the basis in $\mathbb{R}^{n_2}$ and y for points in $\mathbb{R}^{n_2}$.

To prove (a), assume that $\mathscr{S}$ satisfies the weak decomposition property with respect to $\{\mathscr{S}_x, \mathscr{S}_y\}$ and let $f \in \mathrm{BLO}_{\mathscr{S}_x}(\mathbb{R}^n)\cap \mathrm{BLO}_{\mathscr{S}_y}(\mathbb{R}^n)$. Fixing a shape $S\in\mathscr{S}$, write $S = S_1\times S_2$ where $S_1\in\mathscr{S}_x$ and $S_2\in\mathscr{S}_y$. Then,

$$⨍_{S_2}⨍_{S_1}[f(x,y)-\operatorname*{ess\,inf}_S f]\,dx\,dy = ⨍_{S_2}⨍_{S_1}[f(x,y)-\operatorname*{ess\,inf}_{S_1} f_y]\,dx\,dy + ⨍_{S_2}[\operatorname*{ess\,inf}_{S_1} f_y - \operatorname*{ess\,inf}_S f]\,dy.$$

For the first integral, we estimate

$$⨍_{S_2}⨍_{S_1}[f(x,y)-\operatorname*{ess\,inf}_{S_1} f_y]\,dx\,dy \le ⨍_{S_2}\|f_y\|_{\mathrm{BLO}_{\mathscr{S}_x}(\mathbb{R}^{n_1})}\,dy \le \|f\|_{\mathrm{BLO}_{\mathscr{S}_x}(\mathbb{R}^n)}.$$

For the second integral, fixing $\varepsilon>0$, the set E of $(x,y)\in S_1\times S_2$ with $\operatorname*{ess\,inf}_S f > f(x,y)-\varepsilon$ has positive measure. Moreover, the set F of $(x,y)\in S_1\times S_2$ such that $f(x,y)\ge \operatorname*{ess\,inf}_{S_1} f_y$ and $f(x,y)\ge \operatorname*{ess\,inf}_{S_2} f_x$ has full measure, and so $|E\cap F|>0$. Then, taking a point $(x_0,y_0)\in E\cap F$,

$$\begin{aligned}
⨍_{S_2}[\operatorname*{ess\,inf}_{S_1} f_y - \operatorname*{ess\,inf}_S f]\,dy &\le ⨍_{S_2}[f_y(x_0)-f(x_0,y_0)+\varepsilon]\,dy \\
&= ⨍_{S_2}[f_{x_0}(y)-f(x_0,y_0)]\,dy+\varepsilon \\
&\le ⨍_{S_2}[f_{x_0}(y)-\operatorname*{ess\,inf}_{S_2} f_{x_0}]\,dy+\varepsilon \\
&\le \|f_{x_0}\|_{\mathrm{BLO}_{\mathscr{S}_y}(\mathbb{R}^{n_2})}+\varepsilon \le \|f\|_{\mathrm{BLO}_{\mathscr{S}_y}(\mathbb{R}^n)}+\varepsilon.
\end{aligned}$$

Therefore, letting $\varepsilon\to 0^+$, we conclude that $f\in\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ with

$$\|f\|_{\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)} \le \|f\|_{\mathrm{BLO}_{\mathscr{S}_x}(\mathbb{R}^n)} + \|f\|_{\mathrm{BLO}_{\mathscr{S}_y}(\mathbb{R}^n)}.$$

We now come to the proof of (b). Assume that $\mathscr{S}$ satisfies the strong decomposition property with respect to $\{\mathscr{S}_x, \mathscr{S}_y\}$, and that $\mathscr{S}_x$ and $\mathscr{S}_y$ each contain a differentiation basis that differentiates $L^1_{\text{loc}}(\mathbb{R}^{n_1})$ and $L^1_{\text{loc}}(\mathbb{R}^{n_2})$, respectively. Let $f \in \text{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ and fix a shape $S_1 \in \mathscr{S}_x$. Consider

$$g(y) = \fint_{S_1} [f_y(x) - \operatorname*{ess\,inf}_{S_1} f_y]\, dx$$

as a function of y. For any $S_2 \in \mathscr{S}_y$, writing $S = S_1 \times S_2$, we have $\operatorname*{ess\,inf}_{S_1} f_y \geq \operatorname*{ess\,inf}_{S} f$ for almost every y, and so

$$\int_{S_2} g(y)\, dy \leq |S_2| \fint_S [f - \operatorname*{ess\,inf}_S f] \leq |S_2| \|f\|_{\text{BLO}_{\mathscr{S}}(\mathbb{R}^n)},$$

implying that $g \in L^1_{\text{loc}}(\mathbb{R}^{n_2})$. Let $\varepsilon > 0$. Since $\mathscr{S}_y$ contains a differentiation basis, for almost every $y_0 \in \mathbb{R}^{n_2}$ there exists a shape $S_2 \in \mathscr{S}_y$ containing y_0 such that

$$\left| \fint_{S_2} g(y)\, dy - g(y_0) \right| < \varepsilon.$$

Fix such a y_0 and an S_2, and write $S = S_1 \times S_2$. We have that

$$\fint_{S_1} [f_{y_0}(x) - \operatorname*{ess\,inf}_{S_1} f_{y_0}]\, dx = g(y_0) \leq \varepsilon + \fint_{S_2} g(y)\, dy \leq \varepsilon + \|f\|_{\text{BLO}_{\mathscr{S}}(\mathbb{R}^n)}.$$

Taking $\varepsilon \to 0^+$, since S_1 is arbitrary this implies that $f_{y_0} \in \text{BLO}_{\mathscr{S}_x}(\mathbb{R}^{n_1})$ with

$$\|f_{y_0}\|_{\text{BLO}_{\mathscr{S}_x}(\mathbb{R}^{n_1})} \leq \|f\|_{\text{BLO}_{\mathscr{S}}(\mathbb{R}^n)}.$$

The fact that this is true for almost every y_0 implies that $\|f\|_{\text{BLO}_{\mathscr{S}_x}(\mathbb{R}^n)} \leq \|f\|_{\text{BLO}_{\mathscr{S}}(\mathbb{R}^n)}$. Similarly, one can show that $\|f\|_{\text{BLO}_{\mathscr{S}_y}(\mathbb{R}^n)} \leq \|f\|_{\text{BLO}_{\mathscr{S}}(\mathbb{R}^n)}$. Thus we have that $f \in \text{BLO}_{\mathscr{S}_x}(\mathbb{R}^n) \cap \text{BLO}_{\mathscr{S}_y}(\mathbb{R}^n)$ with

$$\max\{\|f\|_{\text{BLO}_{\mathscr{S}_x}(\mathbb{R}^n)}, \|f\|_{\text{BLO}_{\mathscr{S}_y}(\mathbb{R}^n)}\} \leq \|f\|_{\text{BLO}_{\mathscr{S}}(\mathbb{R}^n)}.$$

To prove part (a) for $k > 2$ factors, we assume it holds for $k-1$ factors. Write $X = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_{k-1}}$, $Y = \mathbb{R}^{n_k}$, and set $\mathscr{S}_Y = \mathscr{S}_k$. Write x for the elements of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_{k-1}}$ and y for the elements of $\mathbb{R}^{n_k}$. Denote by $\hat{x}_i$ the result of deleting x_i from $x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_{k-1}}$.

Assume that $\mathscr{S}$ has the weak decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^{k-1}$. As such, we can define the projection of the basis $\mathscr{S}$ onto X, namely

$$\mathscr{S}_X = \{S_1 \times S_2 \times \ldots \times S_{k-1} : S_i \in \mathscr{S}_i, \exists S_k \in \mathscr{S}_k, \prod_{i=1}^{k} S_i \in \mathscr{S}\}. \tag{3.8}$$

This is a basis of shapes on X which, by definition, has the weak decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^{k-1}$. Moreover, $\mathscr{S}$ has the weak decomposition property with respect to $\{\mathscr{S}_X, \mathscr{S}_Y\}$. Beginning by applying the proven case of $k = 2$, we have

$$\|f\|_{\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)} \leq \|f\|_{\mathrm{BLO}_{\mathscr{S}_X}(\mathbb{R}^n)} + \|f\|_{\mathrm{BLO}_{\mathscr{S}_Y}(\mathbb{R}^n)}.$$

Then, we apply the case of $k-1$ factors to X to yield

$$\begin{aligned}\|f\|_{\mathrm{BLO}_{\mathscr{S}_X}(\mathbb{R}^n)} &= \sup_{y\in Y} \|f_y\|_{\mathrm{BLO}_{\mathscr{S}_X}(X)} \leq \sup_{y\in Y} \sum_{i=1}^{k-1} \|f_y\|_{\mathrm{BLO}_{\mathscr{S}_i}(X)} = \sup_{y\in Y} \sum_{i=1}^{k-1} \sup_{\hat{x}_i} \|(f_y)_{\hat{x}_i}\|_{\mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^{n_i})} \\ &\leq \sum_{i=1}^{k-1} \sup_{(\hat{x}_i, y)} \|f_{(\hat{x}_i, y)}\|_{\mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^{n_i})} = \sum_{i=1}^{k-1} \|f\|_{\mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n)}.\end{aligned}$$

Therefore,

$$\|f\|_{\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)} \leq \sum_{i=1}^{k-1} \|f\|_{\mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n)} + \|f\|_{\mathrm{BLO}_{\mathscr{S}_k}(\mathbb{R}^n)} = \sum_{i=1}^{k} \|f\|_{\mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n)}.$$

To prove part (b) for $k > 2$ factors, first note that if $\mathscr{S}$ has the strong decomposition property, then so does $\mathscr{S}_X$ defined by (3.8). We repeat the first part of the proof of (b) for the case $k = 2$ above to reach

$$\|f_{y_0}\|_{\mathrm{BLO}_{\mathscr{S}_X}(X)} \leq \|f\|_{\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)}$$

for some $y_0 \in \mathbb{R}^{n_k}$. Now we repeat the process for the function f_{y_0} instead of f, with $X_1 = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_{k-2}}$ and $Y_1 = \mathbb{R}^{n_{k-1}}$. This gives

$$\|(f_{y_0})_{y_1}\|_{\mathrm{BLO}_{\mathscr{S}_{X_1}}(X_1)} \leq \|f_{y_0}\|_{\mathrm{BLO}_{\mathscr{S}_X}(X)} \leq \|f\|_{\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)} \quad \forall y_1 \in \mathbb{R}^{n_{k-1}}, y_0 \in \mathbb{R}^{n_k}.$$

We continue until we get to $X_{k-1} = \mathbb{R}^{n_1}$, for which $\mathscr{S}_{X_k} = \mathscr{S}_1$, yielding the estimate

$$\|f_{(y_{k-2},\ldots,y_0)}\|_{\mathrm{BLO}_{\mathscr{S}_1}(\mathbb{R}^{n_1})} \leq \cdots \leq \|f_{y_0}\|_{\mathrm{BLO}_{\mathscr{S}_X}(X)} \leq \|f\|_{\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)}$$

for all $(k-1)$-tuples $y = (y_{k-2}, \ldots, y_0) \in \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_k}$. Taking the supremum over all such y, we have that $f \in \mathrm{BLO}_{\mathscr{S}_1}(\mathbb{R}^n)$ with

$$\|f\|_{\mathrm{BLO}_{\mathscr{S}_1}(\mathbb{R}^n)} = \sup_y \|f_y\|_{\mathrm{BLO}_{\mathscr{S}_1}(\mathbb{R}^{n_1})} \leq \|f\|_{\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)}.$$

A similar process for $i = 2, \ldots, k$ shows that $f \in \mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n)$ with

$$\|f\|_{\mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n)} \leq \|f\|_{\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)}.$$

$\square$

3.5 Rectangular bounded mean oscillation

Let $\mathscr{S}$ be a basis of shapes in $\mathbb{R}^n$ and denote by $\mathscr{S}_x, \mathscr{S}_y$ bases of shapes in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, where $n_1 + n_2 = n$. Additionally, we maintain the convention that $\mathscr{S}$ has the strong decomposition property with respect to $\{\mathscr{S}_x, \mathscr{S}_y\}$. Writing x for the coordinates in $\mathbb{R}^{n_1}$ and y for those in $\mathbb{R}^{n_2}$, denote by f_x the function obtained from f by fixing x. Similarly, f_y is the function obtained from f by fixing y.

We begin by defining the rectangular BMO space at this level of generality.

Definition 3.5.1. We say that f is in $\mathrm{BMO}_{\mathrm{rec},\mathscr{S}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if

$$\|f\|_{\mathrm{BMO}_{\mathrm{rec},\mathscr{S}}} := \sup_{S_1 \in \mathscr{S}_x, S_2 \in \mathscr{S}_y} \fint_{S_1} \fint_{S_2} |f(x,y) - (f_x)_{S_2} - (f_y)_{S_1} + f_S| \, dy \, dx < \infty, \tag{3.9}$$

where $S = S_1 \times S_2$.

In the literature, the classical rectangular BMO space corresponds to $\mathscr{S}_x = \mathcal{Q}_x$ and $\mathscr{S}_y = \mathcal{Q}_y$, and so $\mathscr{S}$ is the subfamily of $\mathcal{R}$ that can be written as the product of two cubes. In dimension two, this is the same as $\mathcal{R}$.

Proposition 3.5.2. *If $f \in \mathrm{BMO}_{\mathscr{S}_x}(\mathbb{R}^n) \cup \mathrm{BMO}_{\mathscr{S}_y}(\mathbb{R}^n)$, then $f \in \mathrm{BMO}_{rec,\mathscr{S}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with*

$$\|f\|_{\mathrm{BMO}_{rec,\mathscr{S}}} \leq 2 \min(\|f\|_{\mathrm{BMO}_{\mathscr{S}_x}}, \|f\|_{\mathrm{BMO}_{\mathscr{S}_y}}).$$

Proof. We have

$$\fint_{S_1} \fint_{S_2} |f(x,y) - (f_y)_{S_1}| \, dy \, dx \leq \sup_{y \in S_2} \fint_{S_1} |f_y(x) - (f_y)_{S_1}| \, dx = \|f\|_{\mathrm{BMO}_{\mathscr{S}_x}}$$

and

$$\begin{aligned}
\fint_{S_1} \fint_{S_2} |(f_x)_{S_2} - f_S| \, dy \, dx = \fint_{S_1} |(f_x)_{S_2} - f_S| \, dx &= \fint_{S_1} \left| \fint_{S_2} f_x(y) \, dy - \fint_{S_2} (f_y)_{S_1} \, dy \right| dx \\
&\leq \fint_{S_2} \fint_{S_1} |f_y(x) - (f_y)_{S_1}| \, dx \, dy \\
&\leq \fint_{S_2} \|f_y\|_{\mathrm{BMO}_{\mathscr{S}_x}} \leq \|f\|_{\mathrm{BMO}_{\mathscr{S}_x}}.
\end{aligned}$$

Thus, writing

$$|f(x,y) - (f_x)_{S_2} - (f_y)_{S_1} + f_S| \leq |f(x,y) - (f_y)_{S_1}| + |(f_x)_{S_2} - f_S|,$$

it follows that

$$\fint_{S_1}\fint_{S_2}|f(x,y)-(f_x)_{S_2}-(f_y)_{S_1}+f_S|\,dy\,dx \le 2\|f\|_{\mathrm{BMO}_{\mathscr{S}_x}}.$$

Similarly, one shows that $\|f\|_{\mathrm{BMO}_{\mathrm{rec},\mathscr{S}}} \le 2\|f\|_{\mathrm{BMO}_{\mathscr{S}_y}}$. $\square$

Proposition 3.5.3. *If $f \in \mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$, then $f \in \mathrm{BMO}_{rec,\mathscr{S}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with*

$$\|f\|_{\mathrm{BMO}_{rec,\mathscr{S}}} \le 3\|f\|_{\mathrm{BMO}_{\mathscr{S}}}.$$

Proof. We have

$$\fint_{S_1}\fint_{S_2}|f(x,y)-f_S|\,dy\,dx \le \|f\|_{\mathrm{BMO}_{\mathscr{S}}},$$

$$\fint_{S_1}\fint_{S_2}|(f_x)_{S_2}-f_S|\,dy\,dx = \fint_{S_1}\left|\fint_{S_2} f_x(y)\,dy - f_S\right|\,dx \le \fint_{S_1}\fint_{S_2}|f(x,y)-f_S|\,dy\,dx \le \|f\|_{\mathrm{BMO}_{\mathscr{S}}},$$

and, similarly,

$$\fint_{S_1}\fint_{S_2}|(f_y)_{S_1}-f_S|\,dy\,dx \le \|f\|_{\mathrm{BMO}_{\mathscr{S}}}.$$

Thus, writing

$$f(x,y)-(f_x)_{S_2}-(f_y)_{S_1}+f_S = [f(x,y)-f_S]-[(f_x)_{S_2}-f_S]-[(f_y)_{S_1}-f_S],$$

it follows that

$$\fint_{S_1}\fint_{S_2}|f(x,y)-(f_x)_{S_2}-(f_y)_{S_1}+f_S|\,dy\,dx \le 3\|f\|_{\mathrm{BMO}_{\mathscr{S}}}.$$

$\square$

Remark 3.5.4. In the case where $\mathscr{S}_x, \mathscr{S}_y$ each contain a differentiation basis that differentiates $L^1_{\mathrm{loc}}(\mathbb{R}^{n_1})$ and $L^1_{\mathrm{loc}}(\mathbb{R}^{n_2})$, respectively, another proof is possible using Theorem 3.4.3 and Proposition 3.5.2. We identify $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ with $\mathrm{BMO}_{\mathscr{S}_x}(\mathbb{R}^n)\cap\mathrm{BMO}_{\mathscr{S}_y}(\mathbb{R}^n)$, so that

$$\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n) \subset \mathrm{BMO}_{\mathscr{S}_x}(\mathbb{R}^n)\cup\mathrm{BMO}_{\mathscr{S}_y}(\mathbb{R}^n) \subset \mathrm{BMO}_{\mathrm{rec},\mathscr{S}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}),$$

with $\|f\|_{\mathrm{BMO}_{\mathrm{rec},\mathscr{S}}} \le 4\|f\|_{\mathrm{BMO}_{\mathscr{S}}}$.

Unlike $\mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$, it turns out that $\mathrm{BMO}_{\mathrm{rec},\mathscr{S}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ may not be a lattice. As $\mathrm{BMO}_{\mathrm{rec},\mathscr{S}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is a linear space, this property is equivalent to being closed under taking absolute values.

Example 3.5.5. Consider $f(x,y)=x-y$. We have that $f(x,y)-(f_x)_{S_2}-(f_y)_{S_1}+f_S$ equals

$$(x-y)-\left(x-\fint_{S_2} y\,dy\right)-\left(\fint_{S_1} x\,dx-y\right)+\left(\fint_{S_1} x\,dx-\fint_{S_2} y\,dy\right)=0,$$

and so it follows that $f\in \mathrm{BMO}_{\mathrm{rec},\mathscr{S}}(\mathbb{R}\times\mathbb{R})$ for any basis $\mathscr{S}$.

For the function $h(x,y)=|f(x,y)|=|x-y|$, however, a computation shows that if $S_1=S_2=I_L=[0,L]$ for $L>0$, then

$$\fint_{I_L}\fint_{I_L}|h(x,y)-(h_x)_{I_L}-(h_y)_{I_L}+h_{I_L\times I_L}|\,dx\,dy=\frac{2}{L^2}\int_0^L\int_0^y\left|2y-\frac{x^2+y^2}{L}-\frac{2L}{3}\right|\,dx\,dy$$

by symmetry of the integrand with respect to the line $y=x$. As the integral of the expression inside the absolute value is zero on $I_L\times I_L$, it follows that

$$\frac{2}{L^2}\int_0^L\int_0^y\left|2y-\frac{x^2+y^2}{L}-\frac{2L}{3}\right|\,dx\,dy=\frac{4}{L^2}\iint_R\left[2y-\frac{x^2+y^2}{L}-\frac{2L}{3}\right]\,dx\,dy,$$

where R is the region defined by the conditions $0\le x\le y$, $0\le y\le L$, $2y\ge \frac{x^2+y^2}{L}+\frac{2L}{3}$. This region corresponds to the intersection of the disc $x^2+(y-L)^2\le\frac{L^2}{3}$ and the upper triangle of the square $I_L\times I_L$. Converting to polar coordinates relative to this region, one can compute

$$\begin{aligned}\iint_R\left[2y-\frac{x^2+y^2}{L}-\frac{2L}{3}\right]\,dx\,dy&=\frac{1}{L}\iint_R\left[\frac{L^2}{3}-x^2-(y-L)^2\right]\,dx\,dy\\&=\frac{1}{L}\int_0^{\frac{L}{\sqrt{3}}}\int_0^{\pi/2}\left(\frac{L^2}{3}-r^2\right)r\,d\theta\,dr=\frac{\pi L^3}{72}.\end{aligned}$$

Therefore,

$$\fint_{I_L}\fint_{I_L}|h(x,y)-(h_x)_{I_L}-(h_y)_{I_L}+h_{I_L\times I_L}|\,dx\,dy=\frac{4}{L^2}\times\frac{\pi L^3}{72}=\frac{\pi L}{18}\to\infty \text{ as } L\to\infty,$$

showing that $h\notin \mathrm{BMO}_{\mathrm{rec},\mathcal{R}}(\mathbb{R}\times\mathbb{R})$.

Just as we defined rectangular BMO, there is a possible analogous definition of rectangular BLO, defined by having bounded averages of the form

$$\fint_{S_1}\fint_{S_2}|f(x,y)-\operatorname*{ess\,inf}_{S_2} f_x-\operatorname*{ess\,inf}_{S_1} f_y+\operatorname*{ess\,inf}_{S} f|\,dy\,dx.$$

This definition, however, has a few deficiencies. For one, without the absolute values, the integrand is not necessarily non-negative, which is something one would expect from any class labelled as BLO. Another property of BLO that fails with this definition is being an upper semilattice, as exhibited by the following example.

Example 3.5.6. If $f(x,y) = x$ and $g(x,y) = y$, then, for any shapes S_1, S_2,

$$f(x,y) - \operatorname*{ess\,inf}_{S_2} f_x - \operatorname*{ess\,inf}_{S_1} f_y + \operatorname*{ess\,inf}_{S} f = x - x - \operatorname*{ess\,inf}_{S_1} x + \operatorname*{ess\,inf}_{S_1} x = 0$$

for almost every $x \in S_1$ and

$$g(x,y) - \operatorname*{ess\,inf}_{S_2} g_x - \operatorname*{ess\,inf}_{S_1} g_y + \operatorname*{ess\,inf}_{S} g = y - \operatorname*{ess\,inf}_{S_2} y - y + \operatorname*{ess\,inf}_{S_2} y = 0$$

for almost every $y \in S_2$.

Considering the function $h(x,y) = \max(x,y)$, however, and $S_1 = S_2 = I_L = [0,L]$ for $L > 0$. We have that

$$\fint_{I_L} \fint_{I_L} |h(x,y) - \operatorname*{ess\,inf}_{I_L} h_x - \operatorname*{ess\,inf}_{I_L} h_y + \operatorname*{ess\,inf}_{I_L \times I_L} h| \, dy \, dx$$

equals

$$\frac{1}{L^2} \int_0^L \int_0^L |\max(x,y) - x - y| \, dy \, dx = \frac{1}{L^2} \int_0^L \int_0^L \min(x,y) \, dy \, dx = \frac{1}{L^2} \times \frac{L^3}{3} = \frac{L}{3},$$

which tends to ∞ as $L \to \infty$.

These deficiencies are rectified if the essential infimum of f over $S_1 \times S_2$ is replaced by the minimum of the essential infima of f_x over S_2 and f_y over S_1:

$$\fint_{S_1} \fint_{S_2} |f(x,y) - \operatorname*{ess\,inf}_{S_2} f_x - \operatorname*{ess\,inf}_{S_1} f_y + \min\{\operatorname*{ess\,inf}_{S_2} f_x, \operatorname*{ess\,inf}_{S_y} f_x\}| \, dy \, dx.$$

The identity $\max(a,b) + \min(a,b) = a + b$ gives us that this is equal to

$$\fint_{S_1} \fint_{S_2} [f(x,y) - \max\{\operatorname*{ess\,inf}_{S_2} f_x, \operatorname*{ess\,inf}_{S_1} f_y\}] \, dy \, dx,$$

where the integrand is now clearly non-negative almost everywhere. Boundedness of these averages is the definition we choose for rectangular BLO.

An additional benefit to this definition is that it can be defined at a higher level of generality. As in Section 3.4, we decompose $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_k}$ for $2 \leq k \leq n$ and let $\mathscr{S}_i$ be a basis for $\mathbb{R}^{n_i}$ for each $1 \leq i \leq k$. We continue to assume that $\mathscr{S}$ has a strong decomposition property, but now with respect to $\{\mathscr{S}_i\}_{i=1}^k$. Recall that $\hat{z}_i$ denotes the result of deleting the ith component from $z \in \mathbb{R}^n$ and that $f_{\hat{z}_i}$ denotes the function on $\mathbb{R}^{n_i}$ obtained from f by fixing the other components equal to $\hat{z}_i$.

Definition 3.5.7. We say that f is in $\mathrm{BLO}_{\mathrm{rec},\mathscr{S}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_k})$ if

$$\|f\|_{\mathrm{BLO}_{\mathrm{rec},\mathscr{S}}} := \sup_{S \in \mathscr{S}} \fint_S [f(z) - \max_{1 \le i \le k} \{\operatorname*{ess\,inf}_{S_i} f_{\hat{z}_i}\}]\, dz < \infty, \tag{3.10}$$

where $S = S_1 \times S_2 \times \ldots \times S_k$.

Proposition 3.5.8. $\mathrm{BLO}_{rec,\mathscr{S}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_k})$ *is an upper semilattice with*

$$\|\max(f,g)\|_{\mathrm{BLO}_{rec,\mathscr{S}}} \le \|f\|_{\mathrm{BLO}_{rec,\mathscr{S}}} + \|g\|_{\mathrm{BLO}_{rec,\mathscr{S}}}.$$

Proof. The proof is the same as that of Proposition 3.2.4. □

The following generalisation of Example 3.5.6 illustrates Proposition 3.5.8.

Example 3.5.9. If f is a function of some variable z_{i_1} alone, that is $f(z) = F(z_{i_1})$ for some function F, and g is a function of z_{i_2} alone, that is $g(z) = G(z_{i_2})$ for some function G, then, for any shape S,

$$f(z) - \max_{1 \le i \le k} \{\operatorname*{ess\,inf}_{S_i} f_{\hat{z}_i}\} = F(z_{i_1}) - \max(F(z_{i_1}), \operatorname*{ess\,inf}_{S_{i_1}} F) = 0$$

for almost every $z_{i_1} \in S_{i_1}$ and

$$g(z) - \max_{1 \le i \le k} \{\operatorname*{ess\,inf}_{S_i} g_{\hat{z}_i}\} = G(z_{i_2}) - \max(G(z_{i_2}), \operatorname*{ess\,inf}_{S_{i_2}} G) = 0$$

for almost every $z_{i_2} \in S_{i_2}$. Therefore, $\|f\|_{\mathrm{BLO}_{\mathrm{rec},\mathscr{S}}} = \|g\|_{\mathrm{BLO}_{\mathrm{rec},\mathscr{S}}} = 0$.

Meanwhile, if $h(z) = \max(f(z), g(z)) = \max(F(z_{i_1}), G(z_{i_2}))$, then for any shape S,

$$h(z) - \max_{1 \le i \le k} \{\operatorname*{ess\,inf}_{S_i} h_{\hat{z}_i}\} = \max(F(z_{i_1}), G(z_{i_2})) - \max(F(z_{i_1}), G(z_{i_2})) = 0,$$

and so $\|h\|_{\mathrm{BLO}_{\mathrm{rec},\mathscr{S}}} = 0$.

This example shows that taking functions of one variable and the maximum of two such functions yields examples of zero elements of rectangular BLO. Other sources of examples come from the following two propositions.

Proposition 3.5.10. *If* $f \in \bigcup_{i=1}^k \mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n)$, *then* $f \in \mathrm{BLO}_{rec,\mathscr{S}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_k})$ *with*

$$\|f\|_{\mathrm{BLO}_{rec,\mathscr{S}}} \le \min_{1 \le i \le k} \{\|f\|_{\mathrm{BLO}_{\mathscr{S}_i}}\}.$$

Proof. Write

$$\fint_S [f(z) - \max_{1\le i\le k}\{\operatorname*{ess\,inf}_{S_i} f_{\hat{z}_i}\}]\,dz \le \fint_S [f(z) - \operatorname*{ess\,inf}_{S_i} f_{\hat{z}_i}]\,dz$$
$$\le \fint_{\hat{S}_i} \|f_{\hat{z}_i}\|_{\mathrm{BLO}_{\mathscr{S}_i}}\,dz \le \|f\|_{\mathrm{BLO}_{\mathscr{S}_i}}$$

for each $1 \le i \le k$, where $\hat{S}_i$ is the result of deleting S_i from S. From this it follows that $\mathrm{BLO}_{\mathscr{S}_i}(\mathbb{R}^n) \subset \mathrm{BLO}_{\mathrm{rec},\mathscr{S}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_k})$ for $1 \le i \le k$. □

Proposition 3.5.11. *If $f \in \mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$, then $f \in \mathrm{BLO}_{rec,\mathscr{S}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k})$ with*

$$\|f\|_{\mathrm{BLO}_{rec,\mathscr{S}}} \le \|f\|_{\mathrm{BLO}_{\mathscr{S}}}.$$

Proof. This follows from the fact that

$$\operatorname*{ess\,inf}_S f \le \max_{1\le i\le k}\{\operatorname*{ess\,inf}_{S_i} f_{\hat{z}_i}\}$$

holds almost everywhere. Therefore,

$$\fint_S [f(z) - \max_{1\le i\le k}\{\operatorname*{ess\,inf}_{S_i} f_{\hat{z}_i}\}]\,dz \le \fint_S [f(z) - \operatorname*{ess\,inf}_S f]\,dz \le \|f\|_{\mathrm{BLO}_{\mathscr{S}}}.$$

□

Remark 3.5.12. In the case where each $\mathscr{S}_i$ contains a differentiation basis that differentiates $L^1_{\mathrm{loc}}(\mathbb{R}^{n_i})$, another proof is possible using Theorem 3.4.6 and Proposition 3.5.10, by analogy with Remark 3.5.4.

One way of generating a function in $\mathrm{BLO}_{\mathscr{S}}(\mathbb{R}^n)$ is demonstrated in the following example. This allows us to exhibit a function in $\mathrm{BLO}_{\mathrm{rec},\mathscr{S}}$ with non-zero norm.

Example 3.5.13. Let $g(x) \in \mathrm{BLO}(\mathbb{R})$ and then consider $f(x,y) = g(x-y)$. Writing $\mathcal{I}_x$ for the basis of intervals in the x-direction and analogously for $\mathcal{I}_y$, we have that $f \in \mathrm{BLO}_{\mathcal{I}_x}(\mathbb{R}^2) \cap \mathrm{BLO}_{\mathcal{I}_y}(\mathbb{R}^2)$. From Theorem 3.4.6, it follows that $f \in \mathrm{BLO}_{\mathcal{R}}(\mathbb{R}^2)$. One can check that $\|f\|_{\mathrm{BLO}_{\mathcal{R}}} \le \|g\|_{\mathrm{BLO}}$.

In particular, $f(x,y) = -\log|x-y|$ is in $\mathrm{BLO}_{\mathcal{R}}(\mathbb{R}^2)$ and has non-zero norm. Regarding $\mathbb{R}^2$ as $\mathbb{R}\times\mathbb{R}$ and taking the rectangle $[0,1]\times[1,2]$, one can compute $\|f\|_{\mathrm{BLO}_{\mathrm{rec},\mathcal{R}}} \ge 2\log 2 - 1$.

3.6 Strong product bases

Write $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_k}$ for $2 \le k \le n$ where $n_1 + n_2 + \ldots + n_k = n$. For $z \in \mathbb{R}^n$, denote by $z_i \in \mathbb{R}^{n_i}$ its ith coordinate, according to this decomposition.

Let $\mathscr{S}$ be a basis for $\mathbb{R}^n$ and $\mathscr{S}_i$ be a basis for $\mathbb{R}^{n_i}$ for each $1 \le i \le k$. Assume that $\mathscr{S}$ has the strong decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$, that each $\mathscr{S}_i$ is an engulfing basis with constants c_d^i and c_e^i, and that each $\mathscr{S}_i$ contains a differentiation basis that differentiates $L^1_{\text{loc}}(\mathbb{R}^{n_i})$. We will call such a basis a strong product basis.

Theorem 3.6.1. *Let $\mathscr{S}$ be a strong product basis such that there exists a $p \in (1, \infty)$ for which $M_{\mathscr{S}}$ is bounded on $L^p(\mathbb{R}^n)$ with norm A_p. If $f \in \mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)$, then*

$$\fint_S M_{\mathscr{S}} f(z)\, dz \le c\, \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \fint_S \max_{1 \le i \le k} \left\{ \operatorname*{ess\,inf}_{S_i} (M_{\mathscr{S}} f)_{\hat{z}_i} \right\} dz, \tag{3.11}$$

for all $S \in \mathscr{S}$, where c is a constant depending on p, n, k, A_p, $\{c_d^i\}_{i=1}^k$, $\{c_e^i\}_{i=1}^k$. Assuming that the right-hand side of (3.11) is finite for every shape $S \in \mathscr{S}$, it follows that $M_{\mathscr{S}} f$ is finite almost everywhere and $M_{\mathscr{S}} f \in \mathrm{BLO}_{rec,\mathscr{S}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k})$ with

$$\|M_{\mathscr{S}} f\|_{\mathrm{BLO}_{rec,\mathscr{S}}} \le c\, \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}.$$

Moreover, if $\mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n) \cong \mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$, then $\|M_{\mathscr{S}} f\|_{\mathrm{BLO}_{rec,\mathscr{S}}} \le C\, \|f\|_{\mathrm{BMO}_{\mathscr{S}}}$ holds for all $f \in \mathrm{BMO}_{\mathscr{S}}(\mathbb{R}^n)$ for which $M_{\mathscr{S}} f$ is finite almost everywhere.

Proof. Fix $f \in \mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)$ and $S \in \mathscr{S}$. We write $S = S_1 \times S_2 \times \ldots \times S_k$, where $S_i \in \mathscr{S}_i$. Here we are using the weak decomposition property of $\mathscr{S}$. As each $\mathscr{S}_i$ is an engulfing basis, each S_i has associated to it a shape $\widetilde{S}_i \in \mathscr{S}_i$ as in Definition 3.3.1, and so we write $\widetilde{S}$ for the shape in $\mathscr{S}$ formed by $\widetilde{S}_1 \times \widetilde{S}_2 \times \ldots \times \widetilde{S}_k$. Here we are using the strong decomposition property of $\mathscr{S}$.

For $I \subset \{1, 2, \ldots, k\}$, we denote by I^c the set $\{1, 2, \ldots, k\} \setminus I$. For a fixed shape $S \in \mathscr{S}$ and $I \subset \{1, 2, \ldots, k\}$, consider the family of shapes

$$\mathcal{F}_I(S) = \{T \in \mathscr{S} : T \cap S \ne \emptyset \text{ and } T_i \cap \widetilde{S}_i^c \ne \emptyset \Leftrightarrow i \in I\}. \tag{3.12}$$

This is the family of shapes that intersect S and "stick out" of $\widetilde{S}$ in the directions corresponding to I. The notation indicating dependence on S may be suppressed when it has been fixed and there is little possibility of confusion.

Let x denote the I-coordinates of z, that is those coordinates $\{z_i \in \mathbb{R}^{n_i} : i \in I\}$, and y denote the I^c-coordinates of z, that is $\{z_i \in \mathbb{R}^{n_i} : i \in I^c\}$. When $|I| = 1$, in which case y is all coordinates except $z_i \in \mathbb{R}^{n_i}$ for some $1 \leq i \leq k$, we write $y = \hat{z}_i$ as in previous sections.

Consider the basis $\mathscr{S}_I$ in $X = \prod_{i \in I} \mathbb{R}^{n_i}$ defined by

$$\mathscr{S}_I = \prod_{i \in I} \mathscr{S}_i.$$

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, define

$$\|f\|_{\text{BMO}^p_{\mathscr{S}_I}(\mathbb{R}^n)} = \sup_y \|f_y\|_{\text{BMO}^p_{\mathscr{S}_I}(X)}.$$

Applying Theorem 3.4.3 to $\mathscr{S}_I$ and then to $\mathscr{S}$ which has the strong decomposition property with respect to $\{\mathscr{S}_i\}_{i=1}^k$, we have

$$\|f\|_{\text{BMO}^p_{\mathscr{S}_I}(\mathbb{R}^n)} \leq \sup_y \sum_{i \in I} \|f_y\|_{\text{BMO}^p_{\mathscr{S}_i}(X)} \leq \sum_{i \in I} \|f\|_{\text{BMO}^p_{\mathscr{S}_i}(\mathbb{R}^n)} \leq c_k \|f\|_{\text{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)}, \tag{3.13}$$

where $c_k = 2^{k-1}k$.

Writing

$$M_I f(z) = \sup \left\{ \fint_T |f| : T \in \mathcal{F}_I(S) \,\text{and}\, T \ni z \right\},$$

we have that

$$M_{\mathscr{S}} f(z) = \max_{I \subset \{1,2,\ldots,k\}} M_I f(z)$$

for $z \in S$. As such, we consider each $M_I f$ separately.

<u>Case $I = \emptyset$ or $I^c = \emptyset$</u>: Here $\mathcal{F}_I$ consists of those shapes that do not leave $\widetilde{S}$ in any direction when $I = \emptyset$, and those shapes that leave $\widetilde{S}$ in every direction when $I^c = \emptyset$. These two cases are treated together as the proof proceeds as in the proof of Theorem 3.3.2.

Write $f = g_I + h_I$, where $g_I = (f - f_{\widetilde{S}})\chi_{\widetilde{S}}$ and $h_I = f_{\widetilde{S}}\chi_{\widetilde{S}} + f\chi_{\widetilde{S}^c}$. Then, by the boundedness of $M_{\mathscr{S}}$ on $L^p(\mathbb{R}^n)$,

$$\fint_S M_{\mathscr{S}} g_I \leq \frac{1}{|S|^{1/p}} \|M_{\mathscr{S}} g_I\|_{L^p} \leq \frac{A_p}{|S|^{1/p}} \|g_I\|_{L^p} \leq A_p c_d^{1/p} \left(\fint_{\widetilde{S}} |f - f_{\widetilde{S}}|^p \right)^{1/p},$$

where $c_d = c_d^1 \times c_d^2 \times \cdots \times c_d^k$. Thus,

$$\fint_S M_I g_I \leq \fint_S M_{\mathscr{S}} g_I \leq A_p c_d^{1/p} \|f\|_{\text{BMO}^p_{\mathscr{S}}}. \tag{3.14}$$

Fix a point $z_0 \in S$ and, for the moment, a shape $T \in \mathcal{F}_I$ such that $T \ni z_0$. When $I = \emptyset$, this implies that $T \subset \widetilde{S}$ and so

$$\fint_T |h_I| \leq \fint_{\widetilde{S}} |f| \leq M_{\mathscr{S}} f(z)$$

for every $z \in \widetilde{S}$. In particular, this is true for every $z \in S$ and so

$$\fint_T |h_I| \leq \operatorname*{ess\,inf}_S M_{\mathscr{S}} f.$$

Hence, we have the pointwise bound

$$M_I h_I(z_0) \leq \operatorname*{ess\,inf}_S M_{\mathscr{S}} f. \tag{3.15}$$

When $I^c = \emptyset$, for each $1 \leq i \leq k$ there is a shape $\overline{T}_i \in \mathscr{S}_i$ containing T_i and $\widetilde{S}_i$ such that $|\overline{T}_i| \leq c_e^i |T_i|$. We then create the shape $\overline{T} = \overline{T}_1 \times \overline{T}_2 \times \ldots \times \overline{T}_k$. This satisfies $\overline{T} \supset T \cup \widetilde{S}$ and $|\overline{T}| \leq c_e |T|$, where $c_e = c_e^1 \times c_e^2 \times \ldots \times c_e^k$, and so

$$\begin{aligned}
\fint_T |h_I - f_{\overline{T}}| \leq c_e \fint_{\overline{T}} |h_I - f_{\overline{T}}| &= \frac{c_e}{|\overline{T}|} \left[|\widetilde{S}| |f_{\widetilde{S}} - f_{\overline{T}}| + \int_{\overline{T} \cap \widetilde{S}^c} |f - f_{\overline{T}}| \right] \\
&\leq \frac{c_e}{|\overline{T}|} \left[\int_{\widetilde{S}} |f - f_{\overline{T}}| + \int_{\overline{T} \cap \widetilde{S}^c} |f - f_{\overline{T}}| \right] \\
&= c_e \fint_{\overline{T}} |f - f_{\overline{T}}| \leq c_e \left(\fint_{\overline{T}} |f - f_{\overline{T}}|^p \right)^{1/p} \leq c_e \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}.
\end{aligned}$$

Hence,

$$\fint_T |h_I| \leq \fint_T |h_I - f_{\overline{T}}| + \fint_{\overline{T}} |f| \leq c_e \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + M_{\mathscr{S}} f(z)$$

for every $z \in \overline{T}$, in particular for every $z \in S$, and so

$$\fint_T |h_I| \leq c_e \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \operatorname*{ess\,inf}_S M_{\mathscr{S}} f.$$

Thus, we have the pointwise bound

$$M_I h_I(z_0) \leq c_e \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \operatorname*{ess\,inf}_S M_{\mathscr{S}} f. \tag{3.16}$$

Case $I \neq \emptyset, I^c \neq \emptyset$: Here the shapes in $\mathcal{F}_I$ leave $\widetilde{S}$ only in those directions corresponding to I. Write S_I for $\prod_{i \in I} S_i$ and $\widetilde{S}_I$ for $\prod_{i \in I} \widetilde{S}_i$.

Write $f = g_I + h_I$, where $g_I = (f - (f_y)_{\widetilde{S}_I}) \chi_{\widetilde{S}}$ and $h_I = (f_y)_{\widetilde{S}_I} \chi_{\widetilde{S}} + f \chi_{\widetilde{S}^c}$. Then, by the boundedness of $M_{\mathscr{S}}$ on $L^p(\mathbb{R}^n)$,

$$\fint_S M_{\mathscr{S}} g_I \leq \frac{1}{|S|^{1/p}} \|M_{\mathscr{S}} g_I\|_{L^p} \leq \frac{A_p}{|S|^{1/p}} \|g_I\|_{L^p} = A_p c_d^{1/p} \left(\fint_{\widetilde{S}} |f - (f_y)_{\widetilde{S}_I}|^p \right)^{1/p},$$

where $c_d = c_d^1 \times c_d^2 \times \cdots \times c_d^k$. As

$$\left(\fint_{\widetilde{S}} |f - (f_y)_{\widetilde{S}_I}|^p\right)^{1/p} = \left(\fint_{\widetilde{S}_{I^c}} \left(\fint_{\widetilde{S}_I} |f_y(x) - (f_y)_{\widetilde{S}_I}|^p \, dx\right) dy\right)^{1/p} \leq \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_I}(\mathbb{R}^n)},$$

we have

$$\fint_S M_I g_I \leq \fint_S M_{\mathscr{S}} g_I \leq A_p c_d^{1/p} \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_I}} \leq A_p c_d^{1/p} c_k \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)}, \tag{3.17}$$

where the last inequality follows from (3.13).

Fix a point $z_0 \in S$ and, for the moment, a shape $T \in \mathcal{F}_I$ such that $T \ni z_0$. For each $i \in I$, there is a shape $\overline{T}_i \in \mathscr{S}_i$ containing T_i and $\widetilde{S}_i$ such that $|\overline{T}_i| \leq c_e^i |T_i|$. We then create the shape $\overline{T}_I = \prod_{i \in I} \overline{T}_i$. This satisfies $\overline{T}_I \supset T_I \cup \widetilde{S}_I$ and $|\overline{T}_I| \leq c_e^I |T_I|$, where $c_e^I = \prod_{i \in I} c_e^i$. For $i \notin I$, write $\overline{T}_i = T_i$ and recall that $T_i \subset \widetilde{S}_i$. Then, we form the shape $\overline{T} = \overline{T}_1 \times \overline{T}_2 \times \cdots \times \overline{T}_k$.

Fixing $y \in T_{I^c} \subset \widetilde{S}_{I^c}$, we proceed as in the proof of Theorem 3.3.2, but work only with the directions in I:

$$\begin{aligned}
\fint_{T_I} |(h_I)_y(x) - (f_y)_{\overline{T}_I}| \, dx &\leq c_e^I \fint_{\overline{T}_I} |(h_I)_y(x) - (f_y)_{\overline{T}_I}| \, dx \\
&= \frac{c_e^I}{|\overline{T}_I|} \left[\int_{\widetilde{S}_I} |(f_y)_{\widetilde{S}_I} - (f_y)_{\overline{T}_I}| \, dx + \int_{\overline{T}_I \cap \widetilde{S}_I^c} |f_y(x) - (f_y)_{\overline{T}_I}| \, dx\right] \\
&\leq \frac{c_e^I}{|\overline{T}_I|} \left[\int_{\widetilde{S}_I} |f_y(x) - (f_y)_{\overline{T}_I}| \, dx + \int_{\overline{T}_I \cap \widetilde{S}_I^c} |f_y(x) - (f_y)_{\overline{T}_I}| \, dx\right] \\
&= c_e^I \fint_{\overline{T}_I} |f_y(x) - (f_y)_{\overline{T}_I}| \, dx \leq c_e^I \left(\fint_{\overline{T}_I} |f_y(x) - (f_y)_{\overline{T}_I}|^p \, dx\right)^{1/p} \\
&\leq c_e^I \|f\|_{\mathrm{BMO}^p_{\mathscr{S}_I}(\mathbb{R}^n)} \leq c_e^I c_k \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}(\mathbb{R}^n)}
\end{aligned}$$

by (3.13). Thus,

$$\fint_{T_I} |(h_I)_y(x)| \, dx \leq \fint_{T_I} |(h_I)_y(x) - (f_y)_{\overline{T}_I}| \, dx + \fint_{\overline{T}_I} |f_y(x)| \, dx \leq c_e^I c_k \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \fint_{\overline{T}_I} |f_y(x)| \, dx.$$

From here, integrating over $y \in T_{I^c}$, we have that

$$\fint_T |h_I| \leq c_e^I c_k \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \fint_{\overline{T}} |f| \leq c_e^I c_k \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + M_{\mathscr{S}} f(z)$$

for any $z \in \overline{T}$. This is true, in particular, if the I^c coordinates of z are equal to y_0, where y_0 denotes the I^c-coordinates of z_0, and $x \in S_I$. Thus,

$$M_I h_I(z_0) \leq c_e^I c_k \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \operatorname*{ess\,inf}_{S_I} (M_{\mathscr{S}} f)_{y_0}. \tag{3.18}$$

Combining (3.14) and (3.17) yields

$$\begin{aligned}\fint_S \max_I M_I g_I \leq \sum_I \fint_S M_I g_I \leq 2A_p c_d^{1/p} \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \sum_{I\neq\emptyset, I^c\neq\emptyset} A_p c_d^{1/p} c_k \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} \\ \leq c\, \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}}.\end{aligned} \tag{3.19}$$

We combine (3.15), (3.16), and (3.18) to yield

$$\fint_S \max_I M_I h_I \leq c\, \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \fint_S \max\left\{ \operatorname*{ess\,inf}_S M_{\mathscr{S}} f,\ \max_{I\neq\emptyset, I^c\neq\emptyset} \{\operatorname*{ess\,inf}_{S_I} (M_{\mathscr{S}} f)_y\}\right\}.$$

Since the infimum can only grow as we fix more variables, the inequality

$$\max\left\{ \operatorname*{ess\,inf}_S M_{\mathscr{S}} f,\ \max_{I\neq\emptyset, I^c\neq\emptyset} \{\operatorname*{ess\,inf}_{S_I} (M_{\mathscr{S}} f)_y\}\right\} \leq \max_{1\leq i\leq k} \{\operatorname*{ess\,inf}_{S_i} (M_{\mathscr{S}} f)_{\hat{z}_i}\},$$

holds almost everywhere in S, and so

$$\fint_S \max_I M_I h_I(z)\, dz \leq c\, \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \fint_S \max_{1\leq i\leq k} \{\operatorname*{ess\,inf}_{S_i} (M_{\mathscr{S}} f)_{\hat{z}_i}\}\, dz. \tag{3.20}$$

Therefore, (3.19) and (3.20) imply that

$$\begin{aligned}\fint_S M_{\mathscr{S}} f(z)\, dz = \fint_S \max_I M_I f(z)\, dz &\leq \fint_S \max_I M_I (g_I(z) + h_I(z))\, dz \\ &\leq \fint_S \max_I M_I g_I(z)\, dz + \fint_S \max_I M_I h_I(z)\, dz \\ &\leq c\, \|f\|_{\mathrm{BMO}^p_{\mathscr{S}}} + \fint_S \max_{1\leq i\leq k} \{\operatorname*{ess\,inf}_{S_i} (M_{\mathscr{S}} f)_{\hat{z}_i}\}\, dz.\end{aligned}$$

□

We end by giving two examples of bases that satisfy the conditions of Theorem 3.6.1.

Example 3.6.2. The first example, which is in many ways the model case and the motivation for studying this problem, is $\mathcal{R}$. When $k = n$, and so $n_i = 1$ for every $1 \leq i \leq n$, the basis $\mathcal{R}$ has the strong decomposition property with respect to $\{\mathcal{I}_i\}_{i=1}^n$, where $\mathcal{I}_i$ is the basis of all intervals in $\mathbb{R}$. Each basis $\mathcal{I}_i$ is both a differentiation basis that differentiates $L^1_{\mathrm{loc}}(\mathbb{R})$ and an engulfing basis (one can take $c_d = 2$ and $c_e = 4$). Moreover, the strong maximal function, M_s, is well known to be bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ ([52]). The anisotropic version of the John-Nirenberg inequality due to Korenovskii ([59, 60]) implies that $\mathrm{BMO}^p_{\mathcal{R}}(\mathbb{R}^n) \cong \mathrm{BMO}_{\mathcal{R}}(\mathbb{R}^n)$ for all $1 < p < \infty$. Therefore, M_s maps $\mathrm{BMO}_{\mathcal{R}}(\mathbb{R}^n)$ to $\mathrm{BLO}_{\mathrm{rec},\mathcal{R}}(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R})$.

Example 3.6.3. A second example is when $k = 2$. Denote by $\mathcal{B}_{n-1}$ the basis of all Euclidean balls in $\mathbb{R}^{n-1}$ and by $\mathcal{I}$ the basis of intervals in $\mathbb{R}$. The differentiation and engulfing properties of these bases are known. In $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, define a cylinder to be the product of a ball $B \in \mathcal{B}_{n-1}$ and an interval $I \in \mathcal{I}$. The basis of all such cylinders $\mathcal{C}$ has the strong decomposition property with respect to $\{\mathcal{B}_{n-1}, \mathcal{I}\}$.

By comparing (in the sense of Definition 2.2 in [26]) these shapes to a family of rectangles, the $L^p(\mathbb{R}^n)$ boundedness of $M_{\mathcal{C}}$ for any $1 < p < \infty$ follows from that of M_s. Moreover, it can be shown along the lines of the work of Korenovskii [59, 60] that the John-Nirenberg inequality holds for $\mathcal{C}$, and so $\mathrm{BMO}^p_{\mathcal{C}}(\mathbb{R}^n) \cong \mathrm{BMO}_{\mathcal{C}}(\mathbb{R}^n)$ holds for all $1 < p < \infty$. Therefore, $M_{\mathcal{C}}$ maps $\mathrm{BMO}_{\mathcal{C}}(\mathbb{R}^n)$ to $\mathrm{BLO}_{\mathrm{rec},\mathcal{C}}(\mathbb{R}^{n-1} \times \mathbb{R})$.

Chapter 4

Rearrangement inequalities on spaces defined by mean oscillation

4.1 Introduction

Given a measurable function f on $\mathbb{R}^n$, its decreasing rearrangement is the unique decreasing function f^* on $\mathbb{R}_+ = (0, \infty)$ that is right-continuous and equimeasurable with $|f|$. The concept of the decreasing rearrangement of a function is an important tool in interpolation theory and the study of function spaces. In particular, there is interest in function spaces that are invariant under equimeasurable rearrangements (see [4]). The Lebesgue spaces L^p, $1 \leq p \leq \infty$, are one such example as the norms that define inclusion in these spaces depend only on the distribution of a function.

In 1961, John and Nirenberg introduced the space BMO of locally integrable functions of bounded mean oscillation on cubes ([54]). This space has proven useful as a replacement for L^∞ in contexts such as singular integral operators and Sobolev embedding theorems. It is easy to see, however, that BMO is not invariant under equimeasurable rearrangements[1]. As such, it is an interesting question to ask whether the decreasing rearrangement of a BMO function is in BMO. Throughout this paper, cubes will always be taken to have sides parallel to the axes. In one dimension, cubes are intervals.

In this direction, the work of Bennett-DeVore-Sharpley in [3] implies that the decreasing

[1]See Section III.1 in the appendix.

rearrangement operator $f \mapsto f^*$ is bounded from $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}_+)$ with $\|f^*\|_{\mathrm{BMO}} \leq 2^{n+5}\|f\|_{\mathrm{BMO}}$. Moreover, if one is willing to assume that $f_{Q_0} = 0$ (see Property (O1) in the preliminaries), then $f \mapsto f^*$ is bounded from $\mathrm{BMO}(Q_0)$ to $\mathrm{BMO}(0, |Q_0|)$ with $\|f^*\|_{\mathrm{BMO}} \leq 2^{n+5}\|f\|_{\mathrm{BMO}}$ for any cube $Q_0 \subset \mathbb{R}^n$.

When $n = 1$, the work of Klemes [57] along with subsequent steps taken by Korenovskii [58] shows that $f \mapsto f^*$ is bounded from $\mathrm{BMO}(I_0)$ to $\mathrm{BMO}(0, |I_0|)$ with the sharp inequality $\|f^*\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}}$ for any interval $I_0 \subset \mathbb{R}$.

Later work of Korenovskii ([59]) generalises this to $\mathrm{BMO}_{\mathcal{R}}$, the anisotropic BMO space (also called the strong BMO space, see [26]) of locally integrable functions of bounded mean oscillation on rectangles. As with cubes, rectangles will always have sides parallel to the axes. Korenovskii shows that $f \mapsto f^*$ is bounded from $\mathrm{BMO}_{\mathcal{R}}(R_0)$ to $\mathrm{BMO}(0, |R_0|)$ with the sharp inequality $\|f^*\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}_{\mathcal{R}}}$ for any rectangle $R_0 \subset \mathbb{R}^n$.

In [26], Dafni and Gibara introduced the space $\mathrm{BMO}_{\mathscr{S}}$ of locally integrable functions of bounded mean oscillation, replacing cubes by shapes S coming from a fixed basis $\mathscr{S}$. Section 4.2 presents the relevant definitions. In view of the known results on the boundedness of the decreasing rearrangement on BMO with respect to intervals, cubes, or rectangles, it makes sense to ask

Question 1. *For what bases $\mathscr{S}$ is the decreasing rearrangement bounded on* BMO*?*

This question is partially addressed in Section 4.3. A general theorem, inspired by the proof of Klemes ([57]), shows boundedness for a number of examples. In particular, under some assumptions, we are able to obtain a result for balls in a metric measure space. We are also able to obtain a dimension-free bound for a special family of rectangles with bounded eccentricity and use this to obtain the following improvement for the case of cubes (see Theorem 4.3.10):

Theorem III. *The decreasing rearrangement is bounded from* $\mathrm{BMO}(\mathbb{R}^n)$ *to* $\mathrm{BMO}(\mathbb{R}_+)$ *with*

$$\|f^*\|_{\mathrm{BMO}} \leq 2^{\frac{n+3}{2}}\|f\|_{\mathrm{BMO}}.$$

By its very nature, the decreasing rearrangement operator $f \mapsto f^*$ is non-local and nonlinear. This lack of linearity means that boundedness on a function space does not imply continuity. This leads to the following natural question.

Question 2. *Is the decreasing rearrangement continuous on* BMO*? If not, what is a description of the subset on which it is?*

In Section 4.4, an example is given showing that the decreasing rearrangement can fail to be continuous on BMO. Considerations are then turned to the important subspace of functions of vanishing mean oscillation, VMO. It is shown that (see Theorems 4.4.6 and 4.4.9)

Theorem IV. *The decreasing rearrangement is bounded from* $\mathrm{VMO}(Q_0)$ *to* $\mathrm{VMO}(0, |Q_0|)$*, where* Q_0 *is* $\mathbb{R}^n$ *or a finite cube. Moreover, it is continuous when* Q_0 *is finite, for functions that are normalized to have mean zero.*

Another important rearrangement is the symmetric decreasing rearrangement, which associated a measurable function f on $\mathbb{R}^n$ to a measurable function Sf on $\mathbb{R}^n$ that is radially decreasing in such a way that $|f|$ and Sf are equimeasurable. This rearrangement is important in the study of PDEs and many geometric problems (see [10]). We may define the symmetric decreasing rearrangement by means of the formula $Sf(x) = f^*(\omega_n|x|^n)$ for $x \in \mathbb{R}^n$, where ω_n denotes the measure of the unit ball in $\mathbb{R}^n$.

Given the immediate connection between the decreasing and the symmetric decreasing rearrangements, one may ask whether BMO-boundedness results for f^* can be transferred to Sf.

Question 3. *Is the symmetric decreasing rearrangement bounded on* BMO*?*

This question is answered in the affirmative in Section 4.5. By passing through shapes on which the mean oscillation of f^* and Sf can be directly compared (and, in fact, are found to coincide), one is able to obtain the following (see Theorem 4.5.1), where by $\mathrm{BMO}_{\mathcal{B}}$ it means the locally integrable functions of bounded mean oscillation on Euclidean balls.

Theorem V. *The symmetric decreasing rearrangement is bounded from* $\mathrm{BMO}_{\mathcal{B}}(\mathbb{R}^n)$ *to* $\mathrm{BMO}_{\mathcal{B}}(\mathbb{R}^n)$ *with*

$$\|Sf\|_{\mathrm{BMO}_{\mathcal{B}}} \leq 2^{\frac{n+3}{2}} \omega_n n^{n/2} \|f\|_{\mathrm{BMO}_{\mathcal{B}}}.$$

4.2 Preliminaries

4.2.1 Rearrangements

Let (M, μ) be a positive measure space and consider a measurable function $f : M \to \mathbb{R}$. For $\alpha \geq 0$, write $E_\alpha = \{x \in M : |f(x)| > \alpha\}$. The distribution function of f is defined as $\mu_f(\alpha) = \mu(E_\alpha)$. Note that $\mu_f : [0, \infty) \to [0, \infty]$ is decreasing and right-continuous.

We say that a measurable function f is rearrangeable [2] if μ_f satisfies $\mu_f(\alpha) \to 0$ as $\alpha \to \infty$. For such functions,

Definition 4.2.1. The decreasing rearrangement of f is the function $f^* : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$f^*(s) = \inf\{\alpha \geq 0 : \mu_f(\alpha) \leq s\}.$$

The condition that f is rearrangeable guarantees that the set $\{\alpha \geq 0 : \mu_f(\alpha) \leq s\}$ is non-empty for $s > 0$ and so f^* is finite on its domain. The set $\{\alpha \geq 0 : \mu_f(\alpha) = 0\}$, however, can be empty. If f is bounded, then f^* tends to $\|f\|_{L^\infty}$ as $s \to 0^+$; otherwise, f^* is unbounded at the origin. As is the case with the distribution function, f^* is decreasing and right-continuous.

If $\mu(M) < \infty$, it follows from the definition that $f^*(s) = 0$ for all $s \geq \mu(M)$, and so f^* is supported on $(0, \mu(M))$. Thus we write $M^* = (0, \mu(M))$ and have $f^* : M^* \to \mathbb{R}_+$.

The following standard properties of the decreasing rearrangement will be used throughout this paper.[3] The notation $|\cdot|$ will be used throughout this chapter to denote Lebesgue measure.

Property (R1) The functions f^* and $|f|$ are equimeasurable; that is, for all $\alpha \geq 0$, $\mu_f(\alpha) = m_{f^*}(\alpha)$, where $m_{f^*}(\alpha) = |\{s \in M^* : f^*(s) > \alpha\}|$.

Property (R2) For any $\alpha \geq 0$ such that $\mu_f(\alpha) < \infty$ and $\beta > \alpha$,

$$\mu(\{x \in M : \alpha < |f(x)| < \beta\}) = |\{s \in M^* : \alpha < f^*(s) < \beta\}|.$$

[2]See Section III.2 in the appendix.

[3]See Proposition 5.3.10-5.3.13 in Appendix III.

Property (R3) The decreasing rearrangement $f \mapsto f^*$ is an isometry from $L^p(M)$ to $L^p(M^*)$ for all $1 \leq p \leq \infty$. Furthermore, it is non-expansive, thus continuous.

Property (R4) (Elementary Hardy-Littlewood inequality) For any measurable set $A \subset M$,

$$\int_A |f| \leq \int_{A^*} f^*,$$

where $A^* = (0, \mu(A))$.

The reader is invited to see [81] for more details on the decreasing rearrangement.

4.2.2 Bounded mean oscillation

Let Q_0 be an open (not necessarily finite) cube in $\mathbb{R}^n$. In particular, Q_0 may be $\mathbb{R}_+$. A shape is taken to mean an open set $S \subset \mathbb{R}^n$ such that $0 < |S| < \infty$. A basis of shapes in Q_0, then, is a collection $\mathscr{S}$ of shapes $S \subset Q_0$ forming a cover of Q_0.

Common examples of bases are the collections of all Euclidean balls, $\mathcal{B}$, all cubes with sides parallel to the axes, $\mathcal{Q}$, and all rectangles with sides parallel to the axes, $\mathcal{R}$. In one dimension, these three choices degenerate to the collection of all (finite) open intervals, $\mathcal{I}$.

For a shape $S \subset Q_0$ and a real-valued function on Q_0 that is integrable on a shape S, its mean oscillation is defined as

$$\Omega(f, S) = \frac{1}{|S|} \int_S |f - f_S| = \fint_S |f - f_S|,$$

where f_S is the mean of f on S.

Some properties of mean oscillations are the following (see [26] for the proofs of all but (O4)):

Property (O1) For any constant α and shape S, $\Omega(f + \alpha, S) = \Omega(f, S)$.

Property (O2) For any shape S, $\Omega(|f|, S) \leq 2\Omega(f, S)$.

Property (O3) For any shape S,

$$\Omega(f, S) = \frac{2}{|S|} \int_S (f - f_S)_+ = \frac{2}{|S|} \int_S (f_S - f)_+,$$

where $y_+ = \max(y, 0)$.

Property (O4) [4] For any shape S, if $\widetilde{S}$ is another shape such that $\widetilde{S} \subset S$ and $|S| \le c|\widetilde{S}|$ for some constant c, then

$$\Omega(f, \widetilde{S}) \le c\,\Omega(f, S).$$

Property (O5) For any shape S,

$$\inf_{\alpha} \fint_S |f - \alpha| = \fint_S |f - m|,$$

where the infimum is taken over all constants α and m is a median of f on S (that is, a (possibly non-unique) number such that $|\{x \in S : f(x) > m\}| \le \frac{1}{2}|S|$ and $|\{x \in S : f(x) < m\}| \le \frac{1}{2}|S|$).

Property (O6) For any shape S,

$$\Omega(\widetilde{f}, S) \le \Omega(f, S),$$

where $\widetilde{f}$ is a truncation; i.e. $\widetilde{f} = \min(\max(f, \alpha), \beta)$ for some $-\infty \le \alpha < \beta \le \infty$.

Definition 4.2.2. We say that a function satisfying $f \in L^1(S)$ for all $S \in \mathscr{S}$ is in $\mathrm{BMO}_{\mathscr{S}}(Q_0)$ if

$$\|f\|_{\mathrm{BMO}_{\mathscr{S}}} = \sup \Omega(f, S) < \infty, \tag{4.1}$$

where the supremum is taken over all shapes $S \in \mathscr{S}$.

The notation $\mathrm{BMO}(Q_0)$ will be reserved for the case $\mathscr{S} = \mathcal{Q}$. Note that because of Property (O1), elements of $\mathrm{BMO}_{\mathscr{S}}(Q_0)$ can be considered modulo constants.

We collect here some properties of BMO functions that will prove to be useful in subsequent sections.

Property (B1) [5] If $|Q_0| < \infty$ and $f \in \mathrm{BMO}(Q_0)$, then $f - f_{Q_0} \in L^1(Q_0)$ with $\|f - f_{Q_0}\|_{L^1(Q_0)} \le |Q_0| \|f\|_{\mathrm{BMO}(Q_0)}$.

Property (B2) When $Q_0 = \mathbb{R}_+$ and f is monotone decreasing,

$$\|f\|_{\mathrm{BMO}} = \sup_{t>0} \Omega(f, (0, t)).$$

[4]See Proposition 5.3.14 in Appendix III.
[5]See Proposition 5.3.15 in Appendix III.

Introduced by Coifman-Rochberg ([18]), $\mathrm{BLO}(Q_0)$ is the class of $f \in L^1_{\mathrm{loc}}(Q_0)$ such that

$$\sup_Q ⨍_Q [f - \operatorname*{ess\,inf}_Q f] < \infty,$$

where the supremum is taken over all cubes $Q \subset Q_0$. It is a strict subset of $\mathrm{BMO}(Q_0)$ and it can easily be shown that it is not closed under multiplication by a negative scalar. For non-increasing functions, however, being in $\mathrm{BMO}(\mathbb{R}_+)$ is equivalent to being in $\mathrm{BLO}(\mathbb{R}_+)$. As such, any statement about $f^* \in \mathrm{BMO}(\mathbb{R}_+)$ can be interpreted in the stronger sense that $f^* \in \mathrm{BLO}(\mathbb{R}_+)$.

For a reference on BMO functions, see [60].

4.2.3 Vanishing mean oscillation

An important subspace of $\mathrm{BMO}(Q_0)$ is the space of functions of vanishing mean oscillation, $\mathrm{VMO}(Q_0)$, originally defined by Sarason in [77]. Let $\delta(\cdot)$ denote the Euclidean diameter.

Definition 4.2.3. We say that a function $f \in \mathrm{BMO}(Q_0)$ is in $\mathrm{VMO}(Q_0)$ if

$$\lim_{t \to 0^+} \sup_{\delta(Q) \le t} \Omega(f, Q) = 0, \tag{4.2}$$

where the supremum is taken over all cubes Q with diameter at most t.

Notice that by the geometry of cubes, having vanishing diameter is equivalent to having vanishing measure. As such, the supremum in Definition 4.2.3 could be taken over all cubes Q with measure at most t.

The space $\mathrm{VMO}(Q_0)$ often plays the role of the continuous functions in $\mathrm{BMO}(Q_0)$. In fact, $\mathrm{VMO}(Q_0)$ is the closure of $C_u(Q_0) \cap \mathrm{BMO}(Q_0)$ in the $\mathrm{BMO}(Q_0)$ norm, where $C_u(Q_0)$ is the space of uniformly continuous functions on Q_0. For $\mathrm{VMO}(\mathbb{R}^n)$, there is another characterisation as the subset of $\mathrm{BMO}(\mathbb{R}^n)$ on which translations are continuous with respect to the $\mathrm{BMO}(Q_0)$ norm.

In the case when Q_0 is unbounded, note that there is also a strictly smaller VMO space, sometimes denoted $\mathrm{VMO}(Q_0)$ and other times denoted $\mathrm{CMO}(Q_0)$ (see [7, 85]).

When considering a more general basis than $\mathcal{Q}$, a naive generalisation of VMO that will suit our purposes is the space of functions that have vanishing mean oscillation with respect to measure:

Definition 4.2.4. We say that a function $f \in \mathrm{BMO}_{\mathscr{S}}(Q_0)$ is in $\mathrm{VMO}_{\mathscr{S}}(Q_0)$ if

$$\lim_{t \to 0^+} \sup_{|S| \leq t} \Omega(f, S) = 0, \tag{4.3}$$

where the supremum is taken over all shapes S of measure at most t.

Again, the notation $\mathrm{VMO}(Q_0)$ will be reserved for the case $\mathscr{S} = \mathcal{Q}$.

This differs from the classical definition of VMO in an important way. It usually involves a modulus of continuity that looks at vanishing diameter. Of course, for bases such as $\mathcal{Q}$ and $\mathcal{B}$, vanishing diameter is the same as vanishing measure. For general bases, however, this is not true: vanishing diameter is strictly stronger than vanishing measure. Consider $\mathcal{R}$, in which a sequence of rectangles can all have the same diameter but have measure tending to zero.

Even worse, there are bases for which the condition (4.3) holds vacuously for any $f \in \mathrm{BMO}_{\mathscr{S}}(Q_0)$ because there are no shapes of arbitrarily small measure. An example would be the basis of cubes with sidelength bounded below by some constant.

One way to guarantee that Definition 4.2.4 is non-trivial is by assuming that $\mathscr{S}$ is a differentiation basis; that is, for each $x \in \mathbb{R}^n$ there exists a sequence of shapes $\{S_k\} \subset \mathscr{S}$ each containing x such that $\delta(S_k) \to 0$ as $k \to \infty$.

4.3 Boundedness of the decreasing rearrangement

In this section, we present a general theorem claiming the boundedness of the decreasing rearrangement on BMO under assumptions on the basis $\mathscr{S}$. Let (X, ρ, μ) be a metric measure space. That is, (X, ρ) is a metric space endowed with a non-trivial Borel measure μ. Note that the definitions in the previous section carry over in this context. Also, Properties (O1)-(O6) remain true.

In this section, we restrict our attention to $\mu(X) = \infty$ and rearrangeable $f \in \mathrm{BMO}_{\mathscr{S}}(X)$. It follows that f^* is supported on $\mathbb{R}_+$. By Property (B2), to show that $f^* \in \mathrm{BMO}(\mathbb{R}_+)$, it suffices to consider the mean oscillation of f^* on intervals of the form $(0, t)$ for $t > 0$.

Note that if f is not rearrangeable, defining f^* might lead to $f^* \equiv \infty$ for a function $f \in \mathrm{BMO}_{\mathscr{S}}(X)$. An example is $f(x) = -\log |x|$, a prototypical function in $\mathrm{BMO}(\mathbb{R}^n)$.

Lemma 4.3.1. *Let $0 \le f \in L^\infty(X)$ and $c_* \ge 1$. For any $t > 0$, set $\alpha_t = (f^*)_{(0,t)}$ and assume that there exists a countable collection of pairwise-disjoint shapes $\{S_i\}$ such that*

(i) for all i,

$$\fint_{S_i} f > \alpha_t;$$

(ii) for all i, there exists a shape $\widetilde{S}_i \supset S_i$ for which $\mu(\widetilde{S}_i) \le c_*\mu(S_i)$ and

$$\fint_{\widetilde{S}_i} f \le \alpha_t;$$

(iii) and, $f \le \alpha_t$ μ-almost everywhere on $X \setminus \bigcup \widetilde{S}_i$.

Then,

$$\Omega(f^*, (0,t)) \le c_* \sup_i \Omega(f, \widetilde{S}_i).$$

Note that the collection $\{S_i\}$ will, in general, depend on t.

Proof. Fix a $t > 0$. By the hypothesis of the theorem, we can find a countable collection of pairwise-disjoint shapes $\{S_i\}$ for which conditions (i), (ii), and (iii) hold. Denote by E the union $\bigcup S_i$ and by E_n the finite union $\bigcup_{i=1}^n \widetilde{S}_i$.

By Property (R4) and (i),

$$\fint_0^{\mu(E_n)} f^* \ge \fint_{E_n} f = \frac{1}{\mu(E_n)} \sum_i \mu(S_i) \fint_{S_i} f > \frac{1}{\mu(E_n)} \left(\sum_i \mu(S_i) \right) \alpha_t = \alpha_t = \frac{1}{t}\int_0^t f^*.$$

From the monotonicity of f^* it follows that

$$t \ge \mu(E_n) = \sum_i^n \mu(S_i). \tag{4.4}$$

As this holds for all n, $\mu(E) \le t$.

Denoting by $\widetilde{E}$ the set $\bigcup \widetilde{S}_i$, we find that

$$\int_0^t |f^* - \alpha_t| = 2\int_0^t (f^* - \alpha_t)_+ = 2\int_{\widetilde{E}} (f - \alpha_t)_+.$$

Here we use Properties (R1) and (O3), and (iii). Hence, by (ii),

$$\begin{aligned} 2\int_{\widetilde{E}} (f - \alpha_t)_+ &\le 2\sum_i \int_{\widetilde{S}_i} (f - \alpha_t)_+ \le 2\sum_i \int_{\widetilde{S}_i} (f - f_{\widetilde{S}_i})_+ \\ &= \sum_i \mu(\widetilde{S}_i) \fint_{\widetilde{S}_i} |f - f_{\widetilde{S}_i}| \le c_* \sum_i \mu(S_i)\Omega(f, \widetilde{S}_i) \le c_* \left(\sum_i \mu(S_i) \right) \sup_i \Omega(f, \widetilde{S}_i). \end{aligned}$$

Combining this with (4.4), we reach

$$\Omega(f^*,(0,t)) = \frac{1}{t}\int_0^t |f^* - \alpha_t| \leq c_* \sup_i \Omega(f, \widetilde{S}_i).$$

□

Proposition 4.3.2. *Assume that $\mu(X) = \infty$ and that $\mathscr{S}$ satisfies the hypotheses of Lemma 4.3.1 for every bounded function. If $0 \leq f \in \mathrm{BMO}_{\mathscr{S}}(X)$ is rearrangeable, then $f^* \in \mathrm{BMO}(\mathbb{R}_+)$ with $\|f^*\|_{\mathrm{BMO}} \leq c_+ \|f\|_{\mathrm{BMO}}$.*

Proof. Consider the truncations $f_k := \max\{\min\{f,k\} - k\}$ for $k \geq 1$ of f and fix $t > 0$. As f^* is decreasing, $f^*(t/2)$ is a median of f^* on $(0,t)$. If $k > f^*(t/2)$, then $f^*(t/2)$ is also a median of $(f_k)^*$ on $(0,t)$.

The truncations f_k are bounded and so we may apply the result of Lemma 4.3.1 to each of them. Combining this with Properties (05) and (O6), it follows that

$$\fint_0^t |(f_k)^* - f^*(t/2)| \leq \Omega((f_k)^*,(0,t)) \leq c_* \|f_k\|_{\mathrm{BMO}} \leq c_* \|f\|_{\mathrm{BMO}}$$

for each $k > f^*(t/2)$. As $(f_k)^*$ converges monotonically to f^*, the monotone convergence theorem implies that

$$\fint_0^t |f^* - f^*(t/2)| < \infty,$$

demonstrating that $f^* \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$.

Therefore, another application of the monotone convergence theorem implies that

$$\Omega(f^*,(0,t)) = \lim_{k\to\infty} \Omega((f_k)^*,(0,t)) \leq c_* \|f\|_{\mathrm{BMO}}.$$

By Property (B2), $f^* \in \mathrm{BMO}(\mathbb{R}_+)$ with $\|f^*\|_{\mathrm{BMO}} \leq c_+ \|f\|_{\mathrm{BMO}}$. □

Note that this proposition holds for non-negative functions. To remove this restriction and to obtain the boundedness of the decreasing rearrangement for rearrangeable $f \in \mathrm{BMO}_{\mathscr{S}}(X)$ that attain negative values, we compose the rearrangement with the absolute value. Since $|f|^* = f^*$, we have $\|f^*\|_{\mathrm{BMO}} = \||f|^*\|_{\mathrm{BMO}}$. Then, denoting by $c_{|\cdot|}(\mathscr{S})$ the smallest constant such that $\||f|\|_{\mathrm{BMO}_{\mathscr{S}}} \leq c_{|\cdot|}(\mathscr{S})\|f\|_{\mathrm{BMO}_{\mathscr{S}}}$ for all $f \in \mathrm{BMO}_{\mathscr{S}}$ (see Property (O2)), we have the following corollary.

Corollary 4.3.3. *Let $f \in \mathrm{BMO}_{\mathscr{S}}(X)$ be rearrangeable. Then, under the hypothesis of Proposition 4.3.2, $f^* \in \mathrm{BMO}(\mathbb{R}_+)$ with $\|f^*\|_{\mathrm{BMO}} \leq c_{|\cdot|}(\mathscr{S})c_*\|f\|_{\mathrm{BMO}_{\mathscr{S}}}$.*

In the following subsections, we will consider some examples of settings in which the hypothesis of the theorem holds. We will show that the hypothesis holds by constructing the families $\{S_i\}$ and $\{\widetilde{S}_i\}$ for which the Conditions (i), (ii), and (iii) hold.

4.3.1 Families of balls in metric spaces

Consider a metric measure space (X, ρ, μ) where μ is a non-trivial Borel regular measure. By a ball we will specifically mean that it has positive and finite radius. Furthermore, for each ball we will assume that it comes with a prescribed centre and radius.

When dealing with balls in a metric space, we have the following general form of the Vitali covering lemma (also called the basic covering theorem).

Lemma 4.3.4 ([48])**.** *For every family $\mathcal{F}$ of balls in X of uniformly bounded radii, there exists a pairwise-disjoint subfamily $\mathcal{G}$ such that*

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} 5B. \tag{4.5}$$

We make the following assumptions on the metric measure space (X, ρ, μ): (D) that μ is doubling, (I) that $|X| = \infty$, (C) that the function $r \mapsto |B(x, r)|$ is continuous for all $x \in X$, and (U) that every family $\mathcal{F}$ of balls in X of uniformly bounded measure has uniformly bounded radii (and so the Vitali covering lemma applies for such families).

For a ball $B = B(x, r)$, we write $\lambda B = B(x, \lambda r)$ for $\lambda > 0$. Recall that a doubling measure is one for which there exists a constant $c_d \geq 1$ such that $0 < \mu(2B) \leq c_d\mu(B) < \infty$ holds for all balls. A consequence of this doubling condition is that for any $\lambda \geq 1$ there exists a constant c_λ such that $\mu(\lambda B) \leq c_\lambda\mu(B)$. With this notation, $c_d = c_2$.

Note that the Lebesgue differentiation theorem holds in the setting of a doubling metric measure space ([48]). A further consequence of doubling (see, for instance, [50]) is that any disjoint collection of balls is necessarily countable. Thus, the collection $\mathcal{G}$ from the Vitali covering lemma can be assumed to be countable.

The condition (D) implied, in particular, if the measure μ is lower Ahlfors Q-regular for some $Q > 0$: $c_A r^Q \le \mu\big(B(x,r)\big)$ for some $c_A > 0$ and for all $x \in X$, $0 < r < \operatorname{diam}(X)$.

Consider the basis of all balls in X, denoted by $\mathcal{B}$. Fix $0 \le f \in L^\infty(X)$ and $t > 0$. Consider $E_{\alpha_t} = \{x \in X : f(x) > \alpha_t\}$. By the Lebesgue differentiation theorem, for almost every $x \in E_{\alpha_t}$ there exists an $r_0(x) > 0$ such that

$$\fint_{B(x,r)} f > \alpha_t \text{ for all } r \le r_0(x).$$

By Property (R4), if r is a radius such that $|B(x,r)| = t$, then

$$\fint_{B(x,r)} f \le \alpha_t. \tag{4.6}$$

Let $r_1(x)$ be the smallest such r. Then,

$$\fint_{B(x,r)} f > \alpha_t \text{ for any } r < r_1(x). \tag{4.7}$$

This is true, in particular, for $r = r(x) = \frac{r_1(x)}{5} < r_1(x)$.

Consider the family $\mathcal{F} = \{B(x, r(x))\}$. By monotonicity of measure, each ball in $\mathcal{F}$ has measure at most t, and so the family has uniformly bounded radii, by assumption. Applying the Vitali covering lemma, we obtain a countable pairwise-disjoint subfamily $\mathcal{G}$ for which (4.5) holds. This is the $\{S_i\}$ of Proposition 4.3.2.

Condition (i) holds by (4.7) with $r = r(x) < r_1(x)$. The role of $\{\widetilde{S}_i\}$ will be played by $5\mathcal{G} = \{5B : B \in \mathcal{G}\}$. Condition (ii) holds by (4.6) with $r = 5r(x) = r_1(x)$ as

$$\mu(B(x, 5r(x))) \le c_5 \mu(B(x, r(x))).$$

Condition (iii) is true since, by (4.5),

$$\bigcup_{B\in\mathcal{G}} 5B \supset \bigcup_{B\in\mathcal{F}} B \supset E_{\alpha_t}$$

holds μ almost everywhere.

Therefore, applying Proposition 4.3.2, yields

Theorem 4.3.5. *Let X satisfy conditions (D), (I), (C), and (U). If $f \in \mathrm{BMO}_{\mathcal{B}}(X)$, then $f^* \in \mathrm{BMO}(\mathbb{R}_+)$ with*

$$\|f^*\|_{\mathrm{BMO}} \le c_5\, c_{|\cdot|}(\mathcal{B}) \|f\|_{\mathrm{BMO}_{\mathcal{B}}}.$$

Remark 4.3.6. The constant 5 in the Vitali covering lemma can be decreased to $3+\varepsilon$ for $\varepsilon > 0$ (see, for instance, [6]). Using this refinement, the constant in the previous theorem can be brought down (taking a limit as $\varepsilon \to 0^+$) to $c_3 c_{|\cdot|}(\mathcal{B})$.

Remark 4.3.7. Note that one can take $c_\lambda = c_d \lambda^{\log_2 c_d}$, but this may not be optimal.

4.3.2 Families of rectangles in Euclidean space

Consider, now, $\mathbb{R}^n$ with the Euclidean metric and Lebesgue measure. We will consider bases comprised of rectangles in $\mathbb{R}^n$. It is interesting to consider restrictions on the sidelengths of the rectangles forming the basis has an effect on the boundedness.

On one side of the spectrum, if we impose the restriction that all sidelengths be the same, we have the basis of all cubes, $\mathcal{Q}$. Fixing $t > 0$, we start by dividing $\mathbb{R}^n$ into a mesh of cubes of measure t. By Property (R4), the average of f on each such cube is at least α_t. As such, we may apply the (local) Calderón-Zygmund lemma[6] to each of these cubes. Each application yields a countable collection of cubes satisfying (i) of Lemma 4.3.2. Hence, the union of these collections also satisfies (i) and $f \le \alpha_t$ almost everywhere on the complement. Moreover, by construction, each such cube Q is contained inside a parent cube Q' such that $f_{Q'} \le \alpha_t$ and satisfying $|Q'| = 2^n|Q|$, where the constant 2^n comes from the fact that Q is obtained from Q' by bisecting each of the n sides of Q'. Then, by continuity of the integral, there is a cube $Q \subset \widetilde{Q} \subset Q'$ such that $f_{\widetilde{Q}} = \alpha_t$. Thus, the collection of $\widetilde{Q}$s satisfies (ii). Last, since $f \le \alpha_t$ almost everywhere outside the collection of Qs, it is also true outside the collection of $\widetilde{Q}$s.

Therefore, Lemma 4.3.2 implies

Theorem 4.3.8. *If* $f \in \mathrm{BMO}(\mathbb{R}^n)$, *then* $f^* \in \mathrm{BMO}(\mathbb{R}_+)$ *with*

$$\|f^*\|_{\mathrm{BMO}} \le 2^n c_{|\cdot|}(\mathcal{Q}) \|f\|_{\mathrm{BMO}}.$$

The basis $\mathcal{R}$, where all the sidelengths are arbitrary, represents the other side of the spectrum. In [61], Korenovskii-Lerner-Stokolos were able to obtain a multidimensional analogue of Riesz' rising sun lemma. This implies, by the work of Korenovskii ([59]) that for any

[6]See Proposition 5.3.16 in Appendix III for a proof.

rectangle $R_0 \subset \mathbb{R}^n$ if $f \in \mathrm{BMO}_{\mathcal{R}}(R_0)$, then $f^* \in \mathrm{BMO}(R_0^*)$ with

$$\|f^*\|_{\mathrm{BMO}} \leq \|f\|_{\mathrm{BMO}_{\mathcal{R}}}.$$

This represents the extreme case of Lemma 4.3.2, where one has a single countable family that is both pairwise disjoint and on which the averages of f can be made equal to a prescribed value. This is, of course, a sharp result. Simple examples show that an analogue of the rising sun lemma cannot exist for cubes. This shows a benefit of using rectangles over cubes.

The unfortunate limitation of this result is that arbitrary rectangles can have arbitrary eccentricity, and so we cannot deduce from this result any information about the boundedness of the decreasing rearrangement on $\mathrm{BMO}(\mathbb{R}^n)$.

A compromise between the rigidity of cubes and the absolute freedom of arbitrary rectangles are families of rectangles of bounded eccentricity. Sharpness of the constants is lost, but these rectangles are still comparable to cubes, and so information about these bases can be transferred (see Property (O4)) to the basis of cubes.

By selecting a family of rectangles that is closed under properly chosen bisections, we are able to find an example of a basis for which the constant obtained in Lemma 4.3.2 is independent of the dimension. Consider the basis

$$\mathcal{D} = \{R = I_1 \times I_2 \times \cdots \times I_n : \exists\, a > 0 \text{ s.t. } \ell(I_j) = a2^{j/n} \text{ for each } j = 1, \ldots, n\},$$

where the intervals need not be ordered by increasing length. By bisecting the longest side of such a rectangle, we obtain two congruent rectangles in $\mathcal{D}$ (in particular, with sidelengths $\frac{a}{2^{1/n}}2^{j/n}$). This property, along with the fact that the Lebesgue differentiation holds for $\mathcal{D}$ as these rectangles have bounded eccentricity, allows one to repeat the proof of the Calderón-Zygmund lemma but with a constant of 2.[7]

Theorem 4.3.9. *If $f \in \mathrm{BMO}_{\mathcal{D}}(\mathbb{R}^n)$, then $f^* \in \mathrm{BMO}(\mathbb{R}_+)$ with*

$$\|f^*\|_{\mathrm{BMO}} \leq 2c_{|\cdot|}(\mathcal{D})\|f\|_{\mathrm{BMO}_{\mathcal{D}}}.$$

As mentioned previously, the fact that the rectangles in $\mathcal{D}$ have bounded eccentricity allows us to improve on Corollary 4.3.8 for the basis $\mathcal{Q}$. Every $R \in \mathcal{D}$ can be fit inside a cube

[7]See Proposition 5.3.17 and the preceding discussion in Appendix III.

Q with

$$\frac{|Q|}{|R|} = 2^{\frac{n-1}{2}},$$

and so it follows from the previous theorem and Property (O4) that

Theorem 4.3.10. *If $f \in \mathrm{BMO}(\mathbb{R}^n)$, then $f^* \in \mathrm{BMO}(\mathbb{R}_+)$ with*

$$\|f^*\|_{\mathrm{BMO}} \leq 2^{\frac{n+1}{2}} \min(c_{|\cdot|}(\mathcal{D}), c_{|\cdot|}(\mathcal{Q}))\|f\|_{\mathrm{BMO}}.$$

In particular, as $\min(c_{|\cdot|}(\mathcal{D}), c_{|\cdot|}(\mathcal{Q})) \leq 2$ (see Property (O2)),

$$\|f^*\|_{\mathrm{BMO}} \leq 2^{\frac{n+3}{2}} \|f\|_{\mathrm{BMO}}.$$

4.4 Continuity of the decreasing rearrangement

In this section, we consider the question of continuity of the decreasing rearrangement on spaces defined by mean oscillation. Recall, as explained in the introduction, that rearrangements are fundamentally nonlinear, and so boundedness does not imply continuity.

Our first aim of this section is to show that in spite of the boundedness of the decreasing rearrangement on BMO, it can fail to be continuous. The furnished counter-example is written for $\mathrm{BMO}(0,1)$, but simple modifications allow the argument to carry over for $\mathrm{BMO}(Q_0)$ for various Q_0.

One can wonder about the subset of BMO on which the decreasing rearrangement is continuous. A natural subspace to consider is VMO as it very often plays the role of the continuous functions within BMO.

Then, we show that under some assumptions on $\mathscr{S}$, boundedness of the decreasing rearrangement from $\mathrm{BMO}_{\mathscr{S}}(Q_0)$ to $\mathrm{BMO}(Q_0^*)$ implies boundedness from $\mathrm{VMO}_{\mathscr{S}}(Q_0)$ to $\mathrm{VMO}(Q_0^*)$. An example is provided, however, that shows that a function in $\mathrm{BMO}(Q_0) \setminus \mathrm{VMO}(Q_0)$ can still have a rearrangement in $\mathrm{VMO}(Q_0^*)$.

Last, we build upon this by using an analogue of the Arzelà-Ascoli Theorem for VMO due to Brezis-Nirenberg to show continuity of the decreasing rearrangement from $\mathrm{VMO}(Q_0)$ to $\mathrm{VMO}(Q_0^*)$ in the case where Q_0 is bounded.

4.4.1 Discontinuity on BMO

Theorem 4.4.1. *The decreasing rearrangement is not continuous on* $\mathrm{BMO}(0,1)$*. That is, there exists a sequence* $\{f_k\} \subset \mathrm{BMO}(0,1)$ *and a function* $f \in \mathrm{BMO}(0,1)$ *such that* $f_k \to f$ *in* $\mathrm{BMO}(0,1)$ *but* $f_k^* \not\to f^*$ *in* $\mathrm{BMO}(0,1)$*. As a consequence, no inequality of the form*

$$\|f^* - g^*\|_{\mathrm{BMO}} \le C\|f - g\|_{\mathrm{BMO}}$$

can hold for all $f, g \in \mathrm{BMO}(0,1)$ *for any constant* C*.*

Proof. For $k \ge 1$, define f_k as the function

$$f_k(x) = \begin{cases} 1, & 0 \le x < 1/2 \\ \frac{1}{k}\sqrt{-\log|3-4x|}, & 1/2 \le x \le 1 \end{cases}.$$

and f as the function

$$f(x) = \begin{cases} 1, & 0 \le x < 1/2 \\ 0, & 1/2 \le x \le 1 \end{cases}.$$

We have

$$f_k(x) - f(x) = \begin{cases} 0, & 0 \le x < 1/2 \\ \frac{1}{k}\sqrt{-\log|3-4x|}, & 1/2 \le x \le 1 \end{cases}$$

and $\|f_k - f\|_{\mathrm{BMO}} = \frac{1}{k}\|f_1 - f\|_{\mathrm{BMO}}$, showing that $f_k \to f$ in BMO.

Since f is monotone and right-continuous, it follows that $f^* = f$. The rearrangement of f_k is given by

$$f_k^*(s) = \begin{cases} \frac{1}{k}\sqrt{-\log(2s)}, & 0 \le s < \frac{1}{2e^{k^2}} \\ 1, & \frac{1}{2e^{k^2}} \le s < \frac{1}{2e^{k^2}} + \frac{1}{2} \\ \frac{1}{k}\sqrt{-\log(2s-1)}, & \frac{1}{2e^{k^2}} + \frac{1}{2} \le s \le 1 \end{cases}.$$

We have

$$f_k^*(s) - f^*(s) = \begin{cases} \frac{1}{k}\sqrt{-\log(2s)} - 1, & 0 \le s < \frac{1}{2e^{k^2}} \\ 0, & \frac{1}{2e^{k^2}} \le s < \frac{1}{2} \\ 1, & \frac{1}{2} \le s < \frac{1}{2e^{k^2}} + \frac{1}{2} \\ \frac{1}{k}\sqrt{-\log(2s-1)}, & \frac{1}{2e^{k}} + \frac{1}{2} \le s \le 1 \end{cases}.$$

Note that $f_k^* - f^*$ has a jump discontinuity of height one at $s = \frac{1}{2}$ and so, taking any interval in $(0,1)$ of the form $\left(\frac{1}{2}-\delta, \frac{1}{2}+\delta\right)$ for $\delta < \min\left(\frac{1}{2e^{k^2}}, \frac{1}{2} - \frac{1}{2e^{k^2}}\right)$, we find

$$\|f_k^* - f^*\|_{\mathrm{BMO}} \geq 1/2$$

for all k. $\square$

Remark 4.4.2. Alternatively, one could have reasoned that $\{f_k^*\}$ cannot converge to f^* in $\mathrm{BMO}(0,1)$ as follows. Each f_k^* is in $\mathrm{VMO}(0,1)$ and so, by Property (V2), its limit, if it exists, must be in $\mathrm{VMO}(0,1)$. However, $f^* \notin \mathrm{VMO}(0,1)$ as it has a jump discontinuity.

This example can easily be modified to show that the decreasing rearrangement is not continuous from $\mathrm{BMO}(Q_0)$ to $\mathrm{BMO}(Q_0^*)$ for other Q_0 than $(0,1)$. In particular, setting $g_k(x,y) = f_k(x)$ and $g(x,y) = f(x)$ for $y \in (0,1)^{n-1}$ provides a counter-example when $Q_0 = (0,1)^n$. Extending f_k and f, or g_k and g by zero yields counter-examples for $Q_0 = \mathbb{R}$ and $Q_0 = \mathbb{R}^n$, respectively.

4.4.2 Boundedness on VMO

Before demonstrating the boundedness of the decreasing rearrangement on VMO, we need two technical lemmas.

Lemma 4.4.3. *Let $0 \leq f \in L^1_{loc}(Q_0)$ and $\mathscr{S}$ be a differentiation basis. Assume that for every $t > 0$ and every measurable $A \subset Q_0$ such that $|A| > 0$ and $|A^c| > 0$ there exists a shape $\widetilde{S} \in \mathscr{S}$ with $|\widetilde{S}| < t$ such that*

$$\frac{|A \cap \widetilde{S}|}{|\widetilde{S}|} = \frac{1}{2} = \frac{|A^c \cap \widetilde{S}|}{|\widetilde{S}|}.$$

If f^* has a jump discontinuity of height ℓ, then

$$\inf_{\eta} ⨍_{\widetilde{S}} |f - \eta| = ⨍_{\widetilde{S}} |f - m_{\widetilde{S}}(f)| \geq \frac{\ell}{2}, \tag{4.8}$$

where the infimum is taken over all constants η and $m_{\widetilde{S}}(f)$ is a median of f on $\widetilde{S}$.

Remark 4.4.4. The existence of a basis satisfying the assumption of this lemma is known in many familiar contexts, such as $\mathcal{B}$, $\mathcal{Q}$, and $\mathcal{R}$, due to the Lebesgue density theorem[8].

[8]See Lemma 5.3.18 in Appendix III for the proof that $\mathcal{B}$ satisfies the assumptions of this lemma.

Proof. Fix $t > 0$ and assume that f^* has a jump discontinuity at $s_0 \in X^*$. Recalling that f^* is right-continuous, write

$$\alpha = \lim_{s \to s_0^+} f^*(s), \qquad \beta = \lim_{s \to s_0^-} f^*(s), \qquad \gamma = \frac{\alpha + \beta}{2}.$$

Then $\ell = \beta - \alpha$.

The set $E_\gamma = \{x \in X : f(x) > \gamma\}$ is measurable and satisfies, by equimeasurability of f and f^*, $|E_\gamma| = s_0 > 0$ and $|E_\gamma^c| = |X| - s_0 > 0$.

By assumption, we can find a shape $\widetilde{S}$ with $|\widetilde{S}| < t$ such that

$$\frac{|E_\gamma \cap \widetilde{S}|}{|\widetilde{S}|} = \frac{1}{2} = \frac{|E_\gamma^c \cap \widetilde{S}|}{|\widetilde{S}|}.$$

In other words, γ is a median of f on $\widetilde{S}$. Hence, for any constant c,

$$\fint_{\widetilde{S}} |f - \gamma| = \frac{1}{2} \left(\frac{1}{|E_\gamma \cap \widetilde{S}|} \int_{E_\gamma \cap \widetilde{S}} (f - \gamma) + \frac{1}{|E_\gamma^c \cap \widetilde{S}|} \int_{E_\gamma^c \cap \widetilde{S}} (\gamma - f) \right). \tag{4.9}$$

We proceed by estimating each integral separately.

First, by Property (R2) along with the fact that f^* has a jump at s_0,

$$0 = |\{s \in X^* : \alpha < f^*(s) < \beta\}| = |\{x \in X : \alpha < f(x) < \beta\}|.$$

Writing $F_\beta = \{x \in X : f(x) \geq \beta\}$, it follows that $|E_\gamma \setminus F_\beta| = 0$ and so

$$\frac{1}{|E_\gamma \cap \widetilde{S}|} \int_{E_\gamma \cap \widetilde{S}} (f - \gamma) = \frac{1}{|F_\beta \cap \widetilde{S}|} \int_{F_\beta \cap \widetilde{S}} (f - \gamma) \geq \beta - \gamma = \frac{\beta - \alpha}{2}. \tag{4.10}$$

Similarly, $|E_\alpha^c \setminus E_\gamma^c| = 0$ and so

$$\frac{1}{|E_\gamma^c \cap \widetilde{S}|} \int_{E_\gamma^c \cap \widetilde{S}} (\gamma - f) = \frac{1}{|E_\alpha^c \cap \widetilde{S}|} \int_{E_\alpha^c \cap \widetilde{S}} (\gamma - f) \geq \gamma - \alpha = \frac{\beta - \alpha}{2}. \tag{4.11}$$

Combining (4.9), (4.10), (4.11), we find that

$$\fint_{\widetilde{S}} |f - \gamma| \geq \frac{1}{2} \left(\frac{\beta - \alpha}{2} + \frac{\beta - \alpha}{2} \right) = \frac{\beta - \alpha}{2} = \frac{\ell}{2}.$$

This demonstrates the inequality in (4.8). The equality is Property (O5). □

Lemma 4.4.5. *Let I_0 be a interval with left-endpoint at the origin. If $g \in L^1_{loc}(I_0) \setminus \mathrm{VMO}(I_0)$ is monotone and has no jump discontinuities, then there exists an $\varepsilon > 0$ and a sequence of intervals $\{I_k = (a_k, b_k)\}$ such that $a_k, b_k \to 0$ as $k \to \infty$ but $\Omega(g, I_k) \geq \varepsilon$ for all k.*

Proof. Take a monotone $g \in L^1_{\text{loc}}(I_0) \setminus \text{VMO}(I_0)$ having no jump discontinuities. Since $g \notin \text{VMO}(I_0)$ there exists an $\varepsilon > 0$ and a sequence of intervals $\{I_k\}$ such that $|I_k| \to 0$ as $k \to \infty$ but $\Omega(g, I_k) \geq \varepsilon$ for all k. Denoting by a_k the left-endpoint of the interval I_k, it suffices to show that $a_k \to 0$ as $k \to \infty$.

Assume, by way of contradiction, that $a_k \not\to 0$. Then there exists a subsequence $\{a_{k_m}\}$ and an $\varepsilon' > 0$ such that $|a_{k_m}| \geq \varepsilon'$ for all k_m. This means that $I_{k_m} \subset I_0 \setminus [0, \varepsilon')$. On $I_0 \setminus [0, \varepsilon')$, however, g is uniformly continuous and so one can take k_m sufficiently large so that $|g(x) - g(y)| < \frac{\varepsilon}{2}$ for $x, y \in I_{k_m}$. Then, we have

$$\varepsilon \leq \Omega(g, I_{k_m}) \leq \fint_{I_{k_m}} \fint_{I_{k_m}} |g(x) - g(y)| \, dx dy < \frac{\varepsilon}{2}.$$

□

We write, for a function h, $\text{Tr}(h, \alpha) = \max(h, \alpha) - \alpha$. This is a vertical shift of a truncation from below, and so, by Properties (O1) and (O6), we have that for any shape S, $\Omega(\text{Tr}(h, \alpha), S) \leq \Omega(h, S)$.

Theorem 4.4.6. *Let $\mathscr{S}$ be such that the assumptions of Lemma 4.4.3 hold and $f^* \in \text{BMO}(Q_0^*)$ whenever $f \in \text{BMO}_{\mathscr{S}}(Q_0)$ with $\|f^*\|_{\text{BMO}} \leq c\|f\|_{\text{BMO}_{\mathscr{S}}}$. Then, $f^* \in \text{VMO}(Q_0^*)$ whenever $f \in \text{VMO}_{\mathscr{S}}(Q_0)$.*

Proof. Take $f \in \text{VMO}_{\mathscr{S}}(Q_0)$. By Property (O2), $|f| \in \text{VMO}_{\mathscr{S}}(Q_0)$ and so it follows from Lemma 4.4.3 that f^* has no jump discontinuities (recall that $|f|^* = f^*$).

By way of contradiction, assume that $f^* \in \text{BMO}(Q_0^*) \setminus \text{VMO}(Q_0^*)$. Then, f^* satisfies the conditions of g in Lemma 4.4.5 and so there exists an $\varepsilon > 0$ and a sequence of intervals $\{I_k = (a_k, b_k)\}$ such that $a_k, b_k \to 0$ as $k \to \infty$ but $\Omega(f^*, I_k) \geq \varepsilon$ for all k.

Writing $\alpha_k = f^*(b_k)$, consider the intervals

$$\widetilde{I}_k = \{s \in Q_0^* : \alpha_k < f^*(s)\}$$

and the functions $\text{Tr}(f^*, k) = \text{Tr}(f^*, \alpha_k)$. Three facts must be pointed out as they will be used in what follows. First, note that $|\widetilde{I}_k| \to 0$ as $k \to \infty$. Second, note that $\text{Tr}(f^*, k) = f^*$ modulo constants on both I_k and on $\widetilde{I}_k$. Finally, note that $\text{Tr}(f^*, k) = (\text{Tr}(|f|, k))^*$.

Thus, by the boundedness of $f \mapsto f^*$ from $\mathrm{BMO}_{\mathscr{S}}(Q_0)$ to $\mathrm{BMO}(Q_0^*)$,

$$\begin{aligned}\|\mathrm{Tr}(|f|,k)\|_{\mathrm{BMO}_{\mathscr{S}}} &\geq \frac{\|(\mathrm{Tr}(|f|,k))^*\|_{\mathrm{BMO}}}{c} = \frac{\|\mathrm{Tr}(f^*,k)\|_{\mathrm{BMO}}}{c} \\ &\geq \frac{\Omega(\mathrm{Tr}(f^*,k),I_k)}{c} = \frac{\Omega(f^*,I_k)}{c} \\ &\geq \frac{\varepsilon}{c}\end{aligned}$$

and so there exists a shape S_k such that $\Omega(\mathrm{Tr}(|f|,k),S_k) \geq \frac{\varepsilon}{4c}$.

The fact that $f \in \mathrm{VMO}_{\mathscr{S}}(Q_0)$ implies the existence of a $\delta > 0$ for which $\Omega(f,S) < \frac{\varepsilon}{8c}$ if $|S| \leq \delta$. The goal is to show that $|S_k| \leq \delta$ for sufficiently large k. Then, we would have, for such k, that $\Omega(f,S_k) < \frac{\varepsilon}{8c}$ while, at the same time,

$$\Omega(f,S_k) \geq \frac{\Omega(|f|,S_k)}{2} \geq \frac{\Omega(\mathrm{Tr}(|f|,k),S_k)}{2} \geq \frac{\varepsilon}{8c}$$

by Properties (O2) and (O6). This is a contradiction, implying the desired result.

If $|S_k| \leq |\widetilde{I}_k|$, then we have immediately that $|S_k| \to 0$ as $k \to \infty$ since this is true of $|\widetilde{I}_k|$. If $|S_k| > |\widetilde{I}_k|$, then we have that $\mathrm{Tr}(f^*,k) = 0$ on $(|\widetilde{I}_k|,|S_k|)$ and so, by Property (R4),

$$\begin{aligned}\frac{\varepsilon}{4c} \leq \Omega(\mathrm{Tr}(|f|,k),S_k) &\leq \frac{2}{|S_k|}\int_{S_k}\mathrm{Tr}(|f|,k) \\ &\leq \frac{2}{|S_k|}\int_0^{|S_k|}\mathrm{Tr}(f^*,k) = \frac{2}{|S_k|}\int_0^{|\widetilde{I}_k|}\mathrm{Tr}(f^*,k)\end{aligned}$$

Hence,

$$|S_k| \leq \frac{8c}{\varepsilon}\int_0^{|\widetilde{I}_k|}\mathrm{Tr}(f^*,k). \tag{4.12}$$

The functions $g_k = \mathrm{Tr}(f^*,k)\chi_{(0,|\widetilde{I}_k|)}$ satisfy $g_k \leq f^*\chi_{(0,|\widetilde{I}_1|)} \in L^1((0,|\widetilde{I}_1|))$ pointwise almost everywhere as well as $g_k \to 0$ as $k \to \infty$ pointwise almost everywhere. By Lebesgue's dominated convergence theorem,

$$\int_{\widetilde{I}_1} g_k = \int_0^{|\widetilde{I}_k|}\mathrm{Tr}(f^*,k) \to 0$$

as $k \to \infty$. Therefore, by (4.12) the same must hold for $|S_k|$ and the proof is finished. $\square$

Remark 4.4.7. A function $f \in \mathrm{BMO}_{\mathscr{S}}(Q_0) \setminus \mathrm{VMO}_{\mathscr{S}}(Q_0)$ may have a rearrangement in $\mathrm{VMO}_{\mathscr{S}}(Q_0^*)$. An example is provided by f_1 in the proof of Theorem 4.4.1. The function f_1 is not in $\mathrm{VMO}((0,1))$ as it has a jump discontinuity at $x = 1/2$, while $f_1^* \in \mathrm{VMO}(0,1)$.

4.4.3 Continuity on VMO

In this section, we show that when Q_0 is bounded, the decreasing rearrangement is continuous from $\mathrm{VMO}(Q_0)$ to $\mathrm{VMO}(Q_0^*)$.

For such Q_0, the following result of Brezis and Nirenberg provides us with an analogue of the Arzelà-Ascoli Theorem for $\mathrm{VMO}(Q_0)$. This result is a crucial tool in the proof of continuity for the rearrangement on VMO.

Theorem 4.4.8 (Brezis-Nirenberg [8])**.** *A set $\mathcal{F} \subset \mathrm{VMO}(Q_0)$ is relatively compact if and only if*

$$\lim_{t\to 0^+} \sup_{|Q|\le t} \Omega(f, Q) = 0$$

uniformly in $f \in \mathcal{F}$.

Theorem 4.4.9. *Let Q_0 be a finite cube. Then, the decreasing rearrangement is continuous from $\mathrm{VMO}(Q_0)$ to $\mathrm{VMO}(Q_0^*)$. That is, if $\{f_k\} \subset \mathrm{VMO}(Q_0)$ have mean zero and converge to f in $\mathrm{BMO}(Q_0)$, then $\{f_k^*\} \subset \mathrm{VMO}(Q_0)$ converge to f^* in $\mathrm{BMO}(Q_0)$.*

Note that since VMO is closed in BMO, it follows that $f \in \mathrm{VMO}(Q_0)$ and $f^* \in \mathrm{VMO}(Q_0^*)$.

Proof. Let us consider the application of Theorem 4.4.8 to a sequence $\{f_k\} \subset \mathrm{VMO}(Q_0)$ that converges to $f \in \mathrm{VMO}(Q_0)$ in $\mathrm{BMO}(Q_0)$. This means that, as a set, $\{f_k\} \subset \mathrm{VMO}(Q_0)$ is relatively compact, and so

$$\lim_{t\to 0^+} \sup_{|Q|\le t} \Omega(f_k, Q) = 0 \tag{4.13}$$

uniformly in k.

By Theorem 4.4.6, $f^* \in \mathrm{VMO}(Q_0^*)$ and $f_k^* \in \mathrm{VMO}(Q_0^*)$ for each k. It follows that

$$\lim_{t\to 0^+} \sup_{|I|\le t} \Omega(f_k^*, I) = 0 \tag{4.14}$$

for each k. This limit is not *a priori* uniform in k. However, assume for the moment that it is. Then, applying Theorem 4.4.8 in the other direction, we find that the set $\{f_k^*\} \subset \mathrm{VMO}(Q_0^*)$ is relatively compact and so contains a subsequence $\{f_{k_m}^*\}$ converging in $\mathrm{BMO}(Q_0^*)$ to some function $g \in \mathrm{VMO}(Q_0^*)$.

As each f_k has mean zero and $f_k \to f$ in $\mathrm{BMO}(Q_0)$, we have that $f_k \to f$ in $L^1(Q_0)$ by Property (B1). By the continuity of the decreasing rearrangement on $L^1(Q_0)$ (Property

(R3)), we have that $f_k^* \to f^*$ in $L^1(Q_0^*)$. But by Property (B1), again, $\{f_{k_m}^*\}$ also converges to g in $L^1(Q_0^*)$. By uniqueness of limits in L^1, it must be that $g = f^*$ almost everywhere.

Applying the arguments in the above paragraph to any subsequence $\{f_{k_j}\}$ instead of $\{f_k\}$ yields a subsequence of $\{f_{k_j}^*\}$ that converges to f^*. This means that every subsequence of $\{f_k^*\}$ has a further subsequence converging to f^*, implying that $\{f_k^*\}$ converges to f^*.

Of course, this argument has a flaw: the assumption of uniform convergence in (4.14). To rectify this, we assume, by way of contradiction, that the convergence is not uniform in k. Then there exists an $\varepsilon > 0$, a subsequence which we will relabel as $\{f_k\}$, and intervals $\{I_k = (a_k, b_k)\}$ such that $|I_k| \to 0$ as $k \to \infty$ but $\Omega(f_k^*, I_k) \geq \varepsilon$ for all k.

As in the proof of Lemma 4.4.5, we want to show that $a_k, b_k \to 0$ as $k \to \infty$. As f_k^* and f^* are monotone functions in $\mathrm{VMO}(Q_0^*)$, they are uniformly continuous on $[a, |Q_0|)$ for any $a > 0$. Thus, on $[a, |Q_0|)$, $\{f_k^*\}$ is a sequence of monotone functions converging pointwise almost everywhere to the uniformly continuous limit f^*. This allows us to conclude that the convergence is, in fact, uniform and that the sequence is equicontinuous[9]. That is, we can find $\delta > 0$ such that if $|I| < \delta$ for $I \subset [a, |Q_0|)$, then $\Omega(f_k^*, I) < \varepsilon$ for all k. Hence, $a_k < a$ for k sufficiently large. As this is true for any $a > 0$, it follows that both $a_k, b_k \to 0$ as $k \to \infty$.

The rest of the proof follows the same lines as that of Theorem 4.4.6 with $\mathscr{S} = \mathcal{Q}$. We write $\alpha_k = f_k^*(b_k)$, and consider the intervals

$$\widetilde{I}_k = \{s \in Q_0^* : \alpha_k < f_k^*(s)\}$$

and the functions $\mathrm{Tr}(f_k^*, k) = \mathrm{Tr}(f_k^*, \alpha_k)$. As before, $|\widetilde{I}k| \to 0$ as $k \to \infty$, $\mathrm{Tr}(f_k^*, k) = f_k^*$ modulo constants on both I_k and $\widetilde{I}_k$, and $\mathrm{Tr}(f_k^*, k) = (\mathrm{Tr}(|f_k|, k))^*$.

By the boundedness of $f \mapsto f^*$ from $\mathrm{BMO}(Q_0)$ to $\mathrm{BMO}(Q_0^*)$,

$$\|\mathrm{Tr}(|f_k|, k)\|_{\mathrm{BMO}} \geq \frac{\Omega(f_k^*, I_k)}{c} \geq \frac{\varepsilon}{c}$$

and so there exists a cube Q_k such that $\Omega(\mathrm{Tr}(|f_k|, k), Q_k) \geq \frac{\varepsilon}{4c}$.

From the uniformity of (4.13) there exists a $\delta > 0$ for which $\Omega(f_k, Q) < \frac{\varepsilon}{4c}$ for all k if $|Q| \leq \delta$. If we can show that $|Q_k| \leq \delta$ for sufficiently large k, then we would have, for such k, that $\Omega(f_k, Q_k) < \frac{\varepsilon}{4c}$ while, at the same time, $\Omega(f_k, Q_k) \geq \frac{\varepsilon}{4c}$. This contradiction implies the desired result.

[9]See Proposition 5.3.19 in Appendix III.

If $|Q_k| \le |\widetilde{I}_k|$, then we have immediately that $|Q_k| \to 0$ as $k \to \infty$ since this is true for $|\widetilde{I}_k|$. If $|Q_k| > |\widetilde{I}_k|$, then we have that $\mathrm{Tr}(f_k^*, k) = 0$ on $(|\widetilde{I}_k|, |Q_k|)$ and so, by Property (R4),

$$\frac{\varepsilon}{4c} \le \Omega(\mathrm{Tr}(|f_k|, k), Q_k) \le \frac{2}{|Q_k|} \int_0^{\widetilde{I}_k} \mathrm{Tr}(f_k^*, k).$$

Hence,

$$|Q_k| \le \frac{8c}{\varepsilon} \int_0^{\widetilde{I}_k} \mathrm{Tr}(f_k^*, k). \tag{4.15}$$

Since $\{f_k^*\}$ is convergent in $L^1(Q_0^*)$, it follows that it is uniformly integrable[10]. Setting $g_k = \mathrm{Tr}(f_k^*, k)\chi_{(0,|\widetilde{I}_k|)}$, the sequence $\{g_k\}$ is also uniformly integrable as $g_k \le f_k^*$ pointwise almost everywhere. Moreover, $g_k \to 0$ pointwise almost everywhere, thus the Vitali convergence theorem (see [74]) implies that

$$\int_{Q_0^*} g_k = \int_0^{\widetilde{I}_k} \mathrm{Tr}(f_k^*, k) \to 0$$

as $k \to \infty$. Therefore, by (4.15), $|Q_k| \to 0$ as $k \to \infty$. □

Note that the finiteness of $|Q_0|$ was used crucially in two spots: the application of Theorem 4.4.8 and the application of the Vitali convergence theorem.

4.5 Boundedness of the symmetric decreasing rearrangement

Now we turn out attention to the symmetric decreasing rearrangement. Recall that it may be defined by means of the formula $Sf(x) = f^*(\omega_n|x|^n)$ for $x \in \mathbb{R}^n$, where ω_n denotes the measure of the unit ball in $\mathbb{R}^n$.

In this section, we will consider the question of boundedness of the operator $f \mapsto Sf$ on BMO. Due to the radial symmetry of Sf, it is most natural to consider BMO to be with respect to the basis $\mathcal{B}$ of all Euclidean balls. Of course, by the comparability of balls and cubes in $\mathbb{R}^n$, this is the same space as BMO with respect to the basis $\mathcal{Q}$.

Our main theorem of the section is

[10] See Section III.5 of the appendix for the definition and relevant facts.

Theorem 4.5.1. *If $f \in \mathrm{BMO}_{\mathcal{B}}(\mathbb{R}^n)$ then $Sf \in \mathrm{BMO}_{\mathcal{B}}(\mathbb{R}^n)$ with*

$$\|Sf\|_{\mathrm{BMO}_{\mathcal{B}}} \leq 2^{\frac{n+3}{2}} \omega_n n^{n/2} \|f\|_{\mathrm{BMO}_{\mathcal{B}}}.$$

To this end, we will proceed in four steps. The first step is to compare $\mathrm{BMO}_{\mathcal{A}}$ and $\mathrm{BMO}_{\mathcal{B}}$. Here, $\mathcal{A}$ denotes the basis of all balls and all annuli centred at the origin, along with all sectors thereof, where sectors are taken to mean the intersection with a cone with vertex at the origin.

Lemma 4.5.2. *If $f \in \mathrm{BMO}_{\mathcal{A}}(\mathbb{R}^n)$ then $f \in \mathrm{BMO}_{\mathcal{B}}(\mathbb{R}^n)$ with $\|f\|_{\mathrm{BMO}_{\mathcal{B}}} \leq 2^n \|f\|_{\mathrm{BMO}_{\mathcal{A}}}$.*

Proof. For each ball $B \in \mathcal{B}$ the goal is to find an $A \in \mathcal{A}$ such that $B \subset A$ and $|A| \leq 2^n |B|$. Then, by Property (B5),

$$⨍_B |f - f_B| \leq 2^n ⨍_A |f - f_A|,$$

from where the result follows.

To this end, fix a ball B of radius $r > 0$ and centre x. There are four cases: $x = 0$, $|x| = r$, $|x| < r$, $|x| > r$.

When $x = 0$, one can choose $A = B$ and then the ratio $|A|/|B| = 1$. When $|x| = r$, B fits inside a half-circle A of radius $2r$ centred at the origin. In this case,

$$\frac{|A|}{|B|} = \frac{\frac{1}{2}\omega_n (2r)^n}{\omega_n r^n} = 2^{n-1}.$$

When $|x| < r$, B fits inside the ball A of radius $|x| + r$ centred at the origin. One calculates

$$\frac{|A|}{|B|} = \frac{\omega_n (|x| + r)^n}{\omega_n r^n} \leq \frac{\omega_n (2r)^n}{\omega_n r^n} = 2^n.$$

For the case $|x| > r$, we present the proof in dimension $n = 2$. The ball B fits inside a sector A of aperture $2\arcsin(r/|x|)$ of the annulus with inner radius $|x| - r$ and outer radius $|x| + r$ centred at the origin. Thus,

$$|A| = \pi((|x| + r)^2 - (|x| - r)^2) \frac{2\arcsin(r/|x|)}{2\pi} = 4|x| r \arcsin(r/|x|)$$

and so

$$\frac{|A|}{|B|} = \frac{4|x|\arcsin(r/|x|)}{\pi r} \leq 2.$$

□

The next lemma shows that for radial functions on $\mathbb{R}^n$, being in $\mathrm{BMO}_{\mathcal{A}}(\mathbb{R}^n)$ is the same as having radial profile in $\mathrm{BMO}(\mathbb{R})$.

Lemma 4.5.3. *Let $f \in \mathrm{BMO}(\mathbb{R})$ and define $g(x) = f(\omega_n|x|^n)$. Then $g \in \mathrm{BMO}_{\mathcal{A}}(\mathbb{R}^n)$ with $\|g\|_{\mathrm{BMO}_{\mathcal{A}}} = \|f\|_{\mathrm{BMO}}$.*

Proof. Any shape $A \in \mathcal{A}$ can be described by a radius $r \in (r_1, r_2)$, where $0 \le r_1 < r_2 < \infty$, and a region $S \subset \mathbb{S}^{n-1}$. Then, denoting by $\sigma(\cdot)$ the $(n-1)$–dimensional Lebesgue surface measure, we can compute

$$|A| = \omega_n(r_2^n - r_1^n) \times \frac{\sigma(S)}{\sigma(\mathbb{S}^{n-1})} = \frac{(r_2^n - r_1^n)\sigma(S)}{n}.$$

Changing to polar coordinates, we compute the average of Sf on A as follows:

$$\begin{aligned} \fint_A g(x)\,dx &= \frac{n}{(r_2^n - r_1^n)\sigma(S)} \int_S \int_{r_1}^{r_2} f(\omega_n r^n) r^{n-1}\,drd\sigma \\ &= \frac{n}{(r_2^n - r_1^n)} \int_{r_1}^{r_2} f(\omega_n r^n) r^{n-1}\,dr. \end{aligned}$$

Setting $\rho = \omega_n r^n$, we have

$$\frac{n}{(r_2^n - r_1^n)} \int_{r_1}^{r_2} f(\omega_n r^n) r^{n-1}\,dr = \frac{1}{\omega_n(r_2^n - r_1^n)} \int_{\omega_n r_1^n}^{\omega_n r_2^n} f(\rho)\,d\rho,$$

which is just the average of f on the interval $(\omega_n r_1^n, \omega_n r_2^n)$. Similarly, one can show that $\Omega(g, A) = \Omega\big(f, (\omega_n r_1^n, \omega_n r_2^n)\big)$, whence it follows that $\|g\|_{\mathrm{BMO}_{\mathcal{A}}} \le \|f\|_{\mathrm{BMO}}$.

Conversely, for any interval $I = (a, b) \subset [0, \infty)$, we may identify I with an annulus A of inner radius $(a/\omega_n)^{1/n}$ and outer radius $(b/\omega_n)^{1/n}$, and so the previous calculations show that $\Omega(f, I) = \Omega(g, A)$. Hence, $\|f\|_{\mathrm{BMO}} \le \|g\|_{\mathrm{BMO}_{\mathcal{A}}}$. $\square$

Applying this meeting to the pair f^* and Sf which satisfy $Sf(x) = f^*(\omega_n|x|^n)$ yields $\|Sf\|_{\mathrm{BMO}_{\mathcal{A}}} = \|f^*\|_{\mathrm{BMO}}$, thus providing a relationship between the behaviours of the two rearrangements with respect to mean oscillation.

The third step is to relate cubes to balls.

Lemma 4.5.4. *If $f \in \mathrm{BMO}_{\mathcal{B}}(\mathbb{R}^n)$ then $f \in \mathrm{BMO}(\mathbb{R}^n)$ with $\|f\|_{\mathrm{BMO}} \le \omega_n \left(\frac{\sqrt{n}}{2}\right)^n \|f\|_{\mathrm{BMO}_{\mathcal{B}}}$.*

Proof. For each cube $Q \in \mathcal{Q}$ there is a $B \in \mathcal{B}$ such that $Q \subset B$ and $|B| \leq \omega_n \left(\frac{\sqrt{n}}{2}\right)^n |Q|$. Hence, by Property (B5),

$$\fint_Q |f - f_Q| \leq \omega_n \left(\frac{\sqrt{n}}{2}\right)^n \fint_B |f - f_B|,$$

from where the result follows. $\square$

Now we can piece together these three lemmas, along with Corollary 4.3.10, to prove the main theorem of the section.

Proof of Theorem 4.5.1. Let $f \in \mathrm{BMO}_{\mathcal{B}}(\mathbb{R}^n)$. Then,

$$\begin{aligned} \|Sf\|_{\mathrm{BMO}_{\mathcal{B}}} &\leq 2^n \|Sf\|_{\mathrm{BMO}_{\mathcal{A}}} = 2^n \|f^*\|_{\mathrm{BMO}} \leq 2^n \times 2^{\frac{n+3}{2}} \|f\|_{\mathrm{BMO}} \\ &\leq 2^n \times 2^{\frac{n+3}{2}} \times \omega_n \left(\frac{\sqrt{n}}{2}\right)^n \|f\|_{\mathrm{BMO}_{\mathcal{B}}} = 2^{\frac{n+3}{2}} \omega_n n^{n/2} \|f\|_{\mathrm{BMO}_{\mathcal{B}}}. \end{aligned}$$

Therefore, $Sf \in \mathrm{BMO}_{\mathcal{B}}(\mathbb{R}^n)$. $\square$

Chapter 5

Appendices

The goal of this appendix is to supplement the chapters of this thesis. In some instances, additional details or examples are given; in other instances, proofs are given of results that are mentioned within the text.

Appendix I: BMO on shapes and sharp constants

I.1 In this paper, there are three instances of the Hilbert space structure of an L^2 space being used, which could use some further explanation. One such instance is the following, a part of Proposition 2.3.4.

Proposition 5.1.5. *For any basis $\mathscr{S}$, $L^\infty(\Omega) \subset \mathrm{BMO}^2_{\mathscr{S}}(\Omega)$ with*

$$\|f\|_{\mathrm{BMO}^2_{\mathscr{S}}} \leq \|f\|_{L^\infty}.$$

Proof. Fix $f \in L^\infty(\Omega)$ and a shape $S \in \mathscr{S}$, and consider the Hilbert space $L^2(S, \frac{dx}{|S|})$. Denoting by $\langle \cdot, \cdot \rangle$ the inner product on this L^2 space, we see that for any constant c,

$$\langle f - f_S, c \rangle = \fint_S (f - f_S)\overline{c} = \overline{c}(f_S - f_S) = 0$$

Denoting by $\| \cdot \|_{L^2}$ the norm on $L^2(S, \frac{dx}{|S|})$, this orthogonality, along with the Pythagorean inequality, implies that

$$\|f\|^2_{L^2} = \|(f - f_S) + f_S\|^2_{L^2} = \|f - f_S\|^2_{L^2} + \|f_S\|^2_{L^2}$$

and so

$$\|f - f_S\|_{L^2}^2 = \|f\|_{L^2}^2 - \|f_S\|_{L^2}^2.$$

Writing out what this norm is, we have that

$$\fint_S |f - f_S|^2 = \fint_S |f|^2 - \fint_S |f_S|^2 = \fint_S |f|^2 - |f_S|^2.$$

Since $|f_S|^2 \geq 0$, it follows that

$$\fint_S |f - f_S|^2 \leq \fint_S |f|^2 \leq \fint_S \|f\|_{L^\infty}^2 = \|f\|_{L^\infty}.$$

$\square$

Appendix II: Boundedness for maximal functions and BMO on shapes in the product setting

II.1 We mention some of the basic results related to maximal functions. The notation $\{f > \alpha\}$ and its variants will be used in what follows as shorthand for $\{x \in \mathbb{R}^n : f(x) > \alpha\}$.

Proposition 5.2.6. *Let $\mathscr{S}$ be a basis of shapes in $\mathbb{R}^n$ and $f \in L^1(S)$ for every $S \in \mathscr{S}$. Then, the maximal function $M_{\mathscr{S}} f$ is lower semicontinuous, hence measurable.*

Proof. Fix $\alpha > 0$ and consider the set $E = \{M_{\mathscr{S}} f > \alpha\}$. If $x \in E$, then $M_{\mathscr{S}} f(x) > \alpha$ and so there exists a shape $S \in \mathscr{S}$ containing x such that

$$\fint_S |f| > \alpha.$$

This implies that for any $y \in S$, $M_{\mathscr{S}} f(y) > \alpha$. Since S is open, there is a ball $B_x \subset S$ containing x, and so $M_{\mathscr{S}} f(y) > \alpha$ for every $y \in B_x$. Therefore, $B_x \subset E$. Since this is true for every $x \in E$, it follows that E is open. Hence, $M_{\mathscr{S}} f$ is lower semicontinuous and so measurable. $\square$

Recall that the distribution of a measurable function f on $\mathbb{R}^n$ is defined as $\mu_f(\alpha) = \{|f| > \alpha\}$ for $\alpha > 0$.

Let $1 \leq p < \infty$. A measurable function f is said to be in the weak Lebesgue space, $L^p_w(\mathbb{R}^n)$, if there exists a constant $C \geq 0$ such that $\mu_f(\alpha) \leq \frac{C^p}{\alpha^p}$ for all $\alpha > 0$. We set

$$\|f\|_{L^p_w} = \sup_{\alpha>0} \alpha \mu_f(\alpha)^{1/p}.$$

An operator T is said to be strong-type (p,p) if it is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and is said to be weak-type (p,p) if it is bounded from $L^p(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$. From Chebyshev's inequality, it follows that a strong-type (p,p) operator is weak-type (p,p) with the same bound. This is easy enough to prove by hand: Let T be an operator that is strong-type (p,p) with $\|Tf\|_{L^p} \leq c\,\|f\|_{L^p}$ for all $f \in L^p(\mathbb{R}^n)$. Then, for $f \in L^p(\mathbb{R}^n)$ and $\alpha > 0$,

$$\alpha^p \mu_{Tf}(\alpha) = \int_{\mathbb{R}^n} \alpha^p \chi_{\{|Tf|>\alpha\}}\, dx < \int_{\mathbb{R}^n} |Tf|^p \chi_{\{|Tf|>\alpha\}}\, dx \leq \int_{\mathbb{R}^n} |Tf|^p\, dx,$$

showing that $Tf \in L^p_w(\mathbb{R}^n)$ with $\|Tf\|_{L^p_w} \leq \|Tf\|_{L^p} \leq c\,\|f\|_{L^p}$.

The following Hardy-Littlewood-Wiener theorem is the classical boundedness result in the area of maximal functions. Its proof can be found in many texts, including [78], and is also given here.

Theorem 5.2.7. *The Hardy-Littlewood maximal function $M_{\mathcal{Q}}$ is weak-type $(1,1)$ and strong-type (p,p) for all $1 < p \leq \infty$.*

Note that $M_{\mathcal{Q}}$ is measurable for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ by the previous proposition. Also, we need to know that $L^p(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$. This is trivially true for $p = 1$ and holds for $1 < p \leq \infty$ by an application of Hölder's inequality: for any compact $K \subset \mathbb{R}^n$,

$$\int_K |f| = \int_{\mathbb{R}^n} |f\chi_K| \leq \|f\|_{L^p} \|\chi_K\|_{L^q} = |K|^{1/q} \|f\|_{L^p},$$

where $1 \leq q < \infty$ is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Since $\mathcal{Q} \approx \mathcal{B}$ (see Definition 2.2.2 and the following paragraph), it follows that $M_{\mathcal{Q}} \approx M_{\mathcal{B}}$ pointwise almost everywhere. Hence, $M_{\mathcal{Q}}$ is strong/weak-type (p,p) if and only if $M_{\mathcal{B}}$ is. As such, the proof is provided for $M_{\mathcal{B}}$. In fact, the proof is actually provided for the restricted maximal function $M_{\mathcal{B}_R}$, $R > 0$, where $\mathcal{B}_R$ is the basis of all balls of radius at most R. Then, noticing that the bounds are independent of $R > 0$, we can send $R \to \infty$.

Proof. Fix $\alpha > 0$ and $f \in L^1(\mathbb{R}^n)$. For each $x \in \{M_{\mathcal{B}_R} f > \alpha\}$, there exists a ball $B \ni x$ such that

$$\fint_B |f| > \alpha.$$

From here it follows that

$$|B| \leq \frac{1}{\alpha} \int_B |f|$$

Consider the collection $\mathcal{F}$ of these balls. By assumption they all have radius at most R and they form a cover of $\{M_{\mathcal{B}_R} f > \alpha\}$. By the Vitali covering lemma (see Lemma 4.3.4), there is a countable pairwise-disjoint subcollection $\mathcal{G}$ for which

$$|\{M_{\mathcal{B}_R} f > \alpha\}| \leq \sum_{B \in \mathcal{G}} |5B| \leq 5^n \sum_{B \in \mathcal{G}} |B| \leq 5^n \sum_{B \in \mathcal{G}} \frac{1}{\alpha} \int_B |f| \leq \frac{5^n}{\alpha} \int_{\mathbb{R}^n} |f|.$$

This shows that $M_{\mathcal{B}_R}$ is weak-type $(1, 1)$. Note that countability of the subcollection $\mathcal{G}$ follows either since $\mathbb{R}^n$ is separable, hence Lindelöf, or since Lebesgue measure is doubling.

To show that $M_{\mathcal{B}_R}$ is strong-type (∞, ∞) is very easy: for $f \in L^\infty(\mathbb{R}^n)$,

$$M_{\mathcal{B}_R}(x) = \sup_{B \ni x} \fint_B |f| \leq \|f\|_{L^\infty}$$

for almost every $x \in \mathbb{R}^n$, and so $\|M_{\mathcal{B}_R}\|_{L^\infty} \leq \|f\|_{L^\infty}$.

To show the strong-type (p, p) result for $1 < p < \infty$, there are two options. On one hand, this is exactly what comes from an application of the Marcinkiewicz interpolation theorem since we already know that $M_{\mathcal{B}_R}$ is weak-type $(1, 1)$ and strong-type (∞, ∞).

On the other hand, one can prove the result directly by, in essence, proving the special case of the Marcinkiewicz interpolation theorem mentioned above. To that end, fix $\alpha > 0$ and write $f \in L^p(\mathbb{R}^n)$ as $f = f_1 + f_2$, where $f_1 = f\chi_{\{|f| \leq \alpha/2\}}$ and $f_2 = f\chi_{\{|f| > \alpha/2\}}$. Clearly, $f_1 \in L^\infty(\mathbb{R}^n)$ since it is bounded above by $\alpha/2$ almost everywhere. As $p > 1$, it follows that

$$\frac{2|f(x)|}{\alpha} \leq \left(\frac{2|f(x)|}{\alpha}\right)^p$$

if $|f(x)| > \alpha/2$, and so

$$|f(x)| \leq 2^{p-1}\alpha^{1-p}|f(x)|^p$$

almost everywhere. Thus,

$$\int_{\mathbb{R}^n} |f_2| = \int_{\mathbb{R}^n} |f|\chi_{\{|f| > \alpha/2\}} \leq 2^{p-1}\alpha^{1-p}\|f\|_{L^p}^p, \tag{5.1}$$

showing that $f_2 \in L^1(\mathbb{R}^n)$. Also, sublinearity of the maximal function implies that $M_{\mathcal{B}_R} f \leq M_{\mathcal{B}_R} f_1 + M_{\mathcal{B}_R} f_2 \leq \frac{\alpha}{2} + M_{\mathcal{B}_R} f_2$, and so

$$\{M_{\mathcal{B}_R} f > \alpha\} \subset \{M_{\mathcal{B}_R} f_2 > \alpha/2\}. \tag{5.2}$$

This, along with Cavalieri's principle, the weak-type $(1,1)$ estimate applied to f_2, and Fubini's theorem imply

$$\begin{aligned}
\int_{\mathbb{R}^n} |M_{\mathcal{B}_R} f|^p &= p \int_0^\infty \alpha^{p-1} |\{M_{\mathcal{B}_R} f > \alpha\}| \, d\alpha \\
&\leq p \int_0^\infty \alpha^{p-1} |\{M_{\mathcal{B}_R} f_2 > \alpha/2\}| \, d\alpha \\
&\leq 2 \cdot 5^n p \int_0^\infty \alpha^{p-2} \left(\int_{\mathbb{R}^n} |f_2| \right) d\alpha \\
&= 2 \cdot 5^n p \int_0^\infty \alpha^{p-2} \left(\int_{\mathbb{R}^n} |f| \chi_{\{|f| > \alpha/2\}} \right) d\alpha \\
&= 2 \cdot 5^n p \int_{\mathbb{R}^n} |f(x)| \left(\int_0^{2|f(x)|} \alpha^{p-2} \, d\alpha \right) dx \\
&= \frac{2^p 5^n p}{p-1} \int_{\mathbb{R}^n} |f|^p.
\end{aligned}$$

□

A consequence of this theorem is that $M_{\mathcal{Q}} f$ is finite almost everywhere for $f \in L^p(\mathbb{R}^n)$. This should be compared with the case of $f \in \mathrm{BMO}(\mathbb{R}^n)$, where $M_{\mathcal{Q}} f$ may be identically infinite (for instance, if $f(z) = -\log|z|$), even though $\mathrm{BMO}(\mathbb{R}^n) \subset \bigcap_{1 \leq p < \infty} L^p_{\mathrm{loc}}(\mathbb{R}^n)$. This last statement follows from combining Lemma 2.3.3 with the fact that $\mathrm{BMO}(\mathbb{R}^n) \cong \mathrm{BMO}^p(\mathbb{R}^n)$ for all $1 < p < \infty$ (see Corollary 2.7.7).

The fact that $M_{\mathcal{Q}}$ is weak-type $(1,1)$ does not preclude it from being strong-type $(1,1)$. What does preclude this is the following interesting fact: If $f \in L^1(\mathbb{R}^n)$ is not zero almost everywhere, then $M_{\mathcal{Q}} \notin L^1(\mathbb{R}^n)$. In other words, except for the zero element of $L^1(\mathbb{R}^n)$, the Hardy-Littlewood maximal function maps $L^1(\mathbb{R}^n)$ to outside of $L^1(\mathbb{R}^n)$. To see this, again switching to the basis $\mathcal{B}$, take r large enough so that

$$\alpha = \int_{B(0,r)} |f| > 0.$$

Then, for $|x| > r$, the ball $B(x, r + |x|)$ contains $B(0, r)$, and so monotonicity of the integral gives

$$M_{\mathcal{B}} f(x) \geq \fint_{B(x,r+|x|)} |f| \geq \frac{1}{|B(x, r + |x|)|} \int_{B(0,r)} |f| = \frac{\alpha}{\omega_n (r + |x|)^n},$$

where ω_n is the volume of the n-dimensional unit ball. The right-hand side is a non-integrable function of x, and so $M_{\mathcal{B}} f(x)$ cannot be integrable.

II.2 Throughout the paper, it is used, vaguely put, that given a measurable function f supported on a set of finite measure in $\mathbb{R}^n$, the essential infimum of restrictions of f to lower-dimensional cross-sections grows with the dimension of the cross-section. The simplest manifestation of this fact is the following.

Proposition 5.2.8. *Let S_1, S_2 be measurable subsets of positive and finite measure in $\mathbb{R}$ and f be a measurable real-valued function such that $\operatorname{ess\,inf}_S f > -\infty$, where $S = S_1 \times S_2$. Then,*

$$\min\{\operatorname*{ess\,inf}_{S_2} f_x, \operatorname*{ess\,inf}_{S_1} f_y\} \geq \operatorname*{ess\,inf}_{S} f$$

almost everywhere, where $f_x(y) = f_y(x) = f(x, y)$.

Proof. First, recall that the Lebesgue σ-algebra on $\mathbb{R}^2$ contains the family formed by the product of two sets in the Lebesgue σ-algebra on $\mathbb{R}$. As such, the measurability of S_1 and S_2 implies the measurability of S. Moreover, the Lebesgue measure on $\mathbb{R}^2$ coincides with the product measure of two copies of Lebesgue measure on $\mathbb{R}$, and so $|S| = |S_1||S_2|$.

By the definition of essential infimum, $\{(x, y) \in S : f(x, y) < \operatorname{ess\,inf}_S f\}$ has measure zero. By Fubini's theorem (Lebesgue measure is σ-finite), for almost every $y \in S_2$, the set $\{x \in S_1 : f_y(x) < \operatorname{ess\,inf}_S f\}$ is measurable and has measure zero. By the definition of essential infimum, again, this implies that for almost every $y \in S_2$, $\operatorname{ess\,inf}_{S_1} f_y \geq \operatorname{ess\,inf}_S f$. Hence,

$$|\{(x, y) \in S : \operatorname*{ess\,inf}_{S_1} f_y \geq \operatorname*{ess\,inf}_{S} f\}| = |S_1||\{y \in S_2 : \operatorname*{ess\,inf}_{S_1} f_y \geq \operatorname*{ess\,inf}_{S} f\}| = |S_1||S_2| = |S|.$$

Applying the previous argument in the other direction, we find that for almost every $x \in S_1$, $\operatorname{ess\,inf}_{S_2} f_x \geq \operatorname{ess\,inf}_S f$ and

$$|\{(x, y) \in S : \operatorname*{ess\,inf}_{S_2} f_x \geq \operatorname*{ess\,inf}_{S} f\}| = |S|.$$

Therefore, the set

$$\{(x,y) \in S : \min\{\operatorname*{ess\,inf}_{S_2} f_x, \operatorname*{ess\,inf}_{S_1} f_y\} \geq \operatorname*{ess\,inf}_{S} f\}$$

is the intersection of two sets of full measure and so, has full measure itself. The result follows from this. □

Appendix III: Rearrangement inequalities on spaces defined by mean oscillation

III.1 In the introduction, it is mentioned that BMO is not rearrangement invariant. A specific example is the following. Let f be the function given by $f(x) = -\log(2x)\chi_{(0,1/2)}$ which is in $\mathrm{BMO}(0,1)$. Then the function g defined by $g(x) = -\log(2x-1)\chi_{(1/2,1)}$ is just a translate of f, thus equimeasurable with f. This function fails to be in $\mathrm{BMO}(0,1)$.

III.2 Note that the definition of a function being rearrangeable is not uniform across the literature. Some authors choose only to define the decreasing rearrangement for measurable functions that vanish at infinity. That is, measurable functions for which $\mu_f(\alpha) < \infty$ for all $\alpha > 0$.

First, we show that notion is stronger than that of being rearrangeable as defined in the text.

Proposition 5.3.9. *If f be a measurable real-valued function that vanishes at infinity, then it is rearrangeable.*

Proof. Since $\mu_f(\alpha)$ is decreasing and bounded below by zero, its limit as $\alpha \to \infty$ exists. Consider the sequence $\{\alpha_n\}$ where $\alpha_n = n$. Then, the sets $E_n = \{|f| > n\}$ satisfy $E_{n+1} \subset E_n$, $\mu(E_n) = \mu_f(n)$, and $\bigcap_n E_n = \emptyset$. Since $\mu(E_n) < \infty$ for all n by assumption, continuity of μ from above implies that

$$\lim_{\alpha\to\infty} \mu_f(\alpha) = \lim_{n\to\infty} \mu_f(n) = \lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_n E_n\right) = \mu(\emptyset) = 0.$$

□

This shows that assuming that f vanishes at infinity is stronger than the assumption that μ_f vanishes at infinity. For $1 \le p < \infty$, every function $f \in L^p(\mathbb{R}^n)$ vanishes at infinity. This follow immediately from the fact that $L^p(\mathbb{R}^n) \subset L^p_w(\mathbb{R}^n)$. On the other hand, every function in L^∞ is rearrangeable, but does not necessarily vanish at infinity.

III.3 The preliminaries section contains some unproven assertions that are simple enough to prove, so some are included here.

First, note that the distribution $\mu_f : [0,\infty) \to [0,\infty]$ of a measurable function f is decreasing since, by monotonicity of measure,

$$\alpha \le \beta \Rightarrow \{x \in M : |f(x)| > \beta\} \subset \{x \in M : |f(x)| > \alpha\} \Rightarrow \mu_f(\beta) \le \mu_f(\alpha).$$

This implies, then, that the decreasing rearrangement is decreasing:

$$s_1 \le s_2 \Rightarrow \{\alpha \ge 0 : \mu_f(\alpha) \le s_1\} \subset \{\alpha \ge 0 : \mu_f(\alpha) \le s_2\} \Rightarrow f^*(s_1) \ge f^*(s_2).$$

Similarly, one shows that if $|f| \le |g|$, then $f^* \le g^*$.

Proposition 5.3.10 (Property (R1)). *For all $\alpha \ge 0$, $\mu_f(\alpha) = m_{f^*}(\alpha)$.*

Proof. Fix $\alpha \ge 0$ and write $E^*_\alpha = \{s \in M^* : f^*(s) > \alpha\}$. Then,

$$s \in E^*_\alpha \Leftrightarrow f^*(s) > \alpha \Leftrightarrow \inf\{\beta \ge 0 : \mu_f(\beta) \le s\} > \alpha \Leftrightarrow \mu_f(\alpha) > s,$$

since μ_f is decreasing and right-continuous.

Thus, $E^*_\alpha = \{s \in X^* : \mu_f(\alpha) > s\} = [0, \mu_f(\alpha))$ and so $m_{f^*}(\alpha) = |[0, \mu_f(\alpha))| = \mu_f(\alpha)$. $\square$

Proposition 5.3.11 (Property (R2)). *For any $\alpha \ge 0$ such that $\mu_f(\alpha) < \infty$ and $\beta > \alpha$,*

$$\mu(\{x \in M : \alpha < |f(x)| < \beta\}) = |\{s \in M^* : \alpha < f^*(s) < \beta\}|.$$

Proof. As the distribution is decreasing and right-continuous, we have that

$$\mu(\{x \in M : |f(x)| \ge \beta\}) = \lim_{\eta \to \beta^-} \mu_f(\eta).$$

Defining $\mu_f(\beta^-)$ to be this quantity, it is finite since μ_f is decreasing. Thus,

$$\mu(\{x \in M : \alpha < |f(x)| < \beta\}) = \mu_f(\alpha) - \mu_f(\beta^-).$$

By equimeasurability, $\mu_f(\alpha) = m_{f^*}(\alpha)$. Also,

$$\mu_f(\beta^-) = \lim_{\eta \to \beta^-} \mu_f(\eta) = \lim_{\eta \to \beta^-} m_{f^*}(\eta) = m_{f^*}(\beta^-),$$

keeping the notation consistent. Hence,

$$\mu_f(\alpha) - \mu_f(\beta^-) = m_{f^*}(\alpha) - m_{f^*}(\beta^-) = |\{s \in M^* : \alpha < f^*(s) < \beta\}|.$$

$\square$

Proposition 5.3.12 (Property (R3)). *The decreasing rearrangement is an isometry from $L^p(M)$ to $L^p(M^*)$ for all $1 \leq p < \infty$. Furthermore, it is non-expansive, thus continuous.*

Proof. That it is an isometry follows immediately from Cavalieri's principle and equimeasurability:

$$\int_M |f|^p = \int_0^\infty p\alpha^{p-1}\mu_f(\alpha)\, d\alpha = \int_0^\infty p\alpha^{p-1}\mu_{f^*}(\alpha)\, d\alpha = \int_{M^*} (f^*)^p.$$

To show that the decreasing rearrangement is non-expansive, first note that it suffices to show this for $f, g \geq 0$ almost everywhere as

$$\|f^* - g^*\|_{L^p(M^*)} = \||f|^* - |g|^*\|_{L^p(M^*)} \leq \||f| - |g|\|_{L^p(M)} \leq \|f - g\|_{L^p(M)}$$

by the reverse triangle inequality.

Take $0 \leq f, g \in L^p(M)$. As $f \leq \max(f, g)$ and $g \leq \max(f, g)$ almost everywhere, it follows that $f^* \leq \max(f, g)^*$ and $g^* \leq \max(f, g)^*$, and so $\max(f^*, g^*) \leq \max(f, g)^*$. Then, by equimeasurability,

$$\begin{aligned} \int_M |f - g|^p = \int_M [2\max(f, g) - (f + g)]^p &= \int_{M^*} [2\max(f, g)^* - (f^* + g^*)]^p \\ &\geq \int_{M^*} [2\max(f^*, g^*) - (f^* + g^*)]^p = \int_{M^*} |f^* - g^*|^p. \end{aligned}$$

$\square$

Proposition 5.3.13 (Property (R4)). *For any measurable set $A \subset M$,*

$$\int_A |f| \leq \int_{A^*} f^*,$$

where $A^* = (0, \mu(A))$.

Proof. We consider $(|f|\chi_A)^*$. Since $|f|\chi_A \le |f|$, we have that $\mu_{|f|\chi_A} \le \mu_f$ and so $(|f|\chi_A)^* \le f^*$.

If $\mu(A) = \infty$, then $\mu(M) = \infty$ and so $A^* = M^* = (0, \infty)$. Hence, the previous proposition with $p = 1$ gives

$$\int_A |f| = \int_M |f|\chi_A = \int_0^\infty (|f|\chi_A)^* \le \int_0^\infty f^*.$$

If $\mu(A) < \infty$, then $\mu_{|f|\chi_A} \le |A|$ implies that $(|f|\chi_A)^*(t) = 0$ for all $t \ge |A|$. Therefore, $(|f|\chi_A)^* \le f^*\chi_{A^*}$. Using this and the previous proposition with $p = 1$,

$$\int_A |f| = \int_M |f|\chi_A = \int_{M^*} (|f|\chi_A)^* \le \int_{M^*} f^*\chi_{A^*} = \int_{A^*} f^*.$$

□

Note that the following is an improvement (by a factor of 2) of Proposition 2.4.12.

Proposition 5.3.14 (Property (O4))**.** *For any shape S, if $\widetilde{S}$ is another shape such that $\widetilde{S} \subset S$ and $\mu(S) \le c\mu(\widetilde{S})$ for some constant c, then*

$$\Omega(f, \widetilde{S}) \le c\,\Omega(f, S).$$

Proof. If $f_S \le f_{\widetilde{S}}$, then by Property (O3), or Lemma 2.5.15,

$$⨍_{\widetilde{S}} |f - f_{\widetilde{S}}| = 2⨍_{\widetilde{S}} (f - f_{\widetilde{S}})_+ \le 2c⨍_S (f - f_S)_+ = c⨍_S |f - f_S|.$$

Likewise, if $f_S \ge f_{\widetilde{S}}$, then

$$⨍_{\widetilde{S}} |f - f_{\widetilde{S}}| = 2⨍_{\widetilde{S}} (f_{\widetilde{S}} - f)_+ \le 2c⨍_S (f_S - f)_+ = c⨍_S |f - f_S|.$$

□

Proposition 5.3.15 (Property (B1))**.** *If Q_0 is a cube in $\mathbb{R}^n$ of finite measure and $f \in$* BMO(Q_0)*, then $f - f_{Q_0} \in L^1(Q_0)$ with $\|f - f_{Q_0}\|_{L^1(Q_0)} \le |Q_0|\|f\|_{\mathrm{BMO}(Q_0)}$.*

Proof. This follows from Property (O1) and the fact that $f - f_{Q_0}$ has mean zero:

$$\int_{Q_0} |f - f_{Q_0}| = |Q_0|\Omega(f - f_{Q_0}, Q_0) = |Q_0|\Omega(f, Q_0) \le |Q_0|\|f\|_{\mathrm{BMO}(Q_0)}.$$

□

III.4 The key to the proof of the boundedness of the decreasing rearrangement from $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}_+)$ with constant 2^n is the Calderón-Zygmund lemma. This is a fundamental result in harmonic analysis.

Theorem 5.3.16 (Local Calderón-Zygmund lemma ([78])). *Let $0 \leq f \in L^1(Q_0)$ and $\alpha \geq f_{Q_0}$. Then there exists a countable collection of pairwise-disjoint subcubes $\{Q_i\}$ of Q_0 such that*

$$\alpha < \fint_{Q_i} f \leq 2^n \alpha$$

for each i and $f \leq \alpha$ almost everywhere on $Q_0 \setminus \bigcup Q_i$.

Proof. Bisect Q_0 along each of its n sides and denote by Q' one of the resulting subcubes. Then, one of two situations is possible: either $f_{Q'} > \alpha$, in which case it is placed into the desired collection; or, $f_{Q'} \leq \alpha$, in which case we do not. For these latter cubes, we repeat the process. As a result, we obtain a countable collection of pairwise disjoint cubes $\{Q_i\}$ for which $f_{Q_i} > \alpha$ holds for each i.

By construction, for each i, there is a cube ${Q'}_i \supset Q_i$ satisfying $f_{{Q'}_i} \leq \alpha$. Furthermore, Q_i was obtained by bisecting each side of ${Q'}_i$, so $|{Q'}_i| = 2^n |Q_i|$. Hence,

$$\fint_{Q_i} f \leq 2^n \fint_{{Q'}_i} f \leq 2^n \alpha.$$

By Lebesgue's differentiation theorem, for almost every $x \in Q_0$ we have that

$$f(x) = \lim_{Q} \fint_{Q} f$$

for every sequence of cubes $Q \ni x$ such that $\delta(Q) \to 0$. The construction yields for each $x \in Q_0 \setminus \bigcup Q_i$ precisely such a sequence of cubes with the further property that for each cube, $f_Q \leq \alpha$. Therefore, $f(x) \leq \alpha$ almost everywhere on $Q_0 \setminus \bigcup Q_i$. $\square$

In the text, it is claimed that the same process as above can be used within the context of the basis $\mathcal{D}$. With the proof written out, it becomes clear why. Whereas $\mathcal{Q}$ has the property that each cube can be bisected along each side to yield 2^n congruent subcubes, the basis $\mathcal{D}$ has the property that each rectangle can be bisected along its longest side to yield precisely 2 congruent subrectangles that are still in $\mathcal{D}$.

Also, the rectangles in $\mathcal{D}$ have bounded eccentricity due to the relationship between the sidelengths, guaranteeing that the Lebesgue differentiation theorem holds for these rectangles.

Theorem 5.3.17 (Local Calderón-Zygmund lemma for $\mathcal{D}$). *Let $0 \le f \in L^1(R_0)$, where $R_0 \in \mathcal{D}$, and $\alpha \ge f_{R_0}$. Then there exists a countable collection of pairwise disjoint subrectangles $\{R_i\} \subset \mathcal{D}$ of R_0 such that*

$$\alpha < \fint_{R_i} f \le 2\alpha$$

for each i and $f \le \alpha$ almost everywhere on $R_0 \setminus \bigcup R_i$.

III.5 In Remark 4.4.4, it is mentioned that many familiar bases satisfy the assumptions of the lemma. We give here a proof for $\mathcal{B}$, but the same steps work for other bases, including $\mathcal{Q}$ and $\mathcal{R}$. Note that a key part of the proof is the Lebesgue density theorem. It is well-known that it holds for the bases $\mathcal{Q}$ and $\mathcal{B}$, but it also holds for $\mathcal{R}$ (see [44]).

Lemma 5.3.18. *Let A be a measurable subset of $\mathbb{R}^n$ such that $|A| > 0$ and $|A^c| > 0$. Then, for all $t > 0$ there exists a ball $\widetilde{B}$ with $|\widetilde{B}| < t$ such that*

$$\frac{|A \cap \widetilde{B}|}{|\widetilde{B}|} = \frac{1}{2} = \frac{|A^c \cap \widetilde{B}|}{|\widetilde{B}|}.$$

Proof. Note, first, that measurability of A implies measurability of A^c and that, for any ball B, $|A \cap B| + |A^c \cap B| = |B|$.

Fix $t > 0$ and $\frac{1}{2} < \eta < 1$. By the Lebesgue density theorem, there exist $y \in A$, $z \in A^c$, and $r < t/2$ such that

$$\frac{|A \cap B(y,r)|}{|B(y,r)|} > \eta \qquad \text{and} \qquad \frac{|A^c \cap B(z,r)|}{|B(z,r)|} > \eta.$$

This implies that

$$\frac{|A \cap B(y,r)|}{|B(y,r)|} > \eta \qquad \text{and} \qquad \frac{|A \cap B(z,r)|}{|B(z,r)|} < 1 - \eta.$$

Then, the function

$$f(x) = \frac{|A \cap B(x,r)|}{|B(x,r)|}$$

is continuous and so there exists an $\widetilde{x}$ for which

$$\frac{|A \cap B(\widetilde{x},r)|}{|B(\widetilde{x},r)|} = \frac{1}{2} = \frac{|A^c \cap B(\widetilde{x},r)|}{|B(\widetilde{x},r)|}.$$

$\square$

III.6 In the proof of Theorem 4.4.9, the following fact is used without proof.

Proposition 5.3.19. *If $\{f_k\}$ is a sequence of monotone functions on $[a, b]$ converging pointwise almost everywhere to a uniformly continuous limit f, then the convergence is uniform.*

Proof. Fix $\varepsilon > 0$. By the uniform continuous of f, we may select a partition $a = x_0 < x_1 < \cdots < x_N = b$ of $[a, b]$ such that $|f(y) - f(z)| < \varepsilon/2$ for all $y, z \in [x_i, x_{i+1}]$ for each $i = 0, \ldots, N - 1$.

We impose the additional specification that the partition points are chosen such that $\{f_k(x_i)\}$ converges to $f(x_i)$ for each $i = 0, \ldots, N$. That is, there exist K_i such that if $k \geq K_i$, then $|f_k(x_i) - f(x_i)| < \varepsilon/2$. Write $K = \max_i K_i$.

Fix $x \in [a, b]$ and select i such that $x \in [x_i, x_{i+1}]$. For $k \geq K$, the previous two estimates gives us that

$$f(x) - \varepsilon < f(x_i) - \frac{\varepsilon}{2} < f_k(x_i) \leq f_k(x) \leq f_k(x_{i+1}) < f(x_{i+1}) + \frac{\varepsilon}{2} < f(x) + \varepsilon$$

if f_k is monotone increasing, and the opposite inequality if f_k is monotone decreasing.

In either case, we find that $|f_k(x) - f(x)| < \varepsilon$ for all $k \geq K$ and for all $x \in [a, b]$. □

III.7 At the very end of the proof of Theorem 4.4.9, the concept of uniform integrability is needed. We recall here those aspects of the theory that were used in the text. Let M be a measurable subset of $\mathbb{R}^n$ with the induced Lebesgue measure.

Definition 5.3.20. We say that a family $\mathcal{F}$ of measurable functions on M is uniformly integrable if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $A \subset M$ is measurable and $|A| < \delta$, then

$$\int_A |f| < \varepsilon$$

for all $f \in \mathcal{F}$.

Note that this condition is always true if $\mathcal{F}$ is a singleton comprised of one integrable function.

Proposition 5.3.21. *Let $\{f_k\}$ and $\{g_k\}$ be two uniformly integrable sequences on M. Then, their sum $\{f_k + g_k\}$ is also uniformly integrable on M.*

Proof. Fix $\varepsilon > 0$. By uniform integrability of $\{f_k\}$ there exists a δ_1 such that

$$|A| < \delta_1 \Rightarrow \int_A |f_k| < \frac{\varepsilon}{2}$$

for all k. By uniform integrability of $\{g_k\}$ there exists a δ_2 such that

$$|A| < \delta_2 \Rightarrow \int_A |g_k| < \frac{\varepsilon}{2}$$

for all k. Hence, if $\delta < \min(\delta_1, \delta_2)$, then

$$|A| < \delta \Rightarrow \int_A |f_k + g_k| \leq \int_A |f_k| + \int_A |g_k| < \varepsilon$$

for all k. $\square$

Proposition 5.3.22. *If $\{f_k\}$ is a convergent sequence in $L^1(M)$, then $\{f_k\}$ is uniformly integrable.*

Proof. Fix $\varepsilon > 0$. For each k, we have $f_k - f \in L^1(M)$ and so there exists a $\delta_k > 0$ for which

$$\int_A |f_k - f| < \varepsilon$$

whenever $|A| < \delta_k$. Select K such that $\|f_k - f\|_{L^1} < \varepsilon$ for $k \geq K$ and then set $\delta = \min_{1 \leq i \leq K} \delta_i$.

If $|A| < \delta$, then if $k \geq K$ it follows that

$$\int_A |f_k - f| \leq \|f_k - f\|_{L^1} < \varepsilon$$

and if $k < K$ it follows that $|A| < \delta_k$ and so

$$\int_A |f_k - f| < \varepsilon.$$

Therefore, $\{f_k - f\}$ is uniformly integrable. By Proposition 5.3.21, it follows that $\{f_k\} = \{(f_k - f) + f\}$.is uniformly integrable. $\square$

Proposition 5.3.23. *If $\{f_k\}$ is a uniformly integrable sequence on M and $\{g_k\}$ is another sequence of integrable functions on M such that $|g_k| \leq |f_k|$ for all k, then $\{g_k\}$ is also uniformly integrable.*

Proof. Fix $\varepsilon > 0$. Then, by uniform integrability of $\{f_k\}$, there is a $\delta > 0$ such that $|A| < \delta$ implies

$$\int_A |g_k| \leq \int_A |f_k| < \varepsilon$$

for all k, showing that $\{g_k\}$ is uniformly integrable. $\square$

Theorem 5.3.24 (Vitali convergence theorem [74]). *Let E be of finite measure. Suppose the sequence of functions $\{f_k\}$ is uniformly integrable over E. If $f_k \to f$ pointwise almost everywhere on E, then f is integrable over E and*

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

Chapter 6

List of function spaces

- L^p: the Lebesgue space of p-integrable functions.
- L^p_{loc}: the space of locally p-integrable functions.
- $W^{1,p}$: the Sobolev space of L^p functions with distributional first-order partial derivatives in L^p.
- JN_p: the John-Nirenberg space.
- $\mathrm{BMO}_{\mathscr{S}}$: the space of functions of bounded mean oscillation with respect to a basis of shapes $\mathscr{S}$.
- $\mathrm{BMO}_{\text{rec},\mathscr{S}}$: the rectangular BMO space with respect to a basis of shapes $\mathscr{S}$.
- $\mathrm{BLO}_{\mathscr{S}}$: the class of functions of bounded lower oscillation with respect to a basis of shapes $\mathscr{S}$.
- $\mathrm{BLO}_{\text{rec},\mathscr{S}}$: the rectangular BLO space with respect to a basis of shapes $\mathscr{S}$.
- $\mathrm{VMO}_{\mathscr{S}}$: the space of functions of vanishing mean oscillation with respect to a basis of shapes $\mathscr{S}$.

Chapter 7

Bibliography

[1] D. Aalto, L. Berkovits, O. E. Maasalo, H. Yue. John-Nirenberg lemmas for a doubling measure. *Studia Math.*, 204(1):21-37, 2011.

[2] C. Bennett. Another characterization of BLO. *Proc. Amer. Math. Soc.* 85(4):552–556, 1982.

[3] C. Bennett, R. A. De Vore, R. Sharpley. Weak-L^∞ and BMO. *Ann. Math.*, 113(2):601-611, 1981.

[4] C. Bennett, R. Sharpley. *Interpolation of Operators*. Academic Press, Inc., Boston, MA, 1988.

[5] O. Beznosova, A. Reznikov. Dimension-Free Properties of Strong Muckenhoupt and Reverse Hölder Weights for Radon Measures. *J. Geom. Anal*, 29:1109–1115, 2019.

[6] A. Björn, J. Björn. *Nonlinear potential theory on metric spaces*, volume 17 of *Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2011.

[7] G. Bourdaud. Remarques sur certains sous-espaces de $BMO(\mathbb{R}^n)$ et de $bmo(\mathbb{R}^n)$ (in French). *Ann. Inst. Fourier (Grenoble)*, 52(4):1187–1218, 2002.

[8] H. Brezis, L. Nirenberg. Degree theory and BMO. I. Compact manifolds without boundaries. *Selecta Math. (N.S.)* 1(2):197-263, 1995.

[9] H. Brezis, L. Nirenberg. Degree theory and BMO. II. Compact manifolds with boundaries. With an appendix by the authors and Petru Mironescu. *Selecta Math. (N.S.)* 2(3):309-368, 1996.

[10] A. Burchard. A Short Course on Rearrangement Inequalities. Lecture notes. http://www.math. utoronto.ca/almut/rearrange.pdf, 2009.

[11] A. Butaev, On some refinements of the embedding of critical Sobolev spaces into bmo. *Pacific J. Math.*, 298(1):1-26, 2019.

[12] L. Carleson. A counterexample for measures bounded on H^p on the bi-disc. *Mittag Leffler*, Report No. 7, 1974.

[13] S.-Y. A. Chang, R. Fefferman. A continuous version of duality of H^1 with BMO on the bidisc. *Ann. of Math. (2)*, 112(1):179-201, 1980.

[14] S.-Y. A. Chang, R. Fefferman. Some recent developments in Fourier analysis and H^p theory on product domains. *Bull. Amer. Math. Soc. (N.S.)* 12(1):1-43, 1985.

[15] Y. Z. Chen, K. S. Lau. Some new classes of Hardy spaces. *J. Funct. Anal.* 84: 255-278, 1989.

[16] F. Chiarenza, M. Frasca. Morrey spaces and Hardy-Littlewood maximal function. *Rend. Mat. Appl. (7)* 7(3-4):273-279, 1987.

[17] A. Cianchi, L. Pick. Sobolev embeddings into BMO, VMO, and L_∞. *Ark. Math.*, 36(2):317-340, 1998.

[18] R. R. Coifman. R. Rochberg. Another characterization of BMO. *Proc. Amer. Math. Soc.* 79(2):249–254, 1980.

[19] R. Coifman, G. Weiss. Extensions of Hardy spaces and their use in analysis. Bull. Amer. Math. Soc., 83(4):569-645, 1977.

[20] A. Córdoba, R. Fefferman. On differentiation of integrals. *Proc. Nat. Acad. Sci. U.S.A.*. 74(6):2211-2213, 1977.

[21] A. Córdoba. Maximal functions: a proof of a conjecture of A. Zygmund. *Bull. Amer. Math. Soc. (N.S.)*, 1(1):255-257, 1979.

[22] M. Cotlar, C. Sadosky. Two distinguished subspaces of product BMO and Nehari-AAK theory for Hankel operators on the torus. *Integral Equations and Operator Theory*, 26(3):273-304, 1996.

[23] D. Cruz-Uribe,C. J. Neugebauer. The structure of the reverse Hölder classes. *Trans. Amer. Math. Soc.* 347(8):2941–2960, 1995.

[24] M. Cwikel, Y. Sagher and P. Shvartsman. A new look at the John-Nirenberg and John-Strömberg theorems for BMO. Lecture notes. arXiv:1011.0766v1 [math.FA], 2010.

[25] M. Cwikel, Y. Sagher and P. Shvartsman. A new look at the John-Nirenberg and John-Strömberg theorems for BMO. *J. Funct. Anal.* 263:129–166, 2012.

[26] G. Dafni, R. Gibara. BMO on shapes and sharp constants. *Advances in harmonic analysis and partial differential equations*, volume 748 of *Contemp. Math.* 1–33, Amer. Math. Soc., Providence, RI, 2020.

[27] G. Dafni, T Hytönen, R. Korte, H. Yue. The space JN_p: nontriviality and duality. *J. Funct. Anal.*, 275(3):577-603, 2018.

[28] G. Dafni, H. Yue. Some characterizations of local bmo and h^1 on metric measure spaces. *Anal. Math. Phys.* 2(3):285-318, 2012.

[29] J. Duoandikoetxea. The Hardy-Littlewood maximal function and some of its variants. *Advanced courses of mathematical analysis. II*, 37–56, World Sci. Publ., Hackensack, NJ, 2007.

[30] X. T. Duong, J. Li, Y. Ou, B. D. Wick, D. Yang. Product BMO, little BMO, and Riesz commutators in the Bessel setting. *J. Geom. Anal*, 28(3):2558-2601, 2018.

[31] X. T. Duong, J. Li, B. D. Wick, D. Yang. Commutators, little BMO, and weak factorization. *Ann. Inst. Fourier (Grenoble)*, 68(1):109-129, 2018.

[32] C. Fefferman. Characterizations of bounded mean oscillation. *Bull. Amer. Math. Soc.*, 77:587-588, 1971.

[33] C. Fefferman, E. M. Stein. H^p spaces of several variables. *Acta. Math.*, 129(3-4):137-193, 1972.

[34] R. Fefferman. Fourier analysis in several parameters. *Rev. Mat. Iberoamericana* 2(1-2):89-98, 1986.

[35] S. Ferguson, C. Sadosky. Characterizations of bounded mean oscillation on the polydisk in terms of Hankel operators and Carleson measures. *J. Anal. Math.* 81:239-267, 2000.

[36] J. García-Cuerva. Hardy spaces and Beurling algebras. *J. Lond. Math. Soc.* 39:499-513, 1989.

[37] J. B. Garnett. *Bounded analytic functions*, revised first edition, volume 236 of *Graduate Texts in Mathematics*. Springer, New York, 2007.

[38] A. M. Garsia. *Martingale inequalities: Seminar notes on recent progress*. W. A. Benjamin, Inc., Reading, Mass.-London-Amsterdam, 1973.

[39] A. M. Garsia, E. Rodemich. Monotonicity of certain functionals under rearrangement. *Ann. Inst. Fourier (Grenoble)*, 24(2):67-116, 1974.

[40] D. Goldberg. A local version of real Hardy spaces. *Duke Math. J.* 46(1):27-42, 1979.

[41] B. I. Golubov. On the boundedness of the Hardy and the Hardy-Littlewood operators in the spaces ReH1 and BMO. (in Russian) *Mat. Sb.* 188(7):93-106, 1997; translation in *Sb. Math.* 188(7):1041–1054, 1997.

[42] L. Grafakos. *Modern Fourier Analysis*, third edition, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, 2014.

[43] L. Grafakos, L. Liu, C. Pérez, R. H. Torres. The multilinear strong maximal function. *J. Geom. Anal.* 21(1):118-149, 2011.

[44] M. de Guzmán. *Differentiation of Integrals in R^n*, with appendices by Antonio Córdoba and Robert Fefferman, and two by Roberto Moriyón, volume 481 of Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1975.

[45] M. Guzmán-Partida. CLO spaces and central maximal operators. *Arch. Math. (Brno)* 49(2):119-124, 2013.

[46] P. Hagelstein, T. Luque, I. Parissis. Tauberian conditions, Muckenhoupt weights, and differentiation properties of weighted bases. *Trans. Amer. Math. Soc.* 367(11):7999–8032, 2015.

[47] P. Hagelstein, A. Stokolos. Tauberian conditions for geometric maximal operators. *Trans. Amer. Math. Soc.* 361(6):3031-3040, 2009.

[48] J. Heinonen. *Lectures on analysis on metric spaces*, *Universitext*. Springer-Verlag, New York, 2001.

[49] R. Hurri-Syrjänen, N. Marola, A. Vähäkangas. Aspects of local-to-global results. *Bull. Lond. Math. Soc.*, 46(5):1032-1042, 2014.

[50] T. Hytönen. A framework for non-homogeneous analysis on metric spaces, and the RBMO space of Tolsa. *Publ. Mat.* 54(2):485–504, 2010.

[51] B. Jawerth, A. Torchinsky. The strong maximal function with respect to measures. Studia Math., 80(3):261-285, 1984.

[52] B. Jessen, J. Marcinkiewicz, A. Zygmund. Note on the differentiability of multiple integrals. Fund. Math., 25:217-234, 1935.

[53] F. John. Rotation and strain, *Comm. Pure Appl. Math.*, 14:391-413, 1961.

[54] F. John, L. Nirenberg. On functions of bounded mean oscillation. *Comm. Pure Appl. Math.*, 14:415-426, 1961.

[55] P. W. Jones. Extension theorems for BMO, *Indiana Univ. Math. J.*, 29(1):41-66, 1980.

[56] J.-L. Journé. Calderón-Zygmund operators, pseudodifferential operators and the Cauchy integral of Calderón, volume 994 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[57] I. Klemes. A mean oscillation inequality. *Proc. Amer. Math. Soc.*, 93(3):497-500, 1985.

[58] A. A. Korenovskiĭ. The connection between mean oscillations and exact exponents of summability of functions (in Russian). *Mat. Sb.*, 181(12): 1721-1727, 1990; translation in *Math. USSR-Sb.* 71(2): 561-567, 1992.

[59] A. A. Korenovskiĭ. The Riesz "rising sun" lemma for several variables, and the John-Nirenberg inequality (in Russian). *Mat. Zametki*, 77(1):53-66, 2005; translation in *Math. Notes* 77(1-2): 48–60, 2005.

[60] A. A. Korenovskii. *Mean Oscillations and Equimeasurable Rearrangements of Functions*, volume 4 of *Lecture Notes of the Unione Matematica Italiana* . Springer-Verlag Berlin Heidelberg, 2007.

[61] A. A. Korenovskyy, A. K. Lerner, A. M. Stokolos. On a multidimensional form of F. Riesz's "rising sun" lemma. *Proc. Amer. Math. Soc.*, 133(5):1437-1440, 2005.

[62] J. Lakey. Constructive decomposition of functions of finite central mean oscillation. *Proc. Amer. Math. Soc.* 127: 2375–2384, 1999.

[63] E. Yu. Leonchik. Oscillations of functions and differential-difference properties of singular integrals (in Russian). Diss., Odessa, 2003.

[64] A. K. Lerner. BMO-boundedness of the maximal operator for arbitrary measures. *Israel J. of Math.*, 159:243-252, 2007.

[65] A. K. Lerner. The John-Nirenberg inequality with sharp constants. *C. R. Math. Acad. Sci. Paris*, 351(11-12):463-466, 2013.

[66] E. H. Lieb, M. Loss. *Analysis*, second edition, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[67] S. Lu and D. Yang. The central BMO spaces and Littlewood-Paley operators. *Approx. Theory Appl.* 11: 72–94, 1995.

[68] T. Luque, C. Pérez, E. Rela. Reverse Hölder property for strong weights and general measures. *J. Geom. Anal.*, 27(1):162-182, 2017.

[69] N. Marola, O. Saari. Local to global results for spaces of BMO type. *Math. Z.*, 282(1-2):473-484, 2016.

[70] U. Neri. Some properties of functions with bounded mean oscillation. *Studia Math.*, 61(1):63-75, 1977.

[71] E. N. Nikolidakis. Dyadic-BMO functions, the dyadic Gurov-Reshetnyak condition on $[0,1]^n$ and equimeasurable rearrangements of functions. *Ann. Acad. Sci. Fenn. Math.*, 42(2):551-562, 2017.

[72] W. Ou. The natural maximal operator on BMO. *Proc. Amer. Math. Soc.* 129(10):2919–2921, 2001.

[73] H. M. Reimann, T. Rychener. *Funktionen beschränkter mittlerer Oszillation* (in German), volume 487 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1975.

[74] H. L. Royden, P. M. Fitzpatrick. *Real Analysis*, Fourth Edition, Prentice Hall, 2010.

[75] O. Saari. Poincaré inequalities for the maximal function. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 19(3):1065–1083, 2019.

[76] S. Saks. On the strong derivatives of functions of intervals. *Fund. Math.*, 25:235-252, 1935.

[77] D. Sarason. Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc.* 207:391–405, 1975.

[78] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

[79] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[80] E. M. Stein, R. Shakarchi. *Functional Analysis. Introduction to further topics in analysis*, volume 4 of *Princeton Lectures in Analysis*. Princeton University Press, NJ, 2011.

[81] E. M. Stein, G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*, volume 32 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1971.

[82] A. Stokolos. Zygmund's program: some partial solutions. *Ann. Inst. Fourier (Grenoble)* 55(5):1439–1453, 2005.

[83] A. Stokolos. Properties of the maximal operators associated to bases of rectangles in $\mathbb{R}^3$. *Proc. Edinb. Math. Soc. (2)* 51(2):489–494, 2008.

[84] D. M. Stolyarov, V. I. Vasyunin, P. B. Zatitskiy. Monotonic rearrangements of functions with small mean oscillation. *Studia Math.*, 231(3):257-267, 2015.

[85] A. Uchiyama. On the compactness of operators of Hankel type, *Tôhoku Math. J. (2)* 30(1):163–171, 1978.

[86] A. Vasyunin, A. Vol'berg. Sharp constants in the classical weak form of the John-Nirenberg inequality. *Proc. Long. Math. Soc (3)*, 108(6):1417-1434, 2014.

[87] D. Yang. Local Hardy and BMO spaces on non-homogeneous spaces. *J. Aust. Math. Soc.*, 79(2):149–182, 2005.

[88] W. Zhang. The Boundedness of the Hardy-Littlewood Maximal Function and the Strong Maximal Function on the Space BMO, CMC Senior Theses, 2018.

[89] A. Zygmund. A note on the differentiability of integrals. *Colloq. Math.*, 16:199-204, 1967.