Convergence Results for Rearrangements: Old and New.

by

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Abstract

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The purpose of this thesis is twofold. On the one hand, it aims to give a thorough review and exposition of current best results regarding approximating the symmetric decreasing rearrangement by polarizations and Steiner symmetrizations. These results include those of Van Schaftingen on explicit universal approximation to the symmetric decreasing rearrangement by sequences of polarizations as well as his results on almost sure convergence of rearrangements to the symmetric decreasing rearrangement. They also include those of Klartag and Milman which yield rates of convergence for Steiner symmetrizations of convex bodies. On the other hand, new results are proven. We extend Van Schaftingen's results on almost sure convergence of polarizations and Steiner symmetrizations by showing that the conditions on the random variables can be weakened without affecting almost sure convergence to the symmetric decreasing rearrangement. Lastly, we derive rates of convergence for polarizations and Steiner symmetrizations of Hölder continuous functions.

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Contents

Chapter 1

Introduction

1.1 Steiner Symmetrization and the Isoperimetric Problem

Steiner symmetrization was invented by Jacob Steiner to solve the isoperimetric problem. The isoperimetric problem in the plane says that among all closed and simple curves of fixed length, the circle and only the circle encloses the most area. By using simple scaling arguments, one can show that the isoperimetric problem is equivalent to showing that among all sets of fixed area with simple closed curves as boundaries, the disc and only the disc has minimum boundary length or minimum *perimeter*. Steiner first argued that one could concentrate on convex compact sets (convex bodies) since the convex hull of a non-convex set had larger area but smaller perimeter. He then showed that applying a Steiner symmetrization to a convex body always strictly decreased perimeter while preserving area unless that very same symmetrization did not change the initial convex body. Since Steiner symmetrization preserves convexity and any convex body that is invariant under all Steiner symmetrizations is necessarily a disc then Steiner concluded that the isoperimetric problem was solved. His colleagues quickly objected to his proof claiming that it was incomplete since he had failed to prove that a minimizer existed in the first place. This is analogous to the claim that a smooth function is minimized at a point simply because its derivative vanishes only at that point.

Today, we know that Steiner's proof can be completed by appealing to a convergence result for Steiner symmetrization namely that given any initial convex body in the plane there exists a sequence of Steiner symmetrization that transforms it (in the Hausdorff metric sense) to a disc of equal area. Since perimeter is continuous with respect to the Hausdorff metric on the space of convex bodies, then any convex body of fixed area has perimeter greater or equal to that of a disc of equal area. This shows that a perimeter minimizing body exists and (by Steiner's argument) is uniquely the disc.

The example above highlights one of the important applications of convergence results for rearrangements namely that the Schwartz symmetrization of convex bodies (in any dimension) can be approximated by Steiner symmetrizations of convex bodies. The main focus of this thesis is to give a thorough presentation of old and new convergence results for rearrangements. We focus on three rearrangements and the connection between them: polarization, Schwartz symmetrization and Steiner symmetrization. A recurring theme in this thesis is to approximate Schwartz and Steiner symmetrization by polarization (the simplest rearrangement). It is often the case that one can prove properties of Schwartz and Steiner symmetrization, which seem at first difficult to prove, by approximating these rearrangements by polarizations and proving that the analogous properties for polarization hold (see [3] for many applications of this approach). The content of the thesis is motivated by *three convergence problems for rearrangements* that we now describe in detail.

1.2 Three Convergence Problems for Rearrangements

In what follows, f^σ will denote the polarization of f with respect to the reflection σ and $S_u(f)$ will denote the Steiner symmetrization of f with respect to u. Furthermore, for any finite sequence of unit vectors $\{u_i\}_{i=1}^n$, we let

$$S_{u_1,\ldots,u_n}(f) = \left(\bigcirc_{i=1}^n S_{u_i}\right)(f). \tag{1.1}$$

For more on notation and general background on rearrangements see section 2.1 ("Preliminaries").

Problem 1: Explicit Universal Approximation.

Uniform Convergence for $C_c(\mathbb{R}^d)$

Does there exist explicit sequences of reflections $\{\sigma_i\}_{i=1}^\infty$ and unit vectors $\{u_i\}_{i=1}^\infty$ such that

$$\lim_{n\to\infty} \|f^{\sigma_1\cdots\sigma_n} - f^*\|_\infty = 0 \tag{1.2}$$

and

$$\lim_{n\to\infty} \|S_{u_1,\ldots,u_n}(f) - f^*\|_\infty = 0 \tag{1.3}$$

for *every* $f \in C_c(\mathbb{R}^d)$? In Chapter 3 ("Explicit Universal Approximation"), we give a self-contained exposition of a result of Van Schaftingen's [16] that not only proves the existence of sequences $\{\sigma_i\}_{i=1}^\infty$ satisfying (1.2) but yields an algorithm for constructing such sequences. The reader should note that in the following theorem, Ω denotes the space of reflections across hyperplanes not containing the origin. In subsection 3.1.2 ("Metrics on Ω"), we construct a metric ρ on Ω.

Theorem 1. *[17] Let $\{\sigma_n\}_{n=1}^\infty \subset \Omega$ be a dense subset of Ω with respect to the metric ρ. If $f \in C_c(\mathbb{R}^d)$ and*

$$f_{n+1} = \begin{cases} f^{\sigma_1} & n = 0 \\ f_n^{\sigma_1\cdots\sigma_{n+1}} & n \geq 1 \end{cases}$$

then f_n converges uniformly to f^*. In particular, the entire sequence $f^{\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\cdots}$ con-

verges uniformly to f^* *for every* $f \in C_c(\mathbb{R}^d)$ *and in* $L^p(\mathbb{R}^d)$ *for every* $f \in L^p(\mathbb{R}^d)$ *with* $1 \le p < \infty$.

In order to prove Theorem 1, Van Schaftingen uses certain properties of the polarization rearrangement. However, it can be shown (see Section 3.2) that analogous properties also hold for Steiner symmetrization and consequently we extend Theorem 1 to Steiner symmetrization:

Theorem 2. *Let* $\{u_n\}_{n=1}^{\infty} \subset S^{d-1}$ *be a dense subset of* S^{d-1} *with respect to the Euclidean metric. If* $f \in C_c(\mathbb{R}^d)$ *and*

$$f_{n+1} = \begin{cases} S_{u_1}(f) & n = 0 \\ S_{u_1,\dots,u_{n+1}}(f_n) & n \ge 1 \end{cases}$$

then f_n *converges uniformly to* f^*. *In particular, the entire sequence*

$$\{S_{u_1}(f), S_{u_1,u_2}(f), S_{u_1,u_2,u_1,u_2,u_3}(f), \dots\}$$

converges uniformly to f^* *for every* $f \in C_c(\mathbb{R}^d)$ *and in* $L^p(\mathbb{R}^d)$ *for every* $f \in L^p(\mathbb{R}^d)$ *with* $1 \le p < \infty$.

Convergence in Hausdorff Distance

Let $\delta(\cdot,\cdot)$ denote the Hausdorff distance. Does there exist explicit sequences of reflections $\{\sigma_i\}_{i=1}^{\infty}$ and unit vectors $\{u_i\}_{i=1}^{\infty}$ such that

$$\lim_{n\to\infty} \delta(F^{\sigma_1\cdots,\sigma_n}, F^*) = 0 \tag{1.4}$$

and

$$\lim_{n\to\infty} \delta(S_{u_1,\dots,u_n}(F), F^*) = 0 \tag{1.5}$$

for *every* compact set F ? In section 11 ("Convergence in Hausdorff Distance "), it is shown that the sequences satisfying properties 1.2 and 1.3 also satisfy properties 1.4 and 1.5. The essential idea is to (by using distance functions) express each compact set F as a level set of a compactly supported continuous function. Then, by appealing to the properties 1.2 and 1.3, one can show that 1.4 and 1.5 also holds.

Problem 2: Almost Sure Convergence.

The Basic Problem

In a paper of Mani-Levitska [11], it was shown that if Y_i is a sequence of independent random variables distributed uniformly on S^{d-1} then with probability 1,

$$\lim_{n\to\infty} \delta(S_{Y_1,\dots,Y_n}(K), K^*) = 0 \tag{1.6}$$

for every compact convex set K. The very same author asked whether such a result holds for arbitrary compact sets and this problem became known as the "Mani-Levitska Conjecture".

In Chapter 4 ("Almost Sure Convergence"), we consider the more general problem of understanding which sequences of random variables X_i and Y_i distributed in Ω and S^{d-1} respectively satisfy the properties

$$P\left(\lim_{n\to\infty} \|f^{X_1\cdots X_n} - f^*\|_\infty = 0 \quad \forall f \in C_c(\mathbb{R}^d)\right) = 1 \tag{1.7}$$

and

$$P\left(\lim_{n\to\infty} \|S_{Y_1,\dots,Y_n}(f) - f^*\|_\infty = 0 \quad \forall f \in C_c\mathbb{R}^d)\right) = 1. \tag{1.8}$$

By our previous discussion regarding convergence in Hausdorff distance, we know that if 1.8 holds when Y_i are independent and uniformly distributed in S^{d-1} then the Mani-Levitska conjecture also holds.

Existing Results

This Mani-Levitska conjecture appears to be settled first by Van Schaftingen [17]. We briefly outline his result and the method of proof. Suppose that ρ_1 and ρ_2 are metrics on Ω and S^{d-1} respectively with the property that for every fixed $f \in C_c(\mathbb{R}^d)$, the map $f \mapsto f^\sigma$ and the map $f \mapsto S_u(f)$ is continuous in L^p then the following holds:

Theorem 3. *[17] If X_i and Y_i are sequences of independent random variables distributed in Ω and S^{d-1} respectively satisfying the properties*

$$\liminf_{n\to\infty} P(X_n \in B_{\rho_1}(\sigma,\lambda)) > 0 \tag{1.9}$$

and

$$\liminf_{n\to\infty} P(Y_n \in B_{\rho_2}(u,\lambda)) > 0 \tag{1.10}$$

for all $(\sigma,u) \in \Omega \times S^{d-1}$ and for all $\lambda > 0$ then

$$P\left(\lim_{n\to\infty} \|f^{X_1\cdots X_n} - f^*\|_\infty = 0 \quad \forall f \in C_c(\mathbb{R}^d)\right) = 1 \tag{1.11}$$

and

$$P\left(\lim_{n\to\infty} \|S_{Y_1,\dots,Y_n}(f) - f^*\|_\infty = 0 \quad \forall f \in C_c\mathbb{R}^d)\right) = 1. \tag{1.12}$$

To prove Theorem 3, Van Schaftingen first considered sequences $\{\sigma_i\}_{i=1}^\infty \subset \Omega$ and $\{u_i\}_{i=1}^\infty \subset S^{d-1}$ such that

$$\lim_{n\to\infty} \|f^{\sigma_1\cdots\sigma_n} - f^*\|_\infty = 0 \tag{1.13}$$

and

$$\lim_{n\to\infty} \|S_{u_1,\dots,u_n}(f) - f^*\|_\infty = 0 \tag{1.14}$$

for every $f \in C_c(\mathbb{R}^d)$. Theorems 1 and 2 show that such sequences exist. He then showed that (1.13) and (1.14) also holds for all sequences

$$\{\{\tau_i\}_{i=1}^{\infty} : \forall \epsilon > 0, \forall m \geq 1, \exists k \ni \rho_1(\tau_{k+i}, \sigma_i) \leq \epsilon \quad \forall 1 \leq i \leq m\} \tag{1.15}$$

and

$$\{\{v_i\}_{i=1}^{\infty} : \forall \epsilon > 0, \forall m \geq 1, \exists k \ni \rho_2(v_{k+i}, u_i) \leq \epsilon \quad \forall 1 \leq i \leq m\} \tag{1.16}$$

and that these sequences had probability 1 under the conditions given in Theorem 3.

Volcic [18] also obtained a result about random sequences of Steiner symmetrizations. In that paper, it is claimed that if Y_i is a sequence of pairwise independent random variables distributed uniformly in S^{d-1} then

$$P\left(\lim_{n\to\infty} \|S_{Y_1,\dots,Y_n}(f) - f^*\|_p = 0 \quad \forall f \in L^p(\mathbb{R}^d)\right) = 1 \tag{1.17}$$

for all $1 \leq p < \infty$. Volcic main probabilistic tool is the Borel-Cantelli Lemma which holds for pairwise independent random variables (see [5, pp.50-51]). However, we are not entirely convinced of the correctness of the use of the Borel-Cantelli Lemma in Volcic's paper. As a result, it is not clear whether pairwise independence is enough to obtain 1.17. That being said, the proof of the new results on almost sure convergence that is presented in this thesis were partially inspired by Volcic's paper.

New Results

In Chapter 4, we extend Theorem 3 by showing that the conditions on the random variables X_i and Y_i can be weakened. More precisely, we will show the following:

Theorem 4. *If $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent random variables distributed in Ω with the property that for every bounded sequence of points $\{x_i\}_1^{\infty} \subset \mathbb{R}^d$ and for every $0 < \lambda < |x_i|$ for all $i \geq 1$*

$$\sum_{i=1}^{\infty} \mu_i(x_i, B_i) = \infty \tag{1.18}$$

for every sequence of balls $B_i \subset (|x_i| - \lambda)B^d$ *of uniform radius then*

$$P\left(\lim_{n\to\infty} \|f^{X_1\cdots X_n} - f^*\|_\infty = 0 \quad \forall f \in C_c(\mathbb{R}^d)\right) = 1 \tag{1.19}$$

and

$$P\left(\lim_{n\to\infty} \|f^{X_1\cdots X_n} - f^*\|_p = 0 \quad \forall f \in L^p(\mathbb{R}^d)\right) = 1. \tag{1.20}$$

Theorem 5. *If* Y_i *is a sequence of independent random variables distributed in* S^{d-1} *with the property*

$$\sum_{i=1}^{\infty} P(Y_i \in B(u_i, \lambda)) \tag{1.21}$$

for every sequence $\{u_i\}_{i=1}^{\infty} \subset S^{d-1}$ *and for every* $\lambda > 0$ *then*

$$P\left(\lim_{n\to\infty} \|S_{Y_1,\ldots,Y_n}(f) - f^*\|_\infty = 0 \quad \forall f \in C_c(\mathbb{R}^d)\right) = 1 \tag{1.22}$$

and

$$P\left(\lim_{n\to\infty} \|S_{Y_1,\ldots,Y_n}(f) - f^*\|_p = 0 \quad \forall f \in L^p(\mathbb{R}^d)\right) = 1. \tag{1.23}$$

In the last section of Chapter 4 ("Examples") it is shown that if the conditions 1.9 and 1.10 of Theorem 3 are satisfied then so are the conditions 1.18 and 1.21 of Theorems 4 and 5 (see Propositions 12 and 13). Lastly, to show that Theorems 4 and 5 truly extend Theorem 3, we give examples of sequences of independent random variables X_i and Y_i such that the conditions 1.18 and 1.21 of Theorems 4 and 5 are satisfied but not the conditions 1.9 and 1.10 of Theorem 3.

Problem 3: Rates of Convergence.

Theorems 4 and 5 show that almost every random sequence of polarizations and Steiner symmetrizations will transform (in the uniform convergence sense) any compactly supported continuous function into its corresponding decreasing symmetric rearrangement and will also transform (in the Hausdorff distance sense) any compact set into its corresponding Schwartz symmetrization. A natural question arises: how fast can convergence occur? More precisely, let $\mathcal{C} \subset C_c(\mathbb{R}^d)$ be a collection of functions that is invariant under polarization or Steiner symmetrization i.e., $f^\sigma \in \mathcal{C}$ for all $(f, \sigma) \in \mathcal{C} \times \Omega$ or $S_u(f) \in \mathcal{C}$ for all $(f, u) \in \mathcal{C} \times S^{d-1}$. Given $\epsilon > 0$, what is the minimal number of polarizations (or Steiner symmetrizations) needed to transform every function $f \in \mathcal{C}$ into a new function $f' \in \mathcal{C}$ satisfying the property $\|f - f^*\|_\infty \leq \epsilon$? Similarly, let $\mathcal{A}$ be a collection of compact sets in $\mathbb{R}^d$ that is invariant under polarization or Steiner symmetrization. Given $\epsilon > 0$, what is the minimal number of polarizations (or Steiner symmetrizations) needed to transform every compact set $F \in \mathcal{A}$ into a new compact set F' with the property $\delta(F', F^*) \leq \epsilon$?

Existing Results

To our knowledge, there are no published results on rates of convergence for polarizations. There are, however, some very nice results regarding rates of convergence for Steiner symmetrizations of convex bodies. The following result, due to Klartag [9], is currently the best in the literature:

Theorem 6 (Theorem 1.5)**.** *[9]There exists a numerical constant C such that for every $0 < \epsilon < 1$, we need at most*

$$\lceil C d^4 \log^2(1/\epsilon) \rceil$$

Steiner symmetrizations to transform an arbitrary convex body K with volume κ_d into a convex body K' with the property

$$(1 - \epsilon) B^d \subset K' \subset (1 + \epsilon) B^d.$$

The final Chapter ("Rates of Convergence: Convex Bodies ") is dedicated entirely to giving a self-contained exposition of Theorem 6. Apart from providing all the necessary background in spherical harmonics and differential geometry, many proofs are presented differently and some are simplified. The most notable difference being the presentation of the L^∞ decay of spherical harmonics and related combinatorial estimates (see section 6.2.4 "L^∞ Decay of Spherical Harmonics").

New Results

In Chapter 5, we provide rates of convergence for both polarization and Steiner symmetrizations applied to particular subsets $\mathcal{C}$ of $C_c(\mathbb{R}^d)$. We denote by $C^{0,\alpha}(A)$ the space of all Hölder continuous functions with exponent $0 < \alpha \le 1$ defined on a set $A \subset \mathbb{R}^d$ i.e., the space of all functions f with the property that there exists some positive constant c such that

$$|f(x) - f(y)| \le c|x-y|^\alpha. \tag{1.24}$$

If $f \in C^{0,\alpha}(A)$ then we let

$$[f]_\alpha = \sup\{|f(x) - f(y)||x-y|^{-\alpha} : x \ne y\}. \tag{1.25}$$

In our applications, the domain of f will always be fixed and so there need not be any domain dependence for the symbol $[f]_\alpha$. The following is proved:

Theorem 7. *There exists an explicit constant $C(\alpha, d, \lambda_1, \lambda, \lambda_2)$ such that for all $\epsilon > 0$ there exists at most*

$$\left\lceil C(\alpha, d, \lambda_1, \lambda, \lambda_2)(1/\epsilon)^{(1+(d+1)/\alpha)(1+\frac{d}{\alpha})} \right\rceil \tag{1.26}$$

polarizations that transform any function $f \in C^{0,\alpha}(\lambda B^d)$ with $[f]_\alpha \le \lambda_1$ and $\|f - f^*\|_\infty \le \lambda_2$ into f' with the property

$$\|f' - f^*\|_\infty \le \epsilon. \tag{1.27}$$

Theorem 8. *For all $\epsilon > 0$ there exists at most*

$$\left\lceil C(\alpha, d, \lambda_1, \lambda, \lambda_2)(1/\epsilon)^{(1+(d+1)/\alpha)(1+\frac{d}{\alpha})} \right\rceil \tag{1.28}$$

Steiner symmetrizations that transform any function $f \in C^{0,\alpha}(\lambda B^d)$ *with* $[f]_\alpha \leq \lambda_1$ *and* $\|f - f^*\|_\infty \leq \lambda_2$ *into* f' *with the property*

$$\|f' - f^*\|_\infty \leq \epsilon. \tag{1.29}$$

The constant $C(\alpha, d, \lambda_1, \lambda, \lambda_2)$ is the same in both Theorems and will be computed explicitly in the proof of Theorem 7.

Unlike Theorem 6, the rates of convergence in Theorems 7 and 8 are not exponential. They do, however, constitute a first step towards understanding rates of convergence for rearrangements of particular subsets of $C_c(\mathbb{R}^d)$. Lastly, we believe that Theorems 7 and 8 combined with the method of proof of Proposition 11, can be used to obtain rates of convergence for polarizations and Steiner symmetrizations of compact sets. Unfortunately, we did not have time to investigate this further.

Chapter 2

Fundamental Properties of Rearrangements

2.1 Premilinaries

2.1.1 Notation

Given any metric ρ, $B_\rho(x,\lambda)$ will denote the closed ball of radius λ centered at x i.e., the set of all points whose distance from x (with respect to the metric ρ) is at most λ. For convenience, when ρ is the Euclidean metric we will denote a ball of radius λ centered at x by $B(x,\lambda)$. In addition, B^d and S^{d-1} will denote the unit ball and unit sphere in $\mathbb{R}^d$. The volume and surface area of B^d will be denoted by κ_d and γ_d respectively.

Given any two real numbers a and b, $a \wedge b$ and $a \vee b$ will denote the minimum and maximum of a and b respectively. If a is any real number then $a^+ = a \vee 0$ and $a^- = -a \vee 0$ will denote the positive and negative part of a respectively.

The space of continuous functions of compact support will be denoted by $C_c(\mathbb{R}^d)$. If $f \in C_c(\mathbb{R}^d)$, we let

$$\omega(f,h) = \sup\{|f(x) - f(y)| : |x - y| \le h\} \tag{2.1}$$

denote the *modulus of continuity* of f an define the “inverse”of $\omega(f,h)$ by

$$\omega^{-1}(f,t) = \sup\{h : \omega(f,h) \le t\}. \tag{2.2}$$

$\mathcal{M}$ will denote the sigma-algebra consisting of all Lebesgue measurable subsets of $\mathbb{R}^d$ and m will denote the corresponding Lebesgue measure. We suppress the dependence of the symbols $\mathcal{M}$ and m on the dimension d since it will always be clear from the context which dimension d applies. We also let θ denote the normalized surface area measure on S^{d-1}.

Given any set A, $\chi(A)$ will denote its characteristic function:

$$\chi(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \tag{2.3}$$

. We will often encounter set mappings Ξ on level sets $\{x : f(x) > t\}$ for some t and f. Instead of writing $\Xi(\{x : f(x) > t\})$, we will simply write $\Xi(f > t)$. As an example, if μ is a measure for which $\{x : f(x) > t\}$ is measurable for some t and f then we write $\mu(f > t)$ to denote $\mu(\{x : f(x) > t\})$.

We define Ω to be the collection of all reflections through hyperplanes not containing 0 in $\mathbb{R}^d$. Arbitrary elements of Ω are denoted by σ. Lastly, if σ is a reflection through a hyperplane X_0^σ then this hyperplane splits $\mathbb{R}^d$ into two open half-spaces. The half-space containing zero will be denoted by X_+^σ and the other by X_-^σ.

2.1.2 Introduction to Rearrangements

We now introduce the concept of rearrangements of sets and functions. Our presentation follows that of [3]. A rearrangement T is a map $T : \mathcal{M} \to \mathcal{M}$ that is both *monotone* ($A \subset B$ implies $T(A) \subset T(B)$) and *measure preserving* ($m(T(A)) = m(A)$ for all $A \in \mathcal{M}$).

We say that a non-negative measurable function f *vanishes at infinity* if

$$m(f > t) < \infty \tag{2.4}$$

for all $t > 0$. If f vanishes at infinity then we can define its rearrangement Tf by using the "layer cake principle"

$$Tf(x) = \int_0^\infty \chi_{T(f>t)}(x)dt = \sup\{t : x \in T(f > t)\}. \tag{2.5}$$

We immediately see that

$$\bigcup_{i=1}^{\infty} T(f > t + 1/n) \subset \{Tf > t\} \subset T(f > t) \tag{2.6}$$

for all $t \geq 0$. However, we also have

$$\begin{aligned} m\left(\bigcup_{i=1}^{\infty} T(f > t + 1/n)\right) &= \lim_{n\to\infty} m(T(f > t + 1/n)) \\ &= \lim_{n\to\infty} m(f > t + 1/n) \\ &= m(f > t) \end{aligned}$$

for all $t \geq 0$ and thus

$$m(f > t) = m(Tf > t) \tag{2.7}$$

for all $t \geq 0$.

In this thesis, we will be primarily concerned with three types of rearrangements: polarization, symmetric decreasing rearrangement and Steiner symmetrization.

Polarization

Let $\sigma \in \Omega$ and let f be an arbitrary function. We define the *polarization of f with respect to σ* as

$$f^\sigma(x) = \begin{cases} f(x) \vee f(\sigma(x)) & \text{if } x \in X_+^\sigma \\ f(x) \wedge f(\sigma(x)) & \text{if } x \in X_-^\sigma \\ f(x) & \text{if } x \in X_0^\sigma \end{cases} \tag{2.8}$$

. If A is an arbitrary set then its polarization with respect to σ is simply the polarization of χ_A and is denoted by A^σ:

$$A^\sigma = (\sigma(A \cap X_-^\sigma) \cap A^c) \cup (\sigma(A \cap X_+^\sigma) \cap A) \cup (A \cap X_+^\sigma) \cup (A \cap X_0^\sigma). \tag{2.9}$$

In other words, A^σ is the same as A except that the part of A contained in X_-^σ whose reflection does not lie in A is replaced by its reflection in X_+^σ. As a result, we see that polarization is measure preserving. It is clear from 2.8 that if $f \leq g$ then $f^\sigma \leq g^\sigma$ for all $\sigma \in \Omega$ and in particular polarization is monotone on $\mathcal{M}$ i.e., polarization is a rearrangement. One can check directly that $\{f^\sigma > t\} = \{f > t\}^\sigma$ for all $\sigma \in \Omega$ and in particular (recalling 2.5) f^σ is the rearrangement of f with respect to the polarization rearrangement.

Symmetric Decreasing Rearrangement

For any $A \in \mathcal{M}$ there exists a unique open ball centered at the origin A^* with the same measure as A. A^* is called the *Schwartz symmetrization of A*. If $f(x)$ vanishes at infinity then its rearrangement with respect to the Schwartz rearrangement is denoted by $f^*(x)$. It is clear that $f^*(x)$ is radially decreasing i.e., $f^*(x) \leq f^*(y)$ if $|x| \geq |y|$ and $f(x) = f(y)$ if $|x| = |y|$. In the literature, f^* is also called the *symmetric decreasing rearrangement of*

f. If f vanishes at infinity, we let

$$r_f(t) = \left(\frac{m(f > t)}{\kappa_d}\right)^{1/d} \tag{2.10}$$

denote the radius of the open ball $\{f > t\}^*$. The distribution function of a function f vanishing at infinity is always right continuous and thus so is $r_f(t)$. In particular, we have

$$\{f^* > t\} = \{f > t\}^* \tag{2.11}$$

for all $t \geq 0$ and thus f^* is right continuous.

Steiner Symmetrization

If $u \in S^{d-1}$ and $A \in \mathcal{M}$ then, by Fubini's theorem, the set

$$A_{x'} = \{\langle x, u\rangle : x' + u\langle x, u\rangle \in A\} \tag{2.12}$$

is measurable in $\mathbb{R}$ for almost every $x' \in u^\perp$. If we let $A^*_{x'}$ equal the empty set whenever $A_{x'}$ is non-measurable then we denote by

$$S_u(A) = \bigcup_{x' \in u^\perp} \{x' + uA^*_{x'}\} \tag{2.13}$$

the *Steiner symmetrization of* A *with respect to* u. It is clear that Steiner symmetrization is monotone and, by Fubini's theorem, measure preserving. If f vanishes at infinity we denote by $S_u(f)$ the rearrangement of f with respect to the rearrangement S_u or simply the *Steiner symmetrization of* f *with respect to* u. Lastly, we let

$$S_{u_1,\ldots,u_n}(f) = \left(\bigcirc_{i=1}^n S_{u_i}\right)(f) \tag{2.14}$$

for any finite sequence of unit vectors $\{u_i\}_{i=1}^n$.

Now that we have introduced rearrangements, we warn the reader that whenever we speak of a rearrangement of f then one should always assume that f is both vanishing at infinity and non-negative. There will thus be no need to continuously state that the function must be vanishing at infinity and non-negative prior to considering its rearrangement.

2.2 Rearrangement Inequalities

Proposition 1. *(Polarization Formula for Functionals)If $\sigma \in \Omega$ then*

$$\int_{\mathbb{R}^d} [f^\sigma g^\sigma - fg] dx = \int_{X_\pm^\sigma} [(f(x) - f(\sigma(x)))(g(x) - g(\sigma(x)))]^- dx$$

for any two functions f and g.

Proof. If a_1, a_2, b_1, b_2 are real numbers with $a_1 \leq a_2$ and $b_1 \leq b_2$ then

$$(a_1 b_1 + a_2 b_2) - (a_1 b_2 + a_2 b_1) = (a_1 - a_2)(b_1 - b_2) \geq 0. \tag{2.15}$$

Given $x \in \mathbb{R}^d$, we let

$$\begin{aligned} a_1(x) &= f(x) \wedge f(\sigma(x)) \quad , \quad a_2(x) = f(x) \vee f(\sigma(x)) \\ b_1(x) &= g(x) \wedge g(\sigma(x)) \quad , \quad b_2(x) = g(x) \vee g(\sigma(x)). \end{aligned}$$

Recalling the definition of polarization of functions (see 2.8), one has

$$f^\sigma(x) g^\sigma(x) + f^\sigma(x) g^\sigma(\sigma(x)) = a_1(x) b_1(x) + a_2(x) b_2(x). \tag{2.16}$$

If $(f(x) - f(\sigma(x)))(g(x) - g(\sigma(x))) < 0$ then we must have

$$(f(x) - f(\sigma(x)))(g(\sigma(x)) - g(x)) = (a_1(x) - a_2(x))(b_1(x) - b_2(x)) > 0. \tag{2.17}$$

It is clear that $(f(x) - f(\sigma(x)))\,(g(x) - g(\sigma(x))) \geq 0$ if and only if $(f^\sigma g^\sigma - fg)(x) + (f^\sigma g^\sigma - fg)(\sigma(x)) = 0$. By 2.15,2.16 and 2.17, we must have

$$(f^\sigma g^\sigma - fg)(x) + (f^\sigma g^\sigma - fg)(\sigma(x)) = [(f(x) - f(\sigma(x)))\,(g(x) - g(\sigma(x)))]^-. \qquad (2.18)$$

Integrating both sides of 2.18 over either X_+^σ or X_-^σ and using the fact that σ is an isometry yields

$$\int_{\mathbb{R}^d} [(f^\sigma g^\sigma)(x) - (fg)(x)]dx = \int_{X_\pm^\sigma} [(f(x) - f(\sigma(x)))\,(g(x) - g(\sigma(x)))]^- dx \geq 0.$$

□

Corollary 1. *Let $f \in C_c(\mathbb{R}^d)$ and let g be any integrable and strictly radially decreasing function i.e., $g(x) > g(y)$ if $|x| < |y|$. Then for every $\sigma \in \Omega$, $f^\sigma = f$ if and only if*

$$\int_{\mathbb{R}^d} f(x)g(x)dx = \int_{\mathbb{R}^d} f^\sigma(x)g(x)dx. \qquad (2.19)$$

Proof. If 2.19 holds then by Proposition 1, we have $f(x) \geq f(\sigma(x))$ for all $x \in X_+^\sigma$ which is the same as $f^\sigma = f$. □

Proposition 2. *(Hardy-Littlewood Inequality) If T is a rearrangement then*

$$\int_{\mathbb{R}^d} fg dx \leq \int_{\mathbb{R}^d} T(f)T(g)dx. \qquad (2.20)$$

for any two functions f and g.

Proof. We first note that any non-negative function h has the following "layer cake" representation

$$h(x) = \int_0^\infty \chi_{h>t}(x)dt. \qquad (2.21)$$

In particular, we have by 2.21 and Fubini's theorem

$$\int_{\mathbb{R}^d} f(x)g(x)dx = \int_0^\infty \int_0^\infty m(f > t \cap g > s)dtds. \tag{2.22}$$

It thus suffices to prove that for every pair of non-negative real numbers s and t

$$m(f > t \cap g > s) \leq m(T(f) > t \cap T(g) > s). \tag{2.23}$$

By the measure preserving and monotonicity properties of T, we have

$$m(f > t \cap g > s) = m(T(f > t \cap g > s)) \leq m(T(f > t) \cap T(g > s)). \tag{2.24}$$

However, recalling 2.6 and 2.7, we must have

$$m(T(f > t) \cap T(g > s)) = m(Tf > t \cap Tg > s). \tag{2.25}$$

Using 2.24 and 2.25 yields 2.23 and completes the proof. □

Proposition 3. *(L^p Contraction) If $1 \leq p < \infty$ and T is any rearrangement then*

$$\|T(f) - T(g)\|_p \leq \|f - g\|_p$$

for any $(f, g) \in L^p(\mathbb{R}^d) \times L^p(\mathbb{R}^d)$.

Proof. The idea will be to follow the same line of reasoning that was used to prove the Hardy-Littlewood inequality i.e., we want to show that the validity of the Proposition follows from the inequality 2.23. We first observe that $|f(x) - g(x)|^p$ equals

$$p \int_0^\infty \left[(f(x) - t)^{p-1} \chi_{g \leq t} \chi_{f > t}(x) + (g(x) - t)^{p-1} \chi_{f \leq t} \chi_{g > t}(x) \right] dt. \tag{2.26}$$

We have $\chi_{g\leq t}(x) = 1 - \chi_{g>t}(x)$ and similarly for f. In addition, we have

$$\int_{\mathbb{R}^d}\int_0^\infty p(f(x)-t)^{p-1}\chi_{f>t}(x)dtdx = \|f\|_p^p = \|T(f)\|_p^p \tag{2.27}$$

and similarily for g. By considering the layer cake representation of $(f(x)-t)^{p-1}$ and that of $(g(x)-t)^{p-1}$ and by applying Fubini's theorem, it thus suffices to prove that for any positive real numbers s,t

$$m\left((f-t)^{p-1} > s \cap g > t\right) \leq m\left((T(f)-t)^{p-1} > s \cap T(g) > t\right) \tag{2.28}$$

and

$$m\left((g-t)^{p-1} > s \cap f > t\right) \leq m\left((T(g)-t)^{p-1} > s \cap T(f) > t\right). \tag{2.29}$$

However, 2.28 and 2.29 easily follow from 2.23. □

By taking $p \to \infty$ in the last Proposition, we obtain:

Corollary 2. *(L^∞ Contraction) If T is any rearrangement then*

$$\|T(f) - T(g)\|_\infty \leq \|f-g\|_\infty$$

for all $(f,g) \in C_c(\mathbb{R}^d) \times C_c(\mathbb{R}^d)$.

2.3 Compactness Properties

Proposition 4. *Let $f \in C_c(\mathbb{R}^d)$ with modulus of continuity $\omega(f,h)$ then $\omega(f^\sigma,h) \leq \omega(f,h)$ for all $\sigma \in \Omega$.*

Proof. Let x and y be two points in $\mathbb{R}^d$ with $|x-y| \leq h$. It is clear that $|f^\sigma(x) - f^\sigma(y)|$ equals one of the following four values :

$$\{|f(x)-f(y)|, |f(\sigma(x)) - f(\sigma(y))|, |f(\sigma(x)) - f(y)|, |f(x) - f(\sigma(y))|\}. \tag{2.30}$$

Since σ is an isometry, it suffices to consider only the last two cases in 2.30. If x, y belong to different half spaces $X_{\pm}^{\sigma}$ then $|x - \sigma(y)| = |\sigma(x) - y| < |x - y|$ and so the last two cases of 2.30 are bounded by $\omega(f, h)$. We are thus left with the case where x, y belong to the same half space $X_{\pm}^{\sigma}$ and the last two cases of 2.30 hold. In general for any four real numbers a_1, a_2, b_1, b_2 with $a_1 \leq a_2$ and $b_1 \leq b_2$ we have $|a_2 - b_2| \leq |a_1 - b_2|$ and $|a_1 - b_1| \leq |a_1 - b_2|$. In particular $|f(x) - f(\sigma(y))|$ and $|f(\sigma(x)) - f(y)|$ are bounded by $|f(x) - f(y)| \leq \omega(f, h)$. □

Corollary 3. *Let* $\{\sigma_n\}_{n=1}^{\infty} \subset \Omega$ *then* $\{f^{\sigma_1 \cdots \sigma_n}\}_{n=1}^{\infty}$ *is precompact in* $C_c(\mathbb{R}^d)$ *(with respect to uniform convergence) for every* $f \in C_c(\mathbb{R}^d)$.

Proof. It is clear that each element of $\{f^{\sigma_1 \cdots \sigma_n}\}_{n=1}^{\infty}$ is bounded by $\|f\|_{\infty}$. If f has support in λB^d for some $\lambda > 0$ then

$$\{f^{\sigma_1 \cdots \sigma_n} > 0\} = \{f > 0\}^{\sigma_1 \cdots \sigma_n} \subset (\lambda B^d)^{\sigma_1 \cdots \sigma_n} = \lambda B^d.$$

By the previous Proposition, $\{f^{\sigma_1 \cdots \sigma_n}\}_{n=1}^{\infty}$ is an equicontinuous family and thus, by the Arzelà-Ascoli theorem, the sequence $\{f^{\sigma_1 \cdots \sigma_n}\}_{n=1}^{\infty}$ has a subsequence which converges uniformly to a function f in $C_c(\mathbb{R}^d)$. □

Chapter 3

Explicit Universal Approximation

3.1 Sequences of Polarizations

3.1.1 Characterization of f^* via Polarizations

Proposition 5. *Let $f \in C_c(\mathbb{R}^d)$ then f^* is also in $C_c(\mathbb{R}^d)$. Furtheremore, $f = f^*$ if and only if $f^\sigma = f$ for all $\sigma \in \Omega$.*

Proof. We first show that if $f \in C_c(\mathbb{R}^d)$ then $f^* \in C_c(\mathbb{R}^d)$. If $f^*(r)$ denotes the value of f^* at $|x| = r$ then because $||x|-|y|| \leq |x-y|$, it suffices to show that $f^*(r)$ is continuous. In fact, since $f^*(r)$ is always right continuous it suffices to prove that $f^*(r)$ is left continuous. If this is false then there is a jump at a point $r_1 > 0$ i.e., there exists $\epsilon > 0$ such that for all $r < r_1$, $f^*(r) - f^*(r_1) \geq \epsilon$. In particular, we have

$$m(f^*(r_1) < f(x) < f^*(r_1) + \epsilon) = \varnothing \tag{3.1}$$

and this would contradict the continuity of f so f^* is also continuous.

To prove the second part of the Proposition, we first observe that f is radially decreasing if and only if

$$(f(x) - f(\sigma(x))(|x| - |\sigma(x)|) \geq 0 \tag{3.2}$$

for all x and for all $\sigma \in \Omega$. However, 3.2 is also equivalent to $f^\sigma = f$ for all $\sigma \in \Omega$ and thus $f^\sigma = f$ for all $\sigma \in \Omega$ is equivalent to f being radially decreasing. It thus suffices to show that if f is radially decreasing then $f = f^*$. If this is the case then $\{f > t\}^* = \{f > t\}$ since $\{f > t\}$ is an open ball and this in turn yields

$$f^*(x) = \sup_{t\geq 0} x \in \{f > t\}^* = \sup_{t\geq 0} x \in \{f > t\} = f(x) \tag{3.3}$$

for all x. □

3.1.2 Metrics on Ω

Given any two points x and y with $|x| \neq |y|$, we let $\sigma_{x,y}$ denote the unique reflection that maps x to y. In particular, the map

$$x \mapsto \sigma_{0,x} \tag{3.4}$$

establishes a one to one correspondence between $\mathbb{R}^d - \{0\}$ and Ω. In fact, one has

$$\sigma_{0,x}(y) = y + \left\langle x - 2y, \frac{x}{|x|} \right\rangle \frac{x}{|x|} \tag{3.5}$$

As a consequence, if ρ' is any metric on $\mathbb{R}^d$ then letting $\rho(\sigma_1, \sigma_2) = \rho'(\sigma_1(0), \sigma_2(0))$ yields a metric ρ on Ω. The following Proposition shows that the metric

$$\rho(\sigma_1, \sigma_2) = |\sigma_1(0) - \sigma_2(0)| \tag{3.6}$$

on Ω induces a pleasant continuity property.

Proposition 6. *For every fixed $f \in C_c(\mathbb{R}^d)$ and for every $1 \leq p < \infty$, the map $f \mapsto f^\sigma$ is continuous with respect to the metric $\rho(\sigma_1, \sigma_2) = |\sigma_1(0) - \sigma_2(0)|$ on Ω and the L^p norm on $C_c(\mathbb{R}^d)$.*

Proof. Let $\sigma_1, \sigma_2 \in \Omega$ and suppose that the support of f is contained in λB^d for some $\lambda > 0$. From 3.5, we deduce that $|\sigma_2(x) - \sigma_1(x)|$ is bounded by

$$|\sigma_1(0) - \sigma_2(0)| + 4\lambda \left| \frac{\sigma_1(0)}{|\sigma_1(0)|} - \frac{\sigma_2(0)}{|\sigma_2(0)|} \right| \tag{3.7}$$

for all $x \in \lambda B^d$. If we let

$$A(\sigma_1, \sigma_2, \lambda) = (X_+^{\sigma_1} \cap X_-^{\sigma_2}) \cup (X_-^{\sigma_1} \cap X_+^{\sigma_2}) \cap \lambda B^d$$

then because

$$|(f(x) \wedge f(\sigma(x))) - (f(x) \wedge f(\sigma_2(x)))| \leq |f(\sigma_1(x) - f(\sigma_2(x))|$$

and

$$|(f(x) \vee f(\sigma(x))) - (f(x) \vee f(\sigma_2(x)))| \leq |f(\sigma_1(x)) - f(\sigma_2(x))|$$

we obtain the inequality

$$\|f^{\sigma_1} - f^{\sigma_2}\|_p^p \leq 2\|f\|_\infty^p m(A(\sigma_1, \sigma_2, \lambda)) + \int_{\lambda B^d} |f(\sigma_1(x) - f(\sigma_2(x))|^p dx. \tag{3.8}$$

We note that $A(\sigma_1, \sigma_2, \lambda)$ also equals

$$\{x : |\sigma_1(x)| < |x| < |\sigma_2(x)|, |x| \leq \lambda\} \cup \{x : |\sigma_2(x)| < |x| < |\sigma_1(x)|, |x| \leq \lambda\}. \tag{3.9}$$

By 3.7,3.9 and the uniform continuity of f on λB^d, it is clear that the right-hand side of the inequality 3.8 converges to zero as σ_2 approaches σ_1. □

3.1.3 Uniform Convergence in $C_c(\mathbb{R}^d)$ and Convergence in $L^p(\mathbb{R}^d)$.

proof of Theorem 1. By Corollary 3, the sequence of functions f_n is precompact in $C_c(\mathbb{R}^d)$ and consequently we can find a subsequence $\{\sigma_{n_k}\}_{k=1}^{\infty}$ such that f_{n_k} converges uniformly to h. We wish to show that $h = f^*$. We let

$$I(f) = \int_{\mathbb{R}^d} e^{-|x|} f(x) dx. \tag{3.10}$$

Let $l \geq 1$ and choose k large enough that $n_k \geq l$. We have by the contraction property of rearrangements (Proposition 3)

$$\lim_{k\to\infty} \| f_{n_k}^{\sigma_1 \cdots \sigma_l} - h^{\sigma_1 \cdots \sigma_l} \|_\infty \leq \lim_{k\to\infty} \| f_{n_k} - h \|_\infty = 0. \tag{3.11}$$

By Proposition 1, the sequence $I(f_n)$ is increasing and thus by 3.11

$$I(h^{\sigma_1 \cdots \sigma_l}) = \lim_{k\to\infty} I(f_{n_k}^{\sigma_1 \cdots \sigma_l}) \leq \lim_{k\to\infty} I(f_{n_{k+1}}) = I(h). \tag{3.12}$$

By Propositon 1, this implies $h^{\sigma_l} = h$ for all $l \geq 1$ and by the density of the sequence $\{\sigma_l\}_{l=1}^{\infty}$ combined with Proposition 6, we obtain $h^{\sigma} = h$ for all $\sigma \in \Omega$. By Proposition 5, we conclude that $h = h^*$ and since h was an arbitrary limit point we have that f_n converges uniformly to $h = h^*$. The equimeasurability of the functions f_n is not enough to conclude that $h^* = f^*$. The key is to use the contraction property (Corollary 2) for the symmetric decreasing rearrangement. More precisely, we have

$$\| f^* - h \|_\infty \leq \| f^* - f_n^* \|_\infty + \| f_n^* - h^* \|_\infty = \| f_n^* - h^* \|_\infty \leq \| f_n - h \|_\infty \tag{3.13}$$

for all $n \geq 1$ and by taking limits in 3.13, we obtain the desired result. The fact that $f^{\sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_3 \cdots}$ converges uniformly to f^* follows directly from Corollary 2. If $f \in L^p(\mathbb{R}^d)$ then we can find a sequence of functions $f_k \in C_c(\mathbb{R}^d)$ such that f_k converges in $L^p(\mathbb{R}^d)$

to f. If $\{\tau_n\}_{n=1}^{\infty}$ denotes the sequence $\{\sigma_1, \sigma_2, \sigma_1, \sigma_2, \sigma_3, \ldots\}$ then for all $(k,n) \in \mathbb{N} \times \mathbb{N}$

$$\begin{aligned} \|f^{\tau_1\cdots\tau_n} - f^*\|_p &\leq \|f^{\tau_1\cdots\tau_n} - f_k^{\tau_1\cdots\tau_n}\|_p + \|f_k^{\tau_1\cdots\tau_n} - f_k^*\|_p + \|f_k^* - f^*\|_p. \\ &\leq \|f_k^{\tau_1\cdots\tau_n} - f_k^*\|_p + 2\|f_k - f\|_p. \end{aligned}$$

Sending n to infinity and then k to infinity yields the desired result. □

Corollary 4. *For any $f \in C_c(\mathbb{R}^d)$, $\omega(f^*, h) \leq \omega(f, h)$.*

Proof. By Theorem 1, there exists a sequence $\{\sigma_n\}_{n=1}^{\infty}$ such that $f^{\sigma_1\cdots\sigma_n}$ converges to f^* uniformly for all $f \in C_c(\mathbb{R}^d)$. By Proposition 4, we have

$$\omega(f^*, h) = \lim_{n\to\infty} \omega(f^{\sigma_1\cdots\sigma_n}, h) \leq \omega(f, h).$$

□

If $m(E) < \infty$ then

$$\mathrm{Per}_M(E) = \lim_{\epsilon \downarrow 0} \frac{m(E + \epsilon B^d) - m(E)}{\epsilon} \tag{3.14}$$

denotes the *Minkowski perimeter* of E whenever the limit 3.14 exists.

We will now use Corollary 4 to give a very short proof of the isoperimetric inequality.

Proposition 7. *(Isoperimetric Inequality) For any measurable set E with finite Minkowski perimeter and volume κ_d*

$$Per_M(E) \geq \gamma_d. \tag{3.15}$$

Proof. Consider the function $f(x) = 1 - d(E,x)$ with $d(E,x)$ the distance function from E restricted to $E + B^d$. It is well known that $f(x)$ is a contraction and clearly $f(x) = 1$ if and only if $x \in \overline{E}$. By corollary 4, f^* is also a contraction and in particular

$$\liminf_{t\downarrow 0} \frac{r(1-t) - 1}{t} \geq 1. \tag{3.16}$$

However, we also have

$$\lim_{t\downarrow 0}\frac{r(1-t)-1}{t} = \kappa_d^{-1/d}\lim_{t\downarrow 0}\frac{m(f>1-t)^{1/d}-\kappa_d^{1/d}}{t} = \gamma_d^{-1}\mathrm{Per}_M(E). \tag{3.17}$$

Combining 3.16 and 3.17 completes the proof. □

3.2 Sequences of Steiner Symmetrizations

Proposition 8. *Let* $\{r_n\}_{n=1}^{\infty}$ *be any dense subset of* $\mathbb{R}$ *and form the sequence* $\{\sigma_n = \sigma_{0,r_n u}\}_{n=1}^{\infty} \subset \Omega$. *Then for all* $f \in C_c(\mathbb{R}^d)$, $f^{\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\cdots}$ *converges uniformly to* $S_u(f)$ *and* $\omega(S_u(f),h) \le \omega(f,h)$ *for all* $h > 0$.

Proof. For any $x' \in u^{\perp}$, we let $f_{x'}(r) = f(x'+ru)$. By Theorem 1 (applied to $d=1$), $f^{\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\cdots}(x'+ru)$ converges pointwise to $f_{x'}^*(r)$ for every $x' \in u^{\perp}$. If $x = x' + x''$ with $x'' = \langle x,u\rangle u$ and we let $g(x) = f_{x'}^*(x'')$ then by Corollary 3, $f^{\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\cdots}$ converges uniformly to $g(x)$. Hence it suffices to show that $g(x) = S_u(f)$. Let $t \ge 0$ then

$$\{g(x)>t\} = \{x : f_{x'}^*(x'')>t\} = \{x' + \{f_{x'}(r)>t\}^* u\} = S_u(f>t)$$

and thus

$$g(x) = \sup_{t\ge 0} x \in \{g>t\} = \sup_{t\ge 0} x \in S_u(\{f>t\}) = S_u(f)(x). \tag{3.18}$$

The fact that $\omega(S_u(f),h) \le \omega(f,h)$ for all $h>0$ follows from Proposition 4 and uniform convergence of $f^{\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\cdots}$ to $S_u(f)$. □

Characterization of f^* via Steiner Symmetrizations

Proposition 9. *For any* $f \in C_c(\mathbb{R}^d)$, $f = f^*$ *if and only if* $S_u(f) = f$ *for all* $u \in S^{d-1}$.

Proof. If $f = f^*$ then for all $t \geq 0$ and for all $u \in S^{d-1}$

$$S_u(\{f > t\}) = S_u(\{f > t\}^*) = \{f > t\}^* = \{f > t\}$$

and this implies (see 3.18) $S_u(f) = f$. If $S_u(f) = f$ for all $u \in S^{d-1}$ then for all $t \geq 0$

$$S_u(\{f > t\}) = \{S_u(f) > t\} = \{f > t\}, \quad \forall u \in S^{d-1}.$$

We will show that if $x_1, x_2 \in \partial\{f > t\}$ then $|x_1| = |x_2|$. Let $u_i = x_i/|x_i|$, $x_3 = \frac{1}{2}(u_1 + u_2)$ and let π_{x_3} be the reflection across $x_3^{\perp}$. Since $S_{u_3}(\{f > t\}) = \{f > t\}$ then necessarily $\pi_{x_3}(x_1) = x_2$ which in turn implies that $|x_1| = |x_2|$. In other words, $\{f > t\}$ is an open ball and this implies (see 3.3) that $f = f^*$. □

Corollary 5. *If g is any integrable and strictly radially decreasing function and $f \in C_c(\mathbb{R}^d)$ then for every $u \in S^{d-1}$, $S_u(f) = f$ if and only if*

$$\int_{\mathbb{R}^d} f(x)g(x)dx = \int_{\mathbb{R}^d} S_u(f)(x)g(x)dx. \tag{3.19}$$

Furthermore, $f = f^*$ if and only if 3.19 holds for all $u \in S^{d-1}$.

Proof. By Proposition 8, there exists a sequence $\{\sigma_n\}_{n=1}^{\infty} \in \Omega^{\infty}$ such that $f^{\sigma_1 \cdots \sigma_n}$ converges to $S_u(f)$ uniformly. If 3.19 holds then by Corollary 1, we must have $f^{\sigma_n} = f$ for all $n \geq 1$ which in turn implies that $S_u(f) = f$. If 3.19 holds for all $u \in S^{d-1}$ then by the previous sentence we must have $S_u(f) = f$ for all $u \in S^{d-1}$ and by Lemma 9 we deduce that $f = f^*$. □

Proposition 10. *For every fixed $f \in C_c(\mathbb{R}^d)$ and for every $1 \leq p < \infty$, the map*

$$u \mapsto S_u(f) \tag{3.20}$$

is continuous with respect to the Euclidean norm on S^{d-1} and the $L^p(\mathbb{R}^d)$ norm on $C_c(\mathbb{R}^d)$.

Proof. If $u_1 \in S^{d-1}$ then there exists rotations Q_u such that $Q_u(u_1) = u$ for all $u \in S^{d-1}$ and $Q_u(x)$ converges uniformly to x on compacts whenever u approaches u_1. It is clear that

$$S_u(f) = S_{u_1}(f \circ Q_u^{-1}) \circ Q_u \tag{3.21}$$

for all $u \in S^{d-1}$. In particular, $\|S_u(f) - S_{u_1}(f)\|_p$ is bounded by

$$\|S_{u_1}(f \circ Q_u^{-1}) \circ Q_u - S_{u_1}(f) \circ Q_u\|_p + \|S_{u_1}(f) \circ Q_u - S_{u_1}(f)\|_p. \tag{3.22}$$

By the contraction property of rearrangements (Proposition 3):

$$\|S_{u_1}(f \circ Q_u^{-1}) \circ Q_u - S_{u_1}(f) \circ Q_u\|_p \leq \|f \circ Q_u^{-1} - f\|_p. \tag{3.23}$$

Combining 3.22 and 3.23, we have

$$\begin{aligned} \|S_u(f) - S_{u_1}(f)\|_p &\leq \|f \circ Q_u^{-1} - f\|_p + \|S_{u_1}(f) \circ Q_u - S_{u_1}(f)\|_p \\ &= \|f - f \circ Q_u\|_p + \|S_{u_1}(f) \circ Q_u - S_{u_1}(f)\|_p. \end{aligned} \tag{3.24}$$

As u approaches u_1, $f \circ Q_u$ and $S_{u_1}(f) \circ Q_u$ converge uniformly to f and $S_{u_1}(f)$ respectively and thus the right-hand side of 3.24 approaches zero. □

3.2.1 Uniform Convergence in $C_c(\mathbb{R}^d)$ and Convergence in $L^p(\mathbb{R}^d)$.

proof of Theorem 2. If we examine the proof of Theorem 1, all that was needed were the properties of polarization described in Corollaries 1 and 3 as well as those found in Propositions 5 and 6. However, these properties are analogous to the properties found in Proposition 8, Corollary 5 and Proposition 10 for Steiner symmetrization. As a result, it is clear that the proof of Theorem 1 also works for Steiner symmetrization and we thus obtain Theorem 2. □

3.3 Convergence in Hausdorff Distance

Theorems 1 and 2 show that there exists a sequence of reflections $\{\sigma_n\}_{n=1}^{\infty}$ and a sequence of unit vectors $\{u_n\}_{n=1}^{\infty}$ such that $A^{\sigma_1 \cdots \sigma_n}$ and $S_{u_1,\ldots,u_n}(A)$ converges in symmetric difference to A^* for any $A \in \mathcal{M}$ with $m(A) < \infty$. When F is a compact set, one can obtain a stronger notion of convergence of the sets $F^{\sigma_1 \cdots \sigma_n}$ and $S_{u_1,\ldots,u_n}(F)$ to F^* by replacing convergence in symmetric difference with convergence in the Hausdorff distance. We will let δ define the Hausdorff distance on the collection of all subsets contained in $\mathbb{R}^d$ i.e.,

$$\delta(A_1, A_2) = \inf\{\epsilon : A_1 \subset A_2 + \epsilon B^d, A_2 \subset A_1 + \epsilon B^d\} \tag{3.25}$$

for any two sets A_1, A_2.

Proposition 11. *If $\{T_i\}_{i=1}^{\infty}$ is a sequence of rearrangements and $T_n \circ \cdots \circ T_1(f)$ converges uniformly to f^* for every $f \in C_c(\mathbb{R}^d)$ then $T_n \circ \cdots \circ T_1(F)$ converges to F^* in the Hausdorff distance for any compact set F.*

Proof. As in the proof of Proposition 7, given any compact set F we let $f(x) = 1 - d(F, x)$ with $d(F, x)$ the distance function from F restricted to the set $F + B^d$. Since f is contractive, we have $f_n := T_n \circ \cdots \circ T_1(f)$ converging uniformly to f^*. Clearly $f_n(x) = 1$ if and only if $x \in F_n := T_n \circ \cdots \circ T_1(F)$. Choose x_n in the closure of F^* such that $\delta(F^*, F_n) \geq d(x_n, F_n)$ with $d(x_n, F_n)$ denoting the distance between x_n and F_n. Noting that

$$m(F^* - F_n) = m(F_n - F^*) \tag{3.26}$$

yields

$$m\left(B\left(x_n, \delta(F^*, F_n)\right) \cap F^*\right) \leq m(F_n - B^d). \tag{3.27}$$

If $\epsilon > 0$ and $\|f_n - f^*\|_\infty < \epsilon$ then necessarily

$$F_n \subset \{x : f^*(x) > 1 - \epsilon\} \tag{3.28}$$

and by 3.27:

$$m\left(B\left(x_n, \delta(F^*, F_n)\right) \cap F^*\right) \leq m(f > 1 - \epsilon) - m(F) = m(f > 1 - \epsilon) - m(f = 1). \qquad (3.29)$$

Since f_n converges uniformly to f^* then by 3.29, F_n converges to F^* in Hausdorff distance.

□

Chapter 4

Almost Sure Convergence

4.1 Preliminaries

4.1.1 Random Variables Distributed in Ω

Recalling subsection 3.1.2, we introduced a metric ρ on Ω by letting $\rho(\sigma_1, \sigma_2) = |\sigma_1(0) - \sigma_2(0)|$. Probability measures will always be defined on the Borel sigma-algebra generated by the metric ρ. We will denote this sigma-algebra by $\mathcal{F}$. It is clear that $\mathcal{F}$ contains all sets of the form $\{\sigma : \sigma(x) \in U\}$ with U an open subset of $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. Given any sequence of probability measures μ_i defined on $(\Omega, \mathcal{F})$, there exists, by the Kolmogorov extension theorem [5, pp.471-473] , a unique probability measure P defined on the infinite product space $(\Omega^{\mathbb{N}}, \mathcal{F}^{\mathbb{N}})$ such that

$$P(\sigma_1 \in A_1, \ldots, \sigma_n \in A_n) = \prod_{i=1}^{n} \mu_i(A_i). \tag{4.1}$$

We will denote in bold $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n, \ldots)$ arbitrary elements of $\Omega^{\mathbb{N}}$. If we let $X_i(\boldsymbol{\sigma}) = \sigma_i$ then by 4.1, $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent random variables distributed in Ω with $P(X_i \in A) = \mu_i(A)$ for all $i \geq 1$. Conversely, given any sequence of independent random variables X_i distributed in Ω and defined on some fixed probability space $(S, \mathcal{F}_1, P_1)$, we can construct a probability space $(\Omega^{\mathbb{N}}, \mathcal{F}^{\mathbb{N}}, P)$ by letting

$$P(\sigma_1 \in A_1, \ldots, \sigma_n \in A_n) = \prod_{i=1}^{n} \mu_i(A_i) \tag{4.2}$$

for all $n \geq 1$ with $\mu_i(A) = P_1(X_i \in A)$. We will thus view any sequence of independent random variables X_i distributed in Ω as coordinate maps σ_i defined on probability spaces of the form $(\Omega^{\mathbb{N}}, \mathcal{F}^{\mathbb{N}}, P)$ with $P = \bigotimes_{i=1}^{\infty} \mu_i$ and we will let $\mu_i(x, A)$ denote the probability that σ_i maps x into A i.e.,

$$\mu_i(x, A) = \mu_i(\sigma_i(x) \in A) = P(\sigma_i(x) \in A). \tag{4.3}$$

4.1.2 Random Variables Distributed in S^{d-1}

As for sequences of independent random variables distributed in Ω, we will view any sequence Y_i of independent random variables distributed in S^{d-1} as coordinate maps defined on probability spaces of the form $((S^{d-1})^{\mathbb{N}}, \mathcal{G}^{\mathbb{N}}, P)$ with $\mathcal{G}$ the Borel sigma-algebra generated by the Euclidean metric restricted to S^{d-1} and $P = \bigotimes_{i=1}^{\infty} \mu_i$ for some sequence of probability measures μ_i defined on $(S^{d-1}, \mathcal{G})$. In other words, if $\boldsymbol{u} = (u_1, \ldots, u_n, \ldots)$ denotes arbitrary elements of the infinite product space $(S^{d-1})^{\mathbb{N}}$ then $X_i(\boldsymbol{u}) = u_i$ and

$$P(X_i \in A) = \mu_i(A) \tag{4.4}$$

for every $A \in \mathcal{G}$.

4.2 Random Sequences of Polarizations

Lemma 1. *If $f \in C_c(\mathbb{R}^d)$ and $\epsilon(x) = |f(x) - f^*(x)| > 0$ then there exists $y \in \mathbb{R}^d$ such that*

$$||w| - |z|| \geq \omega^{-1}(f^*, \epsilon(x)/4) \tag{4.5}$$

and

$$(|w| - |z|)(f(w) - f(z)) \geq \omega^{-1}(f^*, \epsilon(x)/4)\epsilon(x)/4 \tag{4.6}$$

for every

$$(w, z) \in B(x, \omega^{-1}(f, \epsilon(x)/8)) \times B\left(y, \omega^{-1}(f, \epsilon(x)/8)\right).$$

Proof. If $t \geq f^*(x)$ then $m(f(y) \leq t, |y| < |x|)$ equals

$$m(f^*(y) \leq t) - m(f(y) \leq t, |y| \geq |x|) \tag{4.7}$$

which is bounded below by

$$m\left(f^*(y) \leq t\right) - m(|y| > |x|) = m(f^*(y) \leq t, |y| \leq |x|). \tag{4.8}$$

Similarily, if $0 \leq t < f^*(x)$ then $m(f(y) > t, |y| \geq |x|)$ equals

$$m(f^*(y) > t) - m(f(y) > t, |y| < |x|) \tag{4.9}$$

which is bounded below by

$$m\left(f^*(y) > t\right) - m(|x| > |y|) = m(f^*(y) > t, |x| \leq |y|). \tag{4.10}$$

If $f^*(x) < t < f(x)$ then by the above

$$m(|y| \leq |x|, f(y) \leq t + s) \geq m(r(t+s) \leq |y| \leq |x|) > m(r(t) \leq |y| \leq |x|) \tag{4.11}$$

for every $t + s \leq f(x)$. This implies that for every such s, there exists $y(s)$ such that $|y(s)| \leq r(t)$ and $f(y(s)) \leq t + s$. By continuity of f, we deduce that there exists $y(t)$ such that $|y(t)| \leq r(t)$ and $f(y(t)) \leq t$. In particular, there exists y such that $|y| \leq r(f(x) - \epsilon(x)/2)$ and $f(y) \leq f(x) - \epsilon(x)/2$. If $|x - w| < \omega^{-1}(f, \epsilon/8)$ and $|y - z| < \omega^{-1}(f, \epsilon/8)$

then by Corollary 4

$$\begin{aligned} |w| - |z| &\geq (|x| - |x - w|) - (|y| + |y - z|) \\ &\geq \omega^{-1}(f^*, \epsilon/2) - 2\omega^{-1}(f, \epsilon/8) \\ &\geq \omega^{-1}(f^*, \epsilon/2) - \omega^{-1}(f, \epsilon/4) \\ &\geq \omega^{-1}(f^*, \epsilon/4) \end{aligned}$$

and

$$\begin{aligned} f(w) - f(z) &\geq (f(x) - f(y)) - (|f(x) - f(w)| + |f(y) - f(z)|) \\ &\geq \epsilon/2 - \epsilon/4 \\ &= \epsilon/4. \end{aligned}$$

The case $f^*(x) - f(x) > 0$ is treated similarily and will be ommited. □

Proof of Theorem 4. For every $f \in C_c(\mathbb{R}^d)$, let $f^{i(\boldsymbol{\sigma})} = f^{\sigma_1 \cdots \sigma_i}$ and

$$I(f) = \int_{\mathbb{R}^d} |x| f(x) dx. \tag{4.12}$$

By Proposition 1

$$\mathbb{E}_P \left[I \left(f^{i-1(\boldsymbol{\sigma})} - f^{i(\boldsymbol{\sigma})} \right) \right]$$

equals

$$\mathbb{E}_P \left[\int_{X_{\pm}^{\sigma_i}} \left[(f^{i-1(\boldsymbol{\sigma})}(x) - f^{i-1(\boldsymbol{\sigma})}(\sigma_i(x)))(|x| - |\sigma_i(x)|) \right]^- dx \right]. \tag{4.13}$$

By the independence of random variables X_i, 4.13 equals

$$\mathbb{E}_P \left[\int_{\Omega} \int_{X_{\pm}^{\sigma}} \left[(f^{i-1(\boldsymbol{\sigma})}(x) - f^{i-1(\boldsymbol{\sigma})}(\sigma(x)))(|x| - |\sigma(x)|) \right]^- dx d\mu_i(\sigma) \right]. \tag{4.14}$$

It is clear from Corollary 1 combined with Proposition 5 that

$$I(f^*) \le I(f^\sigma) \tag{4.15}$$

for all $\sigma \in \Omega$ and for all $f \in C_c(\mathbb{R}^d)$. This implies that

$$\begin{aligned} \sum_{i=1}^{\infty} \mathbb{E}_P\left[I\left(f^{i-1(\boldsymbol{\sigma})} - f^{i(\boldsymbol{\sigma})}\right)\right] &= \lim_{n\to\infty} \mathbb{E}_P\left[I\left(f - f^{n(\boldsymbol{\sigma})}\right)\right] \\ &\le I(f - f^*) \\ &< \infty \end{aligned}$$

for all $f \in C_c(\mathbb{R}^d)$. In particular

$$\sum_{i=1}^{\infty} \int_\Omega \int_{X_\pm^\sigma} \left[(f^{i-1(\boldsymbol{\sigma})}(x) - f^{i-1(\boldsymbol{\sigma})}(\sigma(x)))(|x| - |\sigma(x)|)\right]^- dx d\mu_i(\sigma) < \infty \tag{4.16}$$

almost surely, for all $f \in C_c(\mathbb{R}^d)$. Fix $f \in C_c(\mathbb{R}^d)$ and suppose that $\boldsymbol{\sigma}$ has the property $\|f^{i(\boldsymbol{\sigma})} - f^*\|_\infty \ge \epsilon > 0$ for all $i \ge 1$. This implies that there exists x_i such that

$$\|f^{i(\boldsymbol{\sigma})}(x_i) - f^*(x_i)\|_\infty \ge \epsilon \tag{4.17}$$

for all $i \ge 1$. By Lemma 1, there exists a sequence y_i such that

$$\||w| - |z\|| \ge \omega^{-1}(f^*, \epsilon(x)/4) \tag{4.18}$$

and

$$(|w| - |z|)(f^{i-1(\boldsymbol{\sigma})}(w) - f^{i-1(\boldsymbol{\sigma})}(z)) \ge \omega^{-1}(f^*, \epsilon/4)\epsilon/4 \tag{4.19}$$

for every

$$(w, z) \in B(x_i, \omega^{-1}(f^{i-1(\boldsymbol{\sigma})}, \epsilon/8)) \times B\left(y_i, \omega^{-1}(f^{i-1(\boldsymbol{\sigma})}, \epsilon/8)\right)$$

and for all $i \ge 1$. Corollary 4 shows that polarization does not increase the modulus of continuity and thus by 4.19

$$\int_{\Omega}\int_{X^{\sigma}_{\pm}}\left[(f^{i-1(\boldsymbol{\sigma})}(x)-f^{i-1(\boldsymbol{\sigma})}(\sigma(x)))(|x|-|\sigma(x)|)\right]^{-}dxd\mu_i(\sigma)$$

is bounded below by

$$m\left(1/2\omega^{-1}(f,\epsilon/8)B^d\right)\omega^{-1}(f^*,\epsilon/4)\epsilon/4\cdot\mu_i\left(x_i,B\left(y_i,1/2\omega^{-1}(f,\epsilon/8)\right)\right) \tag{4.20}$$

for all $i \geq 1$. However, by the assumption 1.18 on the random variables X_i as well as 4.18:

$$\sum_{i=1}^{\infty}\mu_i\left(x_i,B\left(y_i,1/2\omega^{-1}(f,\epsilon/8)\right)\right)=\infty. \tag{4.21}$$

In particular, 4.16 is infinite. We have thus shown that for every fixed $f \in C_c(\mathbb{R}^d)$

$$P\left(\lim_{n\to\infty}\|f^{n(\boldsymbol{\sigma})}-f^*\|_{\infty}=0\right)=1. \tag{4.22}$$

Let

$$\{f_k\}_{k=1}^{\infty}\subset C_c(\mathbb{R}^d) \tag{4.23}$$

be a dense sequence (with respect to the L^{∞} norm) and let

$$A=\bigcap_{k=1}^{\infty}\left\{\boldsymbol{\sigma}:\lim_{n\to\infty}\|f_k^{n(\boldsymbol{\sigma})}-f_k^*\|_{\infty}=0\right\}. \tag{4.24}$$

If $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^d)$ with $f \in C_c(\mathbb{R}^d)$ whenever $p=\infty$ then there exists a subsequence $\{k_j\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty}\|f_{k_j}-f\|_p=0. \tag{4.25}$$

If $\boldsymbol{\sigma} \in A$ then by contraction property of rearrangements (Proposition 3)

$$\begin{aligned} \|f^{n(\boldsymbol{\sigma})} - f^*\|_p &\leq \|f^{n(\boldsymbol{\sigma})} - f_{k_j}^{n(\boldsymbol{\sigma})}\|_p + \|f_{k_j}^{n(\boldsymbol{\sigma})} - f_{k_j}^*\|_p + \|f_{k_j}^* - f^*\|_p \\ &\leq 2\|f_{k_j} - f\|_p + \|f_{k_j}^{n(\boldsymbol{\sigma})} - f_{k_j}^*\|_p \end{aligned}$$

for all $(j,n) \in \mathbb{N} \times \mathbb{N}$ and thus

$$\limsup_{n\to\infty} \|f^{n(\boldsymbol{\sigma})} - f^*\|_p \leq 2\|f_{k_j} - f\|_p \tag{4.26}$$

for all $j \geq 1$. Taking j to infinity on the right-hand side of 4.26 yields

$$\lim_{n\to\infty} \|f^{n(\boldsymbol{\sigma})} - f^*\|_p = 0. \tag{4.27}$$

By 4.22

$$\begin{aligned} P(A) &= 1 - P\left(\bigcup_{k=1}^{\infty} \{\boldsymbol{\sigma} : \lim_{n\to\infty} \|f_k^{n(\boldsymbol{\sigma})} - f_k^*\|_\infty > 0\}\right) \\ &\geq 1 - \sum_{i=1}^{\infty} P\left(\lim_{n\to\infty} \|f_k^{n(\boldsymbol{\sigma})} - f_k^*\|_\infty > 0\right) \\ &= 1. \end{aligned}$$

□

4.3 Random Sequences of Steiner Symmetrizations

Proof of Theorem 5. We let

$$S_{i(\boldsymbol{u})}(f) = S_{u_1,\ldots,u_i}(f) \tag{4.28}$$

and

$$I(f) = \int_{\mathbb{R}^d} |x| f(x) dx \tag{4.29}$$

for all $f \in C_c(\mathbb{R}^d)$. By the independence of the random variables Y_i,

$$\mathbb{E}_P\left[I(S_{i-1(\boldsymbol{u})}(f) - S_{i(\boldsymbol{u})}(f))\right]$$

equals

$$\mathbb{E}_P\left[\int_{S^{d-1}} I\left(S_{i-1(\boldsymbol{u})}(f) - S_u(S_{i-1(\boldsymbol{u})}(f))\right) d\mu_i(u)\right]. \tag{4.30}$$

However,

$$\begin{aligned}\sum_{i=1}^{\infty} \mathbb{E}_P\left[I(S_{i-1(\boldsymbol{u})}(f) - S_{i(\boldsymbol{u})}(f))\right] &= \lim_{n\to\infty} \mathbb{E}_P\left[I(f - S_{n(\boldsymbol{u})}(f))\right] \\ &\leq I(f - f^*) \\ &< \infty\end{aligned}$$

and thus

$$\mathbb{E}_P\left[\sum_{i=1}^{\infty}\int_{S^{d-1}} I\left(S_{i-1(\boldsymbol{u})}(f) - S_u(S_{i-1(\boldsymbol{u})}(f))\right) d\mu_i(u)\right] < \infty.$$

In particular,

$$\sum_{i=1}^{\infty}\int_{S^{d-1}} I\left(S_{i-1(\boldsymbol{u})}(f) - S_u(S_{i-1(\boldsymbol{u})}(f))\right) d\mu_i(u) < \infty \tag{4.31}$$

almost surely. If $\boldsymbol{u}$ satisfies $\|S_{i(\boldsymbol{u})}(f) - f^*\|_\infty \geq \epsilon > 0$ for all $i \geq 1$ then by Lemma 1 combined with the fact that Steiner symmetrization does not increase the modulus of continuity (see Proposition 8), there exists a sequence of pairs $(x_i, y_i) \subset \mathbb{R}^d \times \mathbb{R}^d$ such that

$$||w| - |z|| \geq \omega^{-1}(f^*, \epsilon(x)/4) \tag{4.32}$$

and

$$(|w| - |z|)(S_{i-1(\boldsymbol{u})}(f)(w) - S_{i-1(\boldsymbol{u})}(f)(z)) \geq \omega^{-1}(f^*, \epsilon/4)\epsilon/4 \tag{4.33}$$

for every

$$\begin{aligned}(w,z) &\in B(x_i,\omega^{-1}(f,\epsilon/8)) \times B\left(y_i,\omega^{-1}(f,\epsilon/8)\right) \\ &\subset B(x_i,\omega^{-1}(S_{i-1(\boldsymbol{u})}(f),\epsilon/8)) \times B\left(y_i,\omega^{-1}(S_{i-1(\boldsymbol{u})}(f),\epsilon/8)\right).\end{aligned}$$

By Proposition 8,

$$I\left(g - S_u(f)\right) \geq I\left(g - g^{\sigma_{0,ru}}\right) \tag{4.34}$$

for all $r \in \mathbb{R} - \{0\}$, for all $u \in S^{d-1}$ and for all $g \in C_c(\mathbb{R}^d)$. In particular, by using the polarization formula for functionals (Proposition 1), we immediately see from 4.33 and 4.34 that

$$\int_{S^{d-1}} I\left(S_{i-1(\boldsymbol{u})}(f) - S_u(S_{i-1(\boldsymbol{u})}(f))\right) d\mu_i(u)$$

is bounded below by

$$m\left(1/2\omega^{-1}(f,\epsilon/8)B^d\right)\omega^{-1}(f^*,\epsilon/4)\epsilon/4 \cdot \mu_i(A_i) \tag{4.35}$$

with

$$A_i = \left\{\frac{x_i - z}{|x_i - z|} : z \in B(y_i, 1/2\omega^{-1}(f,\epsilon/8))\right\}.$$

However, by the assumption 1.21 on the random variables Y_i as well as 4.32, one has

$$\sum_{i=1}^{\infty} \mu_i(A_i) = \infty \tag{4.36}$$

and thus the left-hand side of 4.31 is infinite. The probability that the left hand side of 4.31 is infinite is zero and thus we may finally conclude that

$$P\left(\lim_{n\to\infty} \|S_{n(\boldsymbol{u})}(f) - f^*\|_\infty = 0\right) = 1 \tag{4.37}$$

for all $f \in C_c(\mathbb{R}^d)$. Mimicking the proof of Theorem 4, we see that 4.37 implies Theorem 5. $\square$

4.4 Examples

We first show that Theorems 4 and 5 are extensions of Theorem 3 by showing that if the conditions 1.9 and 1.10 of Theorem 3 are satisfied then so are the conditions 1.18 and 1.21 of Theorems 4 and 5

Proposition 12. *If X_i is a sequence of independent random variables such that*

$$\liminf_{i\to\infty} P\left(X_i \in B_\rho(\sigma,\lambda)\right) > 0 \tag{4.38}$$

for all $\sigma \in \Omega$ and $\lambda > 0$ then for any $\lambda_1 > 0$

$$\sum_{i=1}^{\infty} \mu_i(x_i, B(y_i,\lambda_1)) = \infty \tag{4.39}$$

for every bounded sequence $\{(x_i,y_i)\}_{i=1}^{\infty} \subset \mathbb{R}^d \times \mathbb{R}^d$.

Proof. Let $\{(x_i,y_i)\}_{i=1}^{\infty}$ be any bounded sequence and let $\lambda_1 > 0$. Let i_k be a subsequence such that (x_{i_k}, y_{i_k}) converges to a pair (x,y). For sufficiently large K

$$\{\sigma : \sigma(x) \in B(y,\lambda_1/2)\} \subset \{\sigma : \sigma(x_{i_k}) \in B(y_{i_k},\lambda_1)\} \tag{4.40}$$

for all $k \geq K$. Recalling that $\sigma_{x,z}$ denotes the unique reflection that maps x to z, it is easily checked that

$$\sigma_{x,z}(w) = \frac{(|x-w|^2 - |z-w|^2)(x-z)}{|x-z|^2} \tag{4.41}$$

for all x, z, w. In particular, the map $z \mapsto \sigma_{x,z}(0) = w$ is continuous on the open set $\mathbb{R}^d - \{x\}$ with continuous inverse $\sigma_{0,w}(x)$. As a consequence, the set $\{\sigma : \sigma(x) \in B(y,\lambda_1/2)\}$ has a non-empty interior with respect to the metric ρ on Ω. By the assumption 4.38, we must have

$$\liminf_{k\to\infty} \mu_{i_k}(x, B(y, \lambda_1/2)) > 0. \tag{4.42}$$

Combining 4.40 and 4.42, we conclude that

$$\sum_{i=1}^{\infty} \mu_i(x, B(y_i, \lambda_1) = \infty. \tag{4.43}$$

□

It is clear that a similar (and easier) proof works for sequences of independent random variables Y_i distributed in S^{d-1}:

Proposition 13. *If Y_i is a sequence of independent random variables distributed in S^{d-1} such that*

$$\liminf_{n\to\infty} \mu_n\left(B(u, \lambda)\right) > 0 \tag{4.44}$$

for every $u \in S^{d-1}$ and for every $\lambda > 0$ then

$$\sum_{i=1}^{\infty} \mu_i\left(B(u_i, \lambda)\right) = \infty \tag{4.45}$$

for every sequence $\{u_i\}_{i=1}^{\infty} \subset S^{d-1}$ and for every $\lambda > 0$.

We wish to give an example of a sequence of independent random variables X_i distributed in Ω such that the condition 1.18 holds but not 4.38. To do so, we will need the following Lemma:

Lemma 2. *If $T_x(y)$ denotes the transformation $T(y) = \sigma_{x,y}(0)$ then the Jacobian $JT_x(y)$ equals*

$$\left(\frac{\left||x|^2 - |y|^2\right|}{|x-y|^2}\right)^{d-1}. \tag{4.46}$$

Proof. Let $y \in \mathbb{R}^d$ with $|y| \neq |x|$ and let $0 < \epsilon < ||x| - |y||$. By using polar coordinates ($y = x + ru$):

$$m(B(y,\epsilon)) = \gamma_d \int_{S^{d-1}} \int_{a(x,u,\epsilon)}^{b(x,u,\epsilon)} r^{d-1} dr d\theta(u) \tag{4.47}$$

with $[a(x,u,\epsilon), b(x,u,\epsilon)] = \{r : x + ru \in B(y,\epsilon)\}$. However, the image of the line segment $\{x + ru : r \in [a(x,u,\epsilon), b(x,u,\epsilon)]\}$ is the line segment $\{T_x(x + a(x,u,\epsilon)) + r : r \in [0, b(x,u,\epsilon) - a(x,u,\epsilon)]\}$ and consequently by using polar coordinates ($y = ru$)

$$m(T_x(B(y,\epsilon))) = \gamma_d \int_{S^{d-1}} \int_{|T(x+a(x,u,\epsilon))|}^{|T(x+a(x,u,\epsilon))|+c(x,u,\epsilon)} r^{d-1} dr d\theta(u) \tag{4.48}$$

with $c(x,u,\epsilon) = \pm(b(x,u,\epsilon) - a(x,u,\epsilon))$ depending on the sign of $\langle u, x\rangle$. Clearly

$$\lim_{\epsilon\to 0} \frac{\int_{|T(x+a(x,u,\epsilon))|}^{|T(x+a(x,u,\epsilon))|+c(x,u,\epsilon)} r^{d-1} dr}{\int_{a(x,u)}^{b(x,u)} r^{d-1} dr} = \left(\frac{|T_x(y)|}{|x-y|}\right)^{d-1} \tag{4.49}$$

and since

$$\sigma_{x,y}(0) = \frac{(|x|^2 - |y|^2)(x-y)}{|x-y|^2} \tag{4.50}$$

then

$$\frac{|T_x(y)|}{|x-y|} = \frac{||x|^2 - |y|^2|}{|x-y|^2}. \tag{4.51}$$

□

For any set $A \in \mathcal{F}$, we denote by A^0 the set $\{\sigma(0) : \sigma \in A\}$ and we let $\phi(y)$ denote the standard Gaussian:

$$\phi(y) = (2\pi)^{-d/2} e^{-|y|^2/2}. \tag{4.52}$$

We define the probability measures

$$\mu_i(A) = \alpha_i^d \int_{A^0} \phi(\alpha_i y) dy \tag{4.53}$$

with α_i positive constants to be determined. Let $\{x_i\}_{i=1}^{\infty}$ be any bounded sequence contained in $\mathbb{R}^d$ and let $0 < \lambda < |x_i| \leq \lambda_1$ for all $i \geq 1$. If $B_i \subset (|x_i| - \lambda)B^d$ for all $i \geq 1$ is a sequence of balls of uniform radius then by Lemma 2

$$\mu_i(x_i, B_i) = \alpha_i^d \int_{B_i} \left(\frac{||x_i|^2 - |y|^2|}{|x_i - y|^2} \right)^{d-1} \phi(\alpha_i y) dy. \tag{4.54}$$

However, we have

$$\frac{||x_i|^2 - |y|^2|}{|x_i - y|^2} \geq \frac{||x_i| - |y||}{|x_i| + |y|} \geq \frac{\lambda}{2\lambda_1} \tag{4.55}$$

for all $y \in B_i$ and for all $i \geq 1$. In addition,

$$\phi(\alpha_i y) \geq (2\pi)^{-d/2} \alpha_i^d (e^{\alpha_i^2})^{-\lambda^2/2}$$

for all $y \in B_i$ and for all $i \geq 1$. As a result there exists a constant c not depending on i such that

$$\mu_i(x_i, B_i) \geq c\alpha_i^d (e^{\alpha_i^2})^{-\lambda^2/2} \tag{4.56}$$

for all $i \geq 1$. Since we want our our measures μ_i to satisfy condition 1.18, we need to choose α_i so that

$$\sum_{i=1}^{\infty} \alpha_i^d (e^{\alpha_i^2})^{-\lambda^2/2} = \infty. \tag{4.57}$$

Letting $\alpha_i = \sqrt{\log\log(i)}$ for $i \geq 3$ yields 4.57. On the other hand, it is clear that

$$\lim_{i\to\infty} \mu_i(B_\rho(\sigma, |\sigma(0)|/2)) = \lim_{i\to\infty} \alpha_i^d \int_{B(\sigma(0), |\sigma(0)|/2)} \phi(\alpha_i y) dy = 0 \tag{4.58}$$

for every $\sigma \in \Omega$. In other words, if X_i is a sequence of independent random variables with $P(X_i \in A) = \mu_i(A)$ for every $A \in \mathcal{F}$ and for every $i \geq 1$ then the condition 1.18 is satisfied but not the condition 4.38.

As for sequence of polarizations, we will try to construct a sequence of random variables Y_i distributed in S^{d-1} that satisfy the condition 1.21 but not the condition 4.44.

Letting

$$\mu_i(B) = \int_{S^{d-1}} \frac{1 - |x_i|^2}{|x_i - u|^d} d\theta(u) \tag{4.59}$$

with $x_i \in \mathbb{R}^d$ and $|x_i| < 1$ for all $i \geq 1$ defines a sequence of probability measures. In fact, μ_i is the "hitting distribution " on S^{d-1} for Brownian motion starting at x_i (see [13, p.102]). Let u_i be any sequence on S^{d-1} and let $\lambda > 0$. We have

$$\sum_{i=1}^{\infty} \mu_i(B(u_i, \lambda)) = \int_{B(u_i,\lambda)} \frac{1 - |x_i|^2}{|x_i - u|^d} d\theta(u) \tag{4.60}$$

$$\geq 2^{-(d-1)} \sum_{i=1}^{\infty} \theta(B(u_i, \lambda) \cap S^{d-1})(1 - |x_i|). \tag{4.61}$$

Since $\theta(B(u_i, \lambda) \cap S^{d-1})$ is equal for all i then choosing $x_i = (1-1/i, 0, \dots, 0)$ yields that the right-hand side of 4.61 diverges and consequently condition 1.21 is satisfied. However, μ_i converges weakly to a point mass at $(1, 0, \dots, 0)$ (see [13, p.31]) and so certainly condition 4.26 is not satisfied. In other words, if Y_i is a sequence of independent random variables with distribution μ_i then condition 1.21 is satisfied but condition 4.26 is not satisfied.

Chapter 5

Rates of Convergence: Hölder Continuous Functions

5.1 Polarization of Hölder Continuous Functions

Let $f \in C^{0,\alpha}(\lambda B^d)$ with $\lambda > 0$ and let

$$\phi(y) = \chi_{|y|\le 2\lambda}(y)(2\lambda\gamma_d)^{-1}|y|^{-(d-1)}. \tag{5.1}$$

For any $A \in \mathcal{F}$, we let

$$\mu(A) = \int_{A^0} \phi(y)dy \tag{5.2}$$

with $A^0 = \{\sigma(0) : \sigma \in A\}$. By construction, μ is a probability measure on Ω. If we let

$$I(f) = \int_{\mathbb{R}^d} |x| f(x) dx \tag{5.3}$$

then the following holds:

Lemma 3. $\mathbb{E}_\mu\left[I(f - f^\sigma)\right]$ *is bounded below by*

$$2^{-(d-1)}\int_{f>f^*}|x|^{-(d-1)}\int_{|y|<|x|}(f(x)-f(y))^-(|x|-|y|)^d\phi(y)dydx.$$

Proof. By the polarization formula for functionals (Proposition 1):

$$\mathbb{E}_\mu\left[I(f-f^\sigma)\right] = \int_\Omega\int_{X_-^\sigma}(|x|-|\sigma(x)|)(f(x)-f(\sigma(x)))^-dxdP(\sigma). \tag{5.4}$$

By Fubini's theorem, the right-hand side of 5.4 equals

$$\int_{\lambda B^d}\int_{|\sigma(x)|<|x|}(|x|-|\sigma(x)|)(f(x)-f(\sigma(x)))^-dP(\sigma)dx. \tag{5.5}$$

By letting $y = \sigma(x)$ and using Lemma 2, 5.5 equals

$$\int_{\lambda B^d}\int_{|y|<|x|}(|x|-|y|)(f(x)-f(y))^-\left(\frac{|x|^2-|y|^2}{|x-y|^2}\right)^{d-1}\phi(y)dydx. \tag{5.6}$$

To complete the proof, note that the triangle inequality yields

$$\frac{|x|^2-|y|^2}{|x-y|^2} \geq \frac{|x|-|y|}{|x|+|y|} \geq \frac{|x|-|y|}{2|x|} \tag{5.7}$$

for $|y| < |x|$. □

Lemma 4. *If* $f(x) > f^*(x)$ *then*

$$\int_{|y|<|x|}(f(x)-f(y))^-(|x|-|y|)^d\phi(y)dy \geq \int_{|y|<|x|}(f(x)-f^*(y))^-(|x|-|y|)^d\phi(y)dy. \tag{5.8}$$

Proof. Fix x such that $f(x) > f^*(x)$ and let $\phi_1(y) = (|x|-|y|)^d\phi(y)$ then it follows from the proof of the Hardy-Littlewood inequality that it suffices to show that

$$m(y : |y| < |x|, f(y) < f(x) - t, \phi_1(y) > s) \tag{5.9}$$

is greater or equal to

$$m(y : |y| < |x|, f^*(y) < f(x) - t, \phi_1(y) > s) \tag{5.10}$$

for every $0 \le t < f(x)$ and $s \ge 0$. From the proof of Lemma 1 (see 4.7), if $0 \le t < f(x)$ then

$$m(y : f(y) < f(x) - t, |y| < |x|) \ge m(y : f^*(y) < f(x) - t, |y| < |x|) \tag{5.11}$$

for every $0 \le t < f(x)$. Since $\{y : \phi(y) > s, |y| < |x|\}$ is an open ball centered at the origin, then it is clear from 5.11 that 5.9 is greater or equal to 5.10. □

To find a lower bound on the right-hand side of 5.8 in terms of $f(x) - f^*(x)$, we will need the following simple Lemma:

Lemma 5. *[8, p.137]If $h_1(r), h_2(r)$ are non-negative increasing and decreasing functions respectively with $h_1(a) = 0$ and $h_2(b) = 0$ then*

$$\sup_{r \in (a,b)} h_1(r)h_2(r) \le \int_a^b -h_1(t)h_2'(t)dt.$$

Proof.

$$\begin{aligned}
\sup_{r \in (a,b)} h_1(r)h_2(r) &= \sup_{r \in (a,b)} h_1(r) \int_r^b -h_2'(t)dt \\
&\le \sup_{r \in (a,b)} \int_r^b -h_1(t)h_2'(t)dt \\
&= \int_a^b -h_1(t)h_2'(t)dt.
\end{aligned}$$

□

Proposition 14.

$$\mathbb{E}_\mu [I(f - f^\sigma)] \ge \gamma_d^{-\frac{d+1}{\alpha}} C_3(\alpha, d, [f]_\alpha, \lambda) \|f - f^*\|_1^{1+\frac{d+1}{\alpha}}$$

with $C_3(\alpha, d, [f]_\alpha, \lambda)$ *equal to*

$$[f]_\alpha^{-(d+1)/\alpha}\lambda^{-d(1+\frac{d+1}{\alpha})}\frac{2^{-(d+1)(1+1/\alpha)}\alpha}{(d+1+\alpha)^2 e}(1/d)^{-\frac{d+1}{\alpha}}.$$

Proof. By Lemmas 3 and 4, $\mathbb{E}_\mu[I(f-f^\sigma)]$ is bounded below by

$$\lambda^{-1}2^{-d}\int_{f>f^*}|x|^{-(d-1)}\gamma_d^{-1}\int_{|y|<|x|}(f(x)-f^*(y))^-(|x|-|y|)^d|y|^{-(d-1)}dydx. \tag{5.12}$$

Applying Lemma 5 to the special case

$$h_1(r) = (f(x)-f^*(r)), h_2(r) = (d+1)^{-1}(|x|-r)^{d+1}$$

shows that

$$\gamma_d^{-1}\int_{|y|<|x|}(f(x)-f^*(y))^-(|x|-|y|)^d|y|^{-(d-1)}dy = \int_{r(f(x))}^{|x|}(f(x)-f^*(r))(|x|-r)^d dr$$

is bounded below by

$$\sup_{r\in(r(f(x)),|x|)}(d+1)^{-1}(f(x)-f^*(r))(|x|-r)^{d+1}. \tag{5.13}$$

It is clear that 5.13 also equals

$$(d+1)^{-1}\sup_{t\in(f^*(x),f(x))}(f(x)-t)(|x|-r(t))^{d+1}. \tag{5.14}$$

Since $f\in C^{0,\alpha}(B^d)$ and the decreasing symmetric rearrangement does not increase the modulus of continuity (see Corollary 4) then $f^*\in C^{0,\alpha}(\lambda B^d)$ with $[f^*]_\alpha \le [f]_\alpha$ and $|x|-r(t)\ge [f^*]_\alpha^{-1/\alpha}(t-f^*(x))^{1/\alpha}$. This implies that 5.14 is bounded below by

$$(d+1)^{-1}[f^*]_\alpha^{-(d+1)/\alpha}\sup_{t\in(f^*(x),f(x))}(t-f^*(x))^{\frac{d+1}{\alpha}}(f(x)-t) \tag{5.15}$$

which equals

$$(d+1)^{-1}\left(\frac{d+1}{\alpha}\right)^{\frac{d+1}{\alpha}}\left(1+\frac{d+1}{\alpha}\right)^{-\left(1+\frac{d+1}{\alpha}\right)}[f^*]_\alpha^{-(d+1)/\alpha}(f(x)-f^*(x))^{\frac{d+1}{\alpha}+1}. \tag{5.16}$$

Noting that

$$(d+1)^{-1}\left(\frac{d+1}{\alpha}\right)^{\frac{d+1}{\alpha}}\left(1+\frac{d+1}{\alpha}\right)^{-\left(1+\frac{d+1}{\alpha}\right)} \geq \frac{e^{-1}\alpha}{(d+1+\alpha)^2} \tag{5.17}$$

gives

$$\mathbb{E}_\mu\left[I(f-f^\sigma)\right] \geq C_1(\alpha, d, [f^*]_\alpha, \lambda)\int_{f>f^*}|x|^{-(d-1)}(f(x)-f^*(x))^{1+\frac{d+1}{\alpha}}\,dx \tag{5.18}$$

with

$$C_1(\alpha, d, [f^*]_\alpha, \lambda) = \lambda^{-1}[f^*]_\alpha^{-(d+1)/\alpha}2^{-d}\frac{e^{-1}\alpha}{(d+1+\alpha)^2}. \tag{5.19}$$

Applying the reverse Hölder inequality [4, p.225] with parameters $p = (1+\frac{d+1}{\alpha})^{-1}$ and $q = -\frac{\alpha}{d+1}$ yields

(5.20)

$$\int_{f>f^*}|x|^{-(d-1)}(f(x)-f^*(x))^{1+\frac{d+1}{\alpha}}\,dx$$

is bounded below by

$$\left(\int_{f>f*}|x|^{\frac{\alpha(d-1)}{d+1}}dx\right)^{-\frac{d+1}{\alpha}}\left(\int_{f>f^*}(f(x)-f^*(x))dx\right)^{1+\frac{d+1}{\alpha}}. \tag{5.21}$$

From the identity

$$\int_{f>f^*}f(x)dx + \int_{f<f^*}f(x)dx = \int_{f>f^*}f^*dx + \int_{f<f^*}f^*dx$$

we deduce that

$$2\int_{f>f^*}(f-f^*)dx = \|f-f^*\|_1. \tag{5.22}$$

By 5.18,5.21 and 5.22, we finally obtain

$$\mathbb{E}_\mu \left[I(f - f^\sigma) \right] \geq \gamma_d^{-\frac{d+1}{\alpha}} C_2(\alpha, d, [f^*]_\alpha, \lambda) \| f - f^* \|_1^{1+\frac{d+1}{\alpha}} \tag{5.23}$$

with $C_2(\alpha, d, [f^*]_\alpha, \lambda)$ equal to

$$[f^*]_\alpha^{-(d+1)/\alpha} \frac{2^{-(d+1)(1+1/\alpha)} \lambda^{-d\left(1+\frac{d+1}{\alpha}\right)} \alpha}{(d+1+\alpha)^2 e} \left(\frac{(d+1)}{\alpha(d-1)+d(d+1)} \right)^{-\frac{d+1}{\alpha}}. \tag{5.24}$$

□

Following section 4.1 ("Random Variables Distributed in Ω"), there exists a unique probability measure P defined on the infinite product space $(\Omega^{\mathbb{N}}, \mathcal{F}^{\mathbb{N}})$ such that

$$P(\sigma_1 \in A_1, \ldots, \sigma_n \in A_n) = \prod_{i=1}^{n} \mu(A_i). \tag{5.25}$$

for every $n \geq 1$. We also let

$$f^{n(\boldsymbol{\sigma})} = f^{\sigma_1 \cdots \sigma_n} \tag{5.26}$$

for all $\boldsymbol{\sigma} \in \Omega^{\mathbb{N}}$.

Proposition 15.

$$\mathbb{E}_P \left[\left(\gamma_d^{-1} \| f^{T(\boldsymbol{\sigma})} - f^* \|_1 \right)^{1+\frac{d+1}{\alpha}} \right] \leq (T(d+1))^{-1} \lambda^{d+1} C_3(\alpha, d, [f^*]_\alpha, \lambda)^{-1} \| f - f^* \|_\infty$$

for all $T \geq 1$.

Proof. We have

$$\mathbb{E}_P \left[I\left(f - f^{T(\boldsymbol{\sigma})}\right) \right] = \sum_{n=1}^{T} \mathbb{E}_P \left[I\left(f^{n-1(\boldsymbol{\sigma})} - f^{n(\boldsymbol{\sigma})} \right) \right] \tag{5.27}$$

for all $T \geq 1$. By Proposition 14 as well as the contraction property of rearrangements

(Proposition 3):

$$\mathbb{E}_P\left[I\left(f^{n-1(\boldsymbol{\sigma})} - f^{n(\boldsymbol{\sigma})}\right)\right] \geq \gamma_d^{-\frac{d+1}{\alpha}} C_3(\alpha, d, [f^*]_\alpha, \lambda)\mathbb{E}_P\left[\|f^{T(\boldsymbol{\sigma})} - f^*\|_1^{1+\frac{d+1}{\alpha}}\right] \tag{5.28}$$

for all $n = 1, \ldots, T$. Using 5.27, we see that 5.28 yields

$$\mathbb{E}_P\left[I\left(f - f^{T(\boldsymbol{\sigma})}\right)\right] \geq T\gamma_d^{-\frac{d+1}{\alpha}} C_3(\alpha, d, [f]_\alpha, \lambda)\mathbb{E}_P\left[\|f^{T(\boldsymbol{\sigma})} - f^*\|_1^{1+\frac{d+1}{\alpha}}\right] \tag{5.29}$$

for all $T \geq 1$. However, we also have (see 4.15)

$$\mathbb{E}_P\left[I\left(f - f^{T(\boldsymbol{\sigma})}\right)\right] \leq I(f - f^*) \leq (d+1)^{-1}\lambda^{d+1}\gamma_d\|f - f^*\|_\infty. \tag{5.30}$$

Combining 5.29 and 5.30 completes the proof. □

Lemma 6. *If $g \in C^{0,\alpha}(\mathbb{R}^d) \cap C_c(\mathbb{R}^d)$ then*

$$[g]_\alpha^{-d/\alpha^2} \frac{\alpha}{d(\alpha + d)} \|g\|_\infty^{1+d/\alpha} \leq \gamma_d^{-1}\|g\|_1 \tag{5.31}$$

Proof. Let $g \in C^{0,\alpha}(\mathbb{R}^d) \cap C_c(\mathbb{R}^d)$ with $[g]_\alpha = 1$. Let x_1 be any point where $|g|$ attains a maximum. By using polar coordinates ($x = x_1 + ru$), we have

$$\gamma_d^{-1}\|g\|_1 = \int_{S^{d-1}} \int_0^\infty |g(ru)| r^{d-1} dr d\theta(u) \tag{5.32}$$

. In particular, there exists u_1 such that

$$\int_0^\infty |g(ru_1)| r^{d-1} dr \leq \gamma_d^{-1}\|g\|_1. \tag{5.33}$$

Since g belongs to $C^{0,\alpha}$ with $[g]_\alpha = 1$ then

$$|g(ru_1)| \geq \|g\|_\infty - r^\alpha \tag{5.34}$$

for all $r \in (0, \|g\|_\infty^{1/\alpha})$ and thus

$$\int_0^{\|g\|_\infty^{1/\alpha}} (\|g\|_\infty - r^\alpha) r^{d-1} dr = \frac{\alpha}{d(\alpha+d)} \|g\|_\infty^{1+d/\alpha} \leq \gamma_d^{-1} \|g\|_1. \tag{5.35}$$

This proves the Lemma for the case $[g]_\alpha = 1$ and the general case follows by scaling. □

Proposition 16. *There exists an explicit constant $C(\alpha, d, [f]_\alpha, \lambda)$ such that*

$$\mathbb{E}_P \left[\|f^{T(\boldsymbol{\sigma})} - f^*\|_\infty \right] \leq \epsilon \tag{5.36}$$

whenever

$$T \geq C(\alpha, d, [f]_\alpha, \lambda)(1/\epsilon)^{(1+\frac{d+1}{\alpha})(1+\frac{d}{\alpha})} \|f - f^*\|_\infty \tag{5.37}$$

for all $\epsilon > 0$ and for all $f \in C^{0,\alpha}(\lambda B^d)$.

Proof. By Lemma 6 and Proposition 15 combined with the fact that

$$[f^*]_\alpha \leq [f^\sigma]_\alpha \leq [f]_\alpha \tag{5.38}$$

for all $\sigma \in \Omega$, we have

$$\mathbb{E}_P \left[\|f^{T(\boldsymbol{\sigma})} - f^*\|_\infty^{(1+\frac{d}{\alpha})(1+\frac{d+1}{\alpha})} \right] \leq T^{-1} C(\alpha, d, [f]_\alpha, \lambda) \|f - f^*\|_\infty$$

with

$$C(\alpha, d, [f]_\alpha, \lambda) = \frac{(\alpha+d)d}{(d+1)\alpha} (2[f]_\alpha)^{d/\alpha^2} \lambda^{d+1} C_3(\alpha, d, [f]_\alpha, \lambda)^{-1}.$$

By Jensen's inequality

$$\mathbb{E}_P \left[\|f^{T(\boldsymbol{\sigma})} - f^*\|_\infty^{(1+\frac{d}{\alpha})(1+\frac{d+1}{\alpha})} \right] \geq \mathbb{E}_P \left[\|f^{T(\boldsymbol{\sigma})} - f^*\|_\infty \right]^{(1+\frac{d}{\alpha})(1+\frac{d+1}{\alpha})}$$

and thus

$$\mathbb{E}_P\left[\|f^{T(\boldsymbol{\sigma})} - f^*\|_\infty\right] \le \left(T^{-1}C(\alpha, d, [f]_\alpha, \lambda)\|f - f^*\|_\infty\right)^{\left[(1+\frac{d}{\alpha})(1+\frac{d+1}{\alpha})\right]^{-1}}.$$

In particular, if

$$T \ge C(\alpha, d, [f]_\alpha, \lambda)\|f - f^*\|_\infty (1/\epsilon)^{(1+\frac{d}{\alpha})(1+\frac{d+1}{\alpha})}$$

then

$$\mathbb{E}_P\left[\|f^{T(\boldsymbol{\sigma})} - f^*\|_\infty\right] \le \epsilon.$$

However, a computation shows that $C(\alpha, d, [f]_\alpha, \lambda)$ equals

$$(1/d)^{\frac{d+1}{\alpha}} \frac{e(\alpha+d)d}{(d+1)\alpha^2}[f]_\alpha^{\frac{d+1}{\alpha}+d/\alpha^2}\lambda^{2d+1+\frac{d(d+1)}{\alpha}}2^{(d+1)(1+1/\alpha)+\frac{d}{\alpha^2}}. \tag{5.39}$$

□

We make a quick remark regarding the constant $C(\alpha, d, [f]_\alpha, \lambda)$. It is clear that there exists a numerical constant c such that

$$\frac{e(\alpha+d)d}{(d+1)\alpha^2}2^{(d+1)(1+1/\alpha)+\frac{d}{\alpha^2}} \le 2^{cd/\alpha^2}$$

and

$$[f]_\alpha^{\frac{d+1}{\alpha}+d/\alpha^2} \le [f]_\alpha^{cd/\alpha^2}.$$

In particular, we see that there exists a numerical constant c such that

$$C(\alpha, d, [f]_\alpha, \lambda) \le (2[f]_\alpha)^{cd/\alpha^2}\lambda^{2d+1+\frac{d(d+1)}{\alpha}}(1/d)^{\frac{d+1}{\alpha}}. \tag{5.40}$$

To complete this section, we note that Theorem 7 is an immediate corollary to Proposition 16.

5.2 Steiner Symmetrization of Hölder Continuous Functions

We will now use the results of the previous section to give a quick proof of Theorem 8. Following section 4.1 ("Random Variables Distributed in S^{d-1}"), there exists a unique probability measure P' defined on the infinite product space $((S^{d-1})^{\mathbb{N}}, \mathcal{G}^{\mathbb{N}})$ such that

$$P'(u_1 \in B_1, \ldots, u_n \in B_n) = \prod_{i=1}^{n} \theta(B_i). \tag{5.41}$$

for every $n \geq 1$. If we let

$$S_{n(\boldsymbol{u})}(f) = S_{u_1,\ldots,u_n}(f) \tag{5.42}$$

for all $n \geq 1$ and for all $\boldsymbol{u} \in \Omega^{\mathbb{N}}$ then we have the following proposition:

Proposition 17.

$$\mathbb{E}_{P'}\left[\|S_{T(\boldsymbol{u})}(f) - f^*\|_\infty\right] \leq \epsilon \tag{5.43}$$

whenever

$$T \geq C(\alpha, d, [f]_\alpha, \lambda)(1/\epsilon)^{(1+\frac{d+1}{\alpha})(1+\frac{d}{\alpha})}\|f - f^*\|_\infty \tag{5.44}$$

for all $\epsilon > 0$ *and for all* $f \in C^{0,\alpha}(\lambda B^d)$.

Proof. Let $f \in C^{0,\alpha}(\lambda B^d)$ for some $\lambda > 0$. It is clear from Proposition 8 that

$$I(f - f^{\sigma_{0,ru}}) \leq I(f - S_u(f)) \tag{5.45}$$

for all $r \in \mathbb{R}^d$ and for all $u \in S^{d-1}$. However, using the notation of the previous section (see 5.2) and 5.45, we obtain

$$\begin{aligned}
\mathbb{E}_\mu\left[I(f-f^\sigma\right] &= \int_{2\lambda B^d} I(f-f^{\sigma_{0,y}})\phi(y)dy \\
&= \int_{S^{d-1}} 1/2\lambda \int_0^{2\lambda} I(f-f^{\sigma_{0,ru}})drd\theta(u) \\
&\le \int_{S^{d-1}} I(f-S_u(f))d\theta(u) \\
&= \mathbb{E}_\theta\left[I(f-S_u(f))\right].
\end{aligned}$$

By Proposition 14:

$$\mathbb{E}_\theta\left[I(f-S_u(f))\right] \ge \gamma_d^{-\frac{d+1}{\alpha}} C_3(\alpha,d,[f]_\alpha,\lambda)\|f-f^*\|_1^{1+\frac{d+1}{\alpha}}. \tag{5.46}$$

This shows that the analog of Proposition 14 for Steiner symmetrization also holds. To prove Proposition 15, all that was needed was Proposition 14 as well as the contraction property (Proposition 3) of rearrangements. Hence, Proposition 15 also holds for Steiner symmetrization with P replaced by P'. The only property of polarization that was needed to prove Proposition 16 from Proposition 15 was

$$[f^*]_\alpha \le [f^\sigma]_\alpha \le [f]_\alpha \tag{5.47}$$

for all $\sigma \in \Omega$. However, by Proposition 8, the analogous statement for Steiner symmetrization also holds i.e.,

$$[f^*]_\alpha \le [S_u(f)]_\alpha \le [f]_\alpha \tag{5.48}$$

for all $u \in S^{d-1}$. □

To completes this section, we note that Theorem 8 is an immediate corollary to Proposition 17.

Chapter 6

Rates of Convergence: Convex Bodies

6.1 Preliminaries

6.1.1 Chapter Outline

We have seen in chapter 4 that almost every random sequence of Steiner symmetrizations will transform any arbitrary compact set F into F^*. In this chapter, we study how fast the transformation can occur when A is convex. More precisely, given $\epsilon > 0$, what is the minimal number of Steiner symmetrizations needed to transform any convex body K with volume κ_d into a new convex body K' with the property

$$(1-\epsilon)B^d \subset K' \subset (1+\epsilon)B^d?$$

As described in the introduction, the purpose of this chapter is to give a self-contained derivation of Theorem 6 due to Klartag. It should be stressed that the proof Theorem 6 depends fundamentally on a previous result due to Klartag and Milman [10]:

Theorem 9. *There exists $3d$ Steiner symmetrizations that transform any convex body K*

with volume κ_d into an isomorphic ball i.e., there exists $3d$ *Steiner symmetrizations that transform any initial convex body K with volume κ_d into a new convex body K' with the property*

$$c_1 B^d \subset K' \subset c_2 B^d \tag{6.1}$$

for some numerical constants c_1 and c_2.

6.1.2 Notation and Preliminary Facts

In what follows, the symbol $\mathbb{E}_\nu(\cdot)$ will always refer to expectation with respect to the probability measure ν defined on some contextual space and V_n will refer to the n dimensional Hausdorff measure. We also let

$$h(K,x) = \sup_{k \in K} \langle x, k \rangle \tag{6.2}$$

and

$$b(K) = 2 \cdot \int_{S^{d-1}} h(K,u) d\theta(u) \tag{6.3}$$

denote the *support function* and *mean width* of K respectively. For any two convex bodies K and L, we have the identity [15, p.53]

$$\| h(K,x) - h(L,x) \|_\infty = \delta(K,L). \tag{6.4}$$

We let

$$R(K) = \inf\{r : K \subset rB^d\} \tag{6.5}$$

and

$$r(K) = \sup\{r : K \subset rB^d\} \tag{6.6}$$

denote the *circumradius* and *inradius* of K respectively.

If $u \in S^{d-1}$ then we define the *Minkowski symmetrization* of K with respect to u by

$M_u(K) = \frac{K+\pi_u(K)}{2}$ with π_u denoting the reflection across the orthogonal complement of u. It is clear from the definition that Minkowski symmetrizaton preserves mean width and always contains Steiner symmetrization i.e., for all $u \in S^{d-1}$

$$S_u(K) \subset M_u(K). \tag{6.7}$$

We first consider the effect of d Minkowski symmetrizations on a convex body K with respect to the d orthonormal column vectors of an arbitrary orthonormal matrix. Consider the discrete cube $\{\pm 1\}^d$ equipped with the uniform probability measure ν and given f in $L^2(S^{d-1})$ define

$$f^{\rho}(u) = \mathbb{E}_{\nu}\left[f\left(\sum_{i=1}^{d} \epsilon_i \left\langle u, \rho^i \right\rangle \rho^i\right)\right] \tag{6.8}$$

with $\rho = (\rho^1, \ldots, \rho^d)$ an element of $\mathcal{O}_d$ - the group of orthogonal matrices. We define S_d^{ρ} to be the subspace of $L^2(S^{d-1})$ of functions f that have the property $f(u) = f\left(\sum_{i=1}^{d} \epsilon_i \left\langle u, \rho^i \right\rangle \rho^i\right)$ for all $\epsilon \in \{\pm 1\}^d$.

Proposition 18. *For all f in $L^2(S^{d-1})$, $proj_{S_d^{\rho}}(f) = f^{\rho}$. In addition, if $M_{\rho}(K)$ is the convex body obtained from applying d Minkowski symmetrizations with respect to the column vectors of ρ to the convex body K then $h(M_{\rho}(K), u) = h^{\rho}(K, u)$.*

Proof. Given $f \in L^2(S^{d-1})$ and $v \in S^{d-1}$, $f^v(u) = 1/2[f(u) + f(\pi_v(u))]$ is the projection of f onto the subspace of $L^2(S^{d-1})$ consisting of functions that are invariant with respect to the reflection π_v. We then obtain $proj_{S_d^{\rho}}(f) = f^{e_1 \cdots e_d} = f^{\rho}$. To prove the second statement of the proposition, note that for any convex body K, $h(M_K(v), u) = 1/2[h(K, u) + h(K, \pi_v(u))]$ and thus $h(K^{\rho}, u) = h^{\rho^1 \cdots \rho^d}(K, u) = h^{\rho}(K, u)$. □

6.1.3 Outline of the Strategy

We now give a brief outline of the strategy that was used by Klartag to derive Theorem 6.

The first step is to show that applying Minkowski symmetrizations with respect to the column vectors of random orthogonal matrices will (on average) transform any initial isomorphic ball into a ball of equal mean width. Using results from the theory of spherical harmonics, it is shown that the convergence described in the previous sentence occurs (on average) exponentially fast. This is done in section 6.2 ("Minkowski Symmetrizatons").

The second step consists of making the link between Minkowski symmetrization and Steiner symmetrization. It is clear that given any convex body its circumradius always bound half its mean width. In fact, one can give a quantitative estimate on the relative size of the circumradius and half the mean width of convex bodies of volume κ_d. The estimate shows that half the mean width is strictly less than circumradius and quantifies the difference. The derivation of this estimate is done in section 6.3 ("Steiner and Minkowski Symmetrization: Making the Link") .

In the third step, Theorem 9 is used and $3d$ Steiner symmetrizations are used to transform any convex body K with volume κ_d into an isomorphic ball (see 6.1).

In the fourth step, the fact that Minkowski symmetrization always contains Steiner symmetrization as well as step 2 is used to deduce that applying Steiner symmetrizations with respect to the column vectors of random orthogonal matrices will (on average) decrease circumradius. The expected decrease is quantified by using the rates of convergence for Minkowski symmetrizations found in step 1. It is shown that (on average) the circumradius decays exponentially fast. This is done in subsection 6.4.1 ("Circumradius").

In the final step, the relationship between circumradius and inradius is considered. It is shown that for convex bodies of volume κ_d, the inradius depends on the circumradius in the sense that if the circumradius is small then the inradius is large (see subsection

6.4.2 ("Inradius")). By using the fourth step, one can finally obtain Theorem 6 (see section 6.4.3 ("Final Results")).

6.2 Minkowski Symmetrizations

6.2.1 Notation and Preliminary Facts

We let $\mathbb{E}_{\mu}^{T}(\cdot)$ denote the expectation with respect to the probability measure $\mu \times \cdots \times \mu$ (T times) defined on the product space $\mathcal{O}_d \times \cdots \times \mathcal{O}_d$ (T times). An arbitrary element of $\mathcal{O}_d \times \cdots \times \mathcal{O}_d$ (T times) will be denoted by $\boldsymbol{\rho}$. We suppress the dependence on T since it will be clear from the context. If $(\rho_1, \ldots, \rho_T) = \boldsymbol{\rho}$ then for any $f \in L^2(S^{d-1})$, we let (recall 6.8)

$$f^{\boldsymbol{\rho}} := f^{\rho_1 \cdots \rho_T}.$$

We will refer to $\mathcal{H}_d^n$ as the space of spherical harmonics of degree n - the restriction to S^{d-1} of harmonic homogeneous polynomials of degree n. The dimension of the vector space $\mathcal{H}_d^n$ will be denoted by $N(d,n)$. If $H(u)$ is a spherical harmonic, we will denote by $H(x)$ the unique harmonic polynomial defined on all of $\mathbb{R}^d$ whose restriction is $H(u)$. The inner product $\langle \cdot, \cdot \rangle$ refers to the integral of the product of functions on S^{d-1} with respect to the measure θ. Finally, μ will denote the normalized Haar measure on $\mathcal{O}_d$ and $\mathcal{O}_d^+$ will denote the group of rotations.

Let K be any convex body with $R(K) \le c$ with $c > 0$ arbitrary. It is shown in [12] that if f is a continuous function on S^{d-1} with modulus of continuity

$$\omega_1(f,s) = \sup\{|f(u_1) - f(u_2)| : \arccos(\langle u_1, u_2 \rangle) \le s\} \tag{6.9}$$

then there exists a polynomial $P_n(u)$ of degree at most n and a constant c_3 not depending on d nor n such that

$$\|f - P_n\|_\infty \le c_3 \omega_1(f, 1/n). \tag{6.10}$$

Since Euclidean distance is bounded by geodesic distance:

$$\omega_1 \left(h(K,u), s \right) \leq cs \tag{6.11}$$

for all $s > 0$. By 6.10, given $\epsilon' > 0$, there exists a polynomial $P_{\epsilon'}(K,u)$ of degree $\lceil \frac{d}{\epsilon'} \rceil$ such that

$$\| h(K,u) - P_{\epsilon'}(K,u) \|_\infty < cc_3\epsilon'. \tag{6.12}$$

It is well known that every polynomial can be written as a sum of spherical harmonics [7, p.70]. More precisely, if $\{H_{ij}(u)\}_{i=1}^{N(d,j)}$ denotes an orthonormal basis for $\mathcal{H}_d^j$ then we have

$$P_{\epsilon'}(K,u) = \sum_{j=0}^{\lceil \frac{d}{\epsilon'} \rceil} Q_j(u) = \sum_{j=0}^{\lceil \frac{d}{\epsilon'} \rceil} \sum_{i=1}^{N(d,j)} \langle P_{\epsilon'}, H_{ij} \rangle H_{ij}(u) \tag{6.13}$$

and in particular

$$Q_0(u) = \int_{S^{d-1}} P_{\epsilon'}(u) d\theta(u). \tag{6.14}$$

To simplify the notation, we will temporarily let $\lceil \frac{d}{\epsilon'} \rceil = n$. We have by 6.12,6.13 and 6.14 combined with the triangle inequality

$$\begin{aligned} \mathbb{E}_\mu^T \left[\left\| h^{\boldsymbol{\rho}}(K,u) - \frac{b(K)}{2} \right\|_\infty \right] &\leq 2cc_3\epsilon' + \mathbb{E}_\mu^T \left[\| P_{\epsilon'}^{\boldsymbol{\rho}}(K,u) - Q_0(u) \|_\infty \right] & (6.15) \\ &\leq 2cc_3\epsilon' + \sum_{j=1}^{n} \mathbb{E}_\mu^T \left[\| Q_j^{\boldsymbol{\rho}} \|_\infty \right]. & (6.16) \end{aligned}$$

Recalling the identity 6.4 and using Proposition 18 as well as 6.16, we obtain the geometric inequality:

$$\mathbb{E}_\mu^T \left[\delta \left(M_{\boldsymbol{\rho}}(K), \frac{b(K)}{2} B^d \right) \right] \leq 2cc_3\epsilon' + \sum_{j=1}^{n} \mathbb{E}_\mu^T \left[\| Q_j^{\boldsymbol{\rho}} \|_\infty \right] \tag{6.17}$$

with $n = \lceil \frac{d}{\epsilon'} \rceil$. However, by Proposition 18, we know that $Q_j^{\boldsymbol{\rho}}$ is the projection of Q_j onto the subspace $\mathcal{H}_d^j \cap S_d^{\rho_1} \cap \cdots \cap S_d^{\rho_d}$. In particular, it is clear that

$$\mathbb{E}_\mu^T \left[\|Q_j^{\boldsymbol{\rho}}\|_2 \right] < \|Q_j\|_2 \tag{6.18}$$

for all $j \geq 1$. In fact, it will be shown using results from the theory spherical harmonics that the left-hand side of 6.18 decays exponentially with respect to T (see subsection 6.2.3 "L^2 Decay of Spherical Harmonics"). From the exponential L^2 decay of spherical harmonics (as described in the previous sentence), we will be able to conclude that for fixed ϵ', the whole expression

$$\sum_{j=1}^{n} \mathbb{E}_\mu^T \left[\|Q_j^{\boldsymbol{\rho}}\|_\infty \right] \tag{6.19}$$

decays exponentially with respect to T (see subsection 6.2.4 "L^∞ Decay of Spherical Harmonics"). In the last subsection 6.2.5 ("Final Results"), we use the results described in the previous sentence to finally conclude that the left-hand side of 6.17 decays exponentially with respect to T.

6.2.2 Tools from Spherical Harmonics

We now present a few results from the theory of spherical harmonics that will be used in subsections 6.2.3 and 6.2.4. The presentation is largely based on Chapter 3 of [7] with a few changes.

Proposition 19. *Let $d \geq 3$ and let $\mathcal{H}$ be a nonzero linear subspace of $\mathcal{H}_d^n$ with the invariance property $H(\rho(u)) \in \mathcal{H}$ for all $H \in \mathcal{H}$ and for all $\rho \in \mathcal{O}_d^+$. There exists a function $Q(t)$ such that for any orthonormal basis $H_1, \ldots, H_m$ of $\mathcal{H}$ and for any two unit vectors u and v, $\sum_{i=1}^m H_i(u)H_i(v) = Q(\langle u, v\rangle)$.*

Proof. Let $H_1, \ldots, H_m$ be any orthonormal basis for $\mathcal{H}$ then for every $H \in \mathcal{H}$, we have

$$H(v) = \sum_{i=1}^{m} \langle H(u), H_i(u) \rangle H_i(v) = \left\langle H(u), \sum_{i=1}^{m} H_i(u)H_i(v) \right\rangle.$$

This shows that $Q_1(u,v) \coloneqq \sum_{i=1}^m H_i(u)H_i(v)$ is invariant under change of basis (this holds for any $d \geq 2$). To finish the proof, we need to show that $Q_1(u,v)$ depends only on $\langle u, v\rangle$ and not on the relative positions of u and v. If we have two pairs of unit vectors say $(u_1,v_1),(u_2,v_2)$ such that $\langle u_1,v_1\rangle = \langle u_2,v_2\rangle$ then there exists a rotation ρ such that $\rho(u_1) = u_2, \rho(v_1) = v_2$. Since $\mathcal{H}$ is invariant under rotations, $H_1 \circ \rho, \dots, H_m \circ \rho$ is also a basis for $\mathcal{H}$ and thus $Q_1(u_1,v_1) = Q_1(\rho(u),\rho(v)) = Q_1(u_2,v_2)$ for all ρ. In particular, we have $Q_1(u,v) = Q_1\left((0,\dots,0),(0,\dots,\sqrt{1-t^2},t)\right) \coloneqq Q(t)$ whenever $\langle u,v\rangle = t$. □

Lemma 7. *Let $\overline{x} = (x_1,\dots,x_{d-1},0)$ and let $T_n(x) = \sum_{i=0}^{\lfloor n/2\rfloor} a_{2i}x_d^{n-2i}|\overline{x}|^{2i}$ with $a_{2(i+1)} = a_{2i}\cdot\frac{(n-2i)(n-2i-1)}{(2i+2)(d+2i-1)}$ and $a_0 = 1$. $H \in \mathcal{H}_d^n$ has the property $H(\rho(u)) = H(u)$ for all rotations ρ that fix e_d if and only if $H(u)$ is a constant multiple of $T_n(u)$.*

Proof. Let $H(u) \in \mathcal{H}_d^n$ have the properties stated in the lemma. Clearly there exists homogeneous polynomials p_i of degree i and real numbers b_i such that

$$H(x) = \sum_{i=0}^{n} b_i x_d^{n-i} p_i(\overline{x}).$$

Let ρ be a rotation leaving e_d fixed and mapping $\overline{x}$ to $(|\overline{x}|,0,\dots,0)$ then $H(x) = H(\rho(x))$ implies

$$H(x) = \sum_{i=0}^{n} b_i\cdot x_d^{n-i}p_i(|\overline{x}|) = \sum_{i=0}^{\lfloor n/2\rfloor} b_{2i}\cdot x_d^{n-2i}\cdot|\overline{x}|^{2i}.$$

We have

$$\begin{aligned}\triangle(H)(x) &= \sum_{i=0}^{\lfloor n/2\rfloor} b_{2i}\cdot\left[\triangle(x_d^{n-2i})\cdot|\overline{x}|^{2i} + x_d^{n-2i}\cdot\triangle(|\overline{x}|^{2i})\right]\\ &= \sum_{i=0}^{\lfloor n/2\rfloor} b_{2i}\cdot\left[(n-2i)(n-2i-1)\cdot x_d^{n-2i-2}\cdot|\overline{x}|^{2i}\right.\\ &\quad\left.+x_d^{n-2i}\cdot\triangle(|\overline{x}|^{2i})\right] \qquad (6.20)\end{aligned}$$

and

$$\begin{aligned}\triangle\left(|\overline{x}|^{2i}\right) &= \sum_{j=1}^{d-1} 2i\left[|\overline{x}|^{2(i-1)} + 2x_j^2(i-2)|\overline{x}|^{2(i-2)}\right] \\ &= 2i(2i+d-3)|\overline{x}|^{2(i-1)}. \end{aligned} \tag{6.21}$$

Combining 6.20 and 6.21 yields

$$0 = \sum_{i=0}^{\lfloor n/2 \rfloor} \left[(n-2i)(n-2i-1)b_{2i} + 2(i+1)(d+2i-1)b_{2i+2}\right] x_d^{n-2i-2}|\overline{x}|^{2i}.$$

Hence we obtain $b_{2i} = b_{2i-2} \cdot \frac{(n-2i)(n-2i-1)}{(2i+2)(d+2i-1)}$ for $i = 1, \ldots, \lfloor n/2 \rfloor$. □

Proposition 20. *For every $d \geq 2$ there exists a unique polynomial $P_d^n(t)$ of degree n with the following property: if $\{H_i\}_{i=1}^{N(d,n)}$ is any orthonormal basis of $\mathcal{H}_d^n$ and u, v are any two unit vectors then $\sum_{i=1}^{N(d,n)} H_i(u)H_i(v) = N(d,n) \cdot P_d^n(\langle u, v\rangle)$.*

Proof. If $d \geq 3$ then $\mathcal{H}_d^n$ is trivially invariant under rotations and thus we may use Proposition 2. Consider $Q(\langle u, e_d\rangle)$ with $Q(t)$ defined as in Proposition 2 then as a function in the variable $u \in S^{d-1}$, it is invariant with respect to rotations leaving e_d fixed. By Lemma 7, $Q(\langle u, e_d\rangle) = C \cdot T_n(u)$. This implies

$$Q(t) = Q\left(\left\langle (0, \ldots, \sqrt{1-t^2}, t), e_d \right\rangle\right) = C \cdot \sum_{i=0}^{\lfloor n/2 \rfloor} a_{2i} \cdot t^{n-2i}(1-t^2)^i.$$

It is clear from Lemma 7 that $a_n \neq 0$ and consequently $Q_n(t)$ is a polynomial of degree n. Letting $P_d^n(t) = Q(t) \cdot 1/N(d,n)$ completes the proof for the case $d \geq 3$. If $d = 2$, it is well known from complex analysis that if $H \in \mathcal{H}_2^n$ there exists a complex number z_1 such that $H(x) = \mathrm{Re}(z_1 z^n)$ with $z = x_1 + ix_2$ and $z_1 = a_1 + b_1 i$. Switching to polar coordinates $(0 \leq \tau < 2\pi)$, we have

$$\begin{aligned} H(u) = H(\cos(\tau), \sin(\tau)) &= \mathrm{Re}\left(z_1 e^{in\tau}\right) \\ &= a_1\cos(n\tau) + b_1\sin(n\tau). \end{aligned}$$

If we let

$$H_1(u) := \cos(n \arccos(u_1)), H_2(u) := \sin(n \arcsin(u_2)) \tag{6.22}$$

then we have

$$\langle H_1, H_2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\tau)\sin(n\tau) d\tau = 0. \tag{6.23}$$

In other words, $\{H_1, H_2\}$ is an orthonormal basis for $\mathcal{H}_2^n$. Let u, v be two unit vectors and set $t = \langle u, v \rangle$ then

$$H_1(u)H_1(v) + H_2(u)H_2(v) = \cos(n \arccos(t)). \tag{6.24}$$

To complete the proof, we let $P_2^n(t) = 1/2 \cdot \cos(n \arccos(t))$. □

Corollary 6. *For any $n \geq 0, P_d^n(1) = 1$ and $|P_d^n(t)| \leq 1$. In addition, if $H \in \mathcal{H}_d^n$ then $\|H\|_\infty \leq \sqrt{N(d,n)} \|H\|_2$.*

Proof. Temporarily let $N(d,n) = N$ then

$$P_d^n(1) = 1/N \cdot \sum_{i=1}^{N} \int_{S^{d-1}} |H(u)|^2 d\theta(u) = 1$$

and by the Cauchy-Schwartz inequality

$$|P_d^n(t)| \leq \sqrt{P_d^n(1)} \sqrt{P_d^n(1)} = 1.$$

To prove the second part of the corollary, let $H \in \mathcal{H}_d^n$ then

$$\begin{aligned} |H(u)| &= \left| \sum_{i=1}^{N} \langle H, H_i \rangle H_i(u) \right| \\ &\leq \sqrt{\sum_{i=1}^{N} \langle H, H_i \rangle^2} \cdot \sqrt{\sum_{i=1}^{N} H_i^2(u)} \\ &= \|H\|_2 \cdot \sqrt{P_d^n(1) \cdot N} \\ &= \sqrt{N} \cdot \|H\|_2. \end{aligned}$$

□

Theorem 10. *(Funck-Hecke Theorem)Let $f(t)$ be a real valued function defined on $[-1,1]$ such that $\int_{-1}^{1} f(t)(1-t^2)^{\frac{d-3}{2}} < \infty$. If H a spherical harmonic of degree n and u a fixed unit vector then the function $f(\langle u,v\rangle)$ is in $L^1(S^{d-1})$ and*

$$\int_{S^{d-1}} f(\langle u,v\rangle)H(v)d\theta(v) = \alpha_{d,n}(f)H(u)$$

with

$$\alpha_{d,n}(f) = \frac{\gamma_{d-1}}{\gamma_d}\int_{-1}^{1} f(t)P_d^n(t)(1-t^2)^{\frac{d-3}{2}}dt.$$

Proof. If $\int_{-1}^{1} f(t)(1-t^2)^{\frac{d-3}{2}} < \infty$ then we have the following transformation formula

$$\int_{S^{d-1}} f(\langle u,v\rangle)d\theta(v) = \frac{\gamma_{d-1}}{\gamma_d}\int_{-1}^{1} f(t)(1-t^2)^{\frac{d-3}{2}}dt \tag{6.25}$$

$$= \int_{-1}^{1} f(t)d\mu_*(t) \tag{6.26}$$

with the measure μ_* implicitly defined by 6.25 and 6.26. In particular, if $f(t)$ is in $L^1_{\mu_*}$ then $f(\langle u,v\rangle)$ is in $L^1(S^{d-1})$ for all unit vectors u. We will now show that the polynomials $P_d^n(t)$ are mutually orthogonal with respect to the inner product

$$\langle g_1,g_2\rangle_* := \int_{-1}^{1} g_1(t)g_2(t)d\mu_*(t).$$

Let m,n be two nonegative integers and let $\{H_i\}_{i=1}^{N(d,m)}$ be an orthonormal basis for $\mathcal{H}_d^m$ and $\{H'_j\}_{j=1}^{N(d,n)}$ an orthonormal basis for $\mathcal{H}_d^n$ then by Proposition 20 and the transformation formula 6.26

$$N(d,m)N(d,n)\langle P_d^m,P_d^n\rangle_* = \sum_{j=1}^{N(d,n)}\sum_{i=1}^{N(d,m)} H_i(u)H'_j(u)\left\langle H_i,H'_j\right\rangle. \tag{6.27}$$

If $m = n$ then by corollary 6

$$N^2(d,n)\langle P_d^n, P_d^n\rangle_* = N(d,n)P_d^n(1) = N(d,n)$$

and if $m \neq n$ then by the Green-Gauss theorem

$$\begin{aligned} 0 &= \int_{B^d}[H_m(x) \triangle H_n(x) - \triangle H_m(x)H_n(x)]dx \\ &= \gamma_d \int_{S^{d-1}}[H_m(u)\langle\nabla H_n(u),u\rangle - \langle\nabla H_m(u),u\rangle H_n(u)]d\theta(u). \end{aligned} \tag{6.28}$$

A basic computation shows that if $P(x)$ is a homogeneous polynomial of degree k then

$$\langle\nabla P(x), x\rangle = kP(x) \quad \forall x.$$

As a consequence, the right-hand side of 6.28 simplifies to

$$\gamma_d(n-m)\langle H_m, H_n\rangle$$

and thus $\langle H_m, H_n\rangle = 0$ whenever $m \neq n$. In particular, the right-hand side of 6.27 vanishes whenever $m \neq n$. The orthogonality of the polynomials $P_d^n(t)$ of differing degrees imply that for every polynomial $P(t)$ of degree N

$$P(t) = \sum_{i=0}^{N} \frac{\langle P, P_d^i\rangle_*}{N(d,i)} P_d^i(t).$$

If $f(t)$ is in $L^1_{\mu_*}$ then it is well known that there exists a sequence of polynomials $Q_N(t)$ of degree at most N such that $Q_N(t)$ converges in $L^1_{\mu_*}$ to $f(t)$ as N tends to infinity.

This gives

$$\begin{aligned}\lim_{N\to\infty}\langle Q_N(\langle u,v\rangle),H(v)\rangle &= \lim_{N\to\infty}\sum_{i=1}^{N}\frac{\left\langle Q_N,P_d^i\right\rangle_*}{N(d,i)}\left\langle P_d^i(\langle u,v\rangle),H(v)\right\rangle\\ &= H(u)\cdot\lim_{N\to\infty}\langle Q_N,P_d^n\rangle_*\\ &= H(u)\,\langle f,P_d^n\rangle_*\,.\end{aligned}$$

Noting that

$$\lim_{N\to\infty}\langle|(Q_N-f)(\langle u,v\rangle)|,|H(v)|\rangle\le\lim_{N\to\infty}\|H\|_\infty\cdot\int_{-1}^{1}|f(t)-Q_N(t)|d\mu_*(t)=0$$

implies

$$H(u)\,\langle f,P_d^n\rangle_*=\lim_{N\to\infty}\langle Q_N(\langle u,v\rangle),H(v)\rangle=\langle f(\langle u,v\rangle),H(v)\rangle\,.$$

□

We say that a function f defined on S^{d-1} is zonal with pole $p\in S^{d-1}$ if it depends solely on its distance from p. For any unit vector u, the distance between u and p depends only $\langle u,p\rangle$. We thus deduce that a function f is zonal with pole p if and only if there exists a function f_1 defined on $[-1,1]$ such that $f(u)=f_1(\langle u,p\rangle)$. The next proposition characterizes zonal spherical harmonics completely.

Proposition 21. *A spherical harmonic H of degree n is a zonal harmonic with pole p if and only if it is a constant multiple of $P_d^n(\langle u,p\rangle)$.*

Proof. We know that there exists a continuous and bounded function $h(t)$ such that $H(u)=h(\langle u,p\rangle)$. We have $H(u)=\sum_{i=1}^{N(d,n)}\langle H,H_i\rangle\,H_i(u)$ and by Theorem 10

$$\langle H,H_i\rangle=\int_{S^{d-1}}H_i(u)h(\langle u,p\rangle)d\theta(u)=\alpha_{d,n}(h)H_i(p).$$

This implies

$$H(u) = \alpha_{d,n}(h) \sum_{i=1}^{N(d,n)} H_i(u)H_i(p) = \alpha_{d,n}(h) \cdot N(d,n) \cdot P_d^n(\langle u, p \rangle).$$

□

Corollary 7. *If $p \in S^{d-1}$ and F is a continuous function on S^{d-1} then for all $H \in \mathcal{H}_d^n$, $\int_{\mathcal{O}_d} F(\rho(p))H(\rho(u))d\mu(\rho) = \langle F, H \rangle P_d^n(\langle u, p \rangle)$.*

Proof. If $\rho \in \mathcal{O}_d$ then $H(\rho(u)) \in \mathcal{H}_d^n$ and thus

$$\int_{\mathcal{O}_d} F(\rho(p))H(\rho(u))d\mu(\rho) = \sum_{i=1}^{N_d} H_i(u) \int_{\mathcal{O}_d} \langle H(\rho(u), H_i(u) \rangle F(\rho(p))d\mu(\rho).$$

This shows that $\int_{\mathcal{O}_d} F(\rho(p))H(\rho(u))d\mu(\rho)$ is in $\mathcal{H}_d^n$ and since it is zonal with pole p it is a constant multiple C of $P_d^n(\langle u, p \rangle)$. Recalling that $P_d^n(1) = 1$ gives

$$\begin{aligned} C &= \int_{\mathcal{O}_d} F(\rho(p))H(\rho(p))d\mu(\rho) \\ &= \int_{\mathcal{O}_d} \int_{S^{d-1}} F(\rho(p))H(\rho(p))d\theta(p)d\mu(\rho) \\ &= \langle F, H \rangle . \end{aligned}$$

□

6.2.3 L^2 Decay of Spherical Harmonics

We will now study the expected rate of decay of the L^2 norm of spherical harmonics under projections onto the random subspaces $S_d^n(\rho)$ generated by the Haar measure. We make the following remark: if $\{H_1', \dots, H_m'\}$ is an orthonormal basis for $S_d^n(I) \cap \mathcal{H}_n^d$ then $\{H_1' \circ \rho^{-1}, \dots, H_m' \circ \rho^{-1}\}$ is an orthonormal subset of $S_d^n(\rho) \cap \mathcal{H}_d^n$ and conversely if one is given an orthonormal basis of $S_d^n(\rho) \cap \mathcal{H}_d^n$ then composing the basis elements with ρ yields an orthonormal subset of $S_d^n(I) \cap \mathcal{H}_n^d$. In particular, the subspaces $S_d^n(\rho) \cap \mathcal{H}_d^n$ have a common dimension which we denote by $M(d,n)$.

Proposition 22. *For any* $H \in \mathcal{H}_d^n$, $\mathbb{E}_\mu\left[\|H^\rho\|_2^2\right] = \frac{M(d,n)}{N(d,n)}\|H\|_2^2$.

Proof. Temporarily let $M(d,n) = M$ and $N(d,n) = N$. Suppose $\{H_1' \ldots, H_M'\}$ denotes an orthonormal basis for $S_d^n(I)$. If $H \in \mathcal{H}_d^n$ then by the remark above

$$\begin{aligned} \int_{\mathcal{O}_d} \|H^\rho\|_2^2 d\mu(\rho) &= \sum_{i=1}^M \int_{\mathcal{O}_d} \left\langle H, H_i' \circ \rho^{-1}\right\rangle^2 d\mu(\rho) \\ &= \sum_{i=1}^M \int_{\mathcal{O}_d} \langle H \circ \rho, H_i'\rangle^2 d\mu(\rho). \end{aligned} \tag{6.29}$$

The right-hand side of equation 6.29 can be rewritten as

$$\sum_{i=1}^M \int_{S^{d-1}} \int_{S^{d-1}} \int_{\mathcal{O}_d} H_i'(u)H_i'(p)H(\rho(u))H(\rho(p))d\mu(\rho)d\theta(u)d\theta(p)$$

which in turn equals (by using Corollary 7)

$$\|H\|_2^2 \sum_{i=1}^M \int_{S^{d-1}} \int_{S^{d-1}} H_i'(u)H_i'(p)P_d^n(\langle u,p\rangle)d\theta(u)d\theta(p). \tag{6.30}$$

By Proposition 20, we have

$$\int_{S^{d-1}} P_d^n(\langle u,p\rangle)H_i'(u)d\theta(u) = H_i'(p) \cdot 1/N$$

and consequently 6.30 equals

$$\|H\|_2^2 \sum_{i=1}^M \|H_i'\|_2^2 \cdot 1/N = \frac{M(d,n)}{N(d,n)} \cdot \|H\|_2^2. \tag{6.31}$$

□

We will now compute $M(d,n)$ and $N(d,n)$ by using standard results from linear algebra and combinatorics.

Proposition 23. *$M(d,n) = 0$ if n is odd and $M(d,n) = \binom{d+\frac{n}{2}-2}{d-2}$ if n is even. In addition, $N(d,n) = \binom{d+n-2}{d-2}\frac{d+2n-2}{d+n-2}$.*

Proof. If n is odd and $H \in S_d^n(I)$ then necessarily $H(u) = H(-u) = -H(u) = 0$ and thus $M(d,n) = 0$. If n is even then $H \in S_d^n(I)$ if and only the powers of $H(x_1,\dots,x_d)$ in each variable are even. If $E(d,n)$ denotes the space of homogeneous polynomials of degree n in d variables whose powers in each variable are even then the space $S_d^n(I) \cap \mathcal{H}_d^n$ is precisely the restriction to S^{d-1} of the kernel of the Laplacian restricted to $E(d,n)$. Since the image of $E(d,n)$ under the Laplacian is precisely $E(d,n-2)$ then

$$M(d,n) = \dim(E(d,n)) - \dim(E(d,n-2)). \tag{6.32}$$

The dimension of $E(d,n)$ is the same as the number of partitions of $n/2$ into at most d parts and this in turn can be computed by reading off the coefficient of $x^{n/2}$ in the generating function

$$(1+x+x^2+\cdots)^d = (1-x)^{-d} = \sum_{k=0}^{\infty}\binom{d+k-1}{d-1}x^k. \tag{6.33}$$

Hence

$$M_d^n = \binom{d+\frac{n}{2}-1}{d-1} - \binom{d+\frac{n}{2}-2}{d-1} = \binom{d+\frac{n}{2}-2}{d-2}. \tag{6.34}$$

By using the same arguments used to compute $M(d,n)$ one gets

$$N(d,n) = \binom{d+n-1}{d-1} - \binom{d+n-3}{d-3} = \binom{d+n-2}{d-2}\frac{d+2n-2}{d+n-2}. \tag{6.35}$$

□

6.2.4 L^∞ Decay of Spherical Harmonics

Returning to the geometry, we recall that given any convex body K with $R(K) \le c$, we have the geometric inequality

$$\mathbb{E}_\mu^T\left[\delta\left(M_{\boldsymbol{\rho}}(K), \frac{b(K)}{2}B^d\right)\right] \le 2cc_3\epsilon' + \sum_{j=1}^{n} \mathbb{E}_\mu^T\left[\|Q_j^{\boldsymbol{\rho}}\|_\infty\right] \tag{6.36}$$

with $n = \lceil \frac{d}{\epsilon'} \rceil$. By Corollary 6 and Jensen's inequality (applied to the concave function $x^{1/2}$):

$$\begin{aligned} \sum_{j=1}^{n} \mathbb{E}_\mu^T\left[\|Q_j^{\boldsymbol{\rho}}\|_\infty\right] &\le \sum_{j=1}^{n} N(d,j)^{1/2}\mathbb{E}_\mu^T\left[\|Q_j^{\boldsymbol{\rho}}\|_2\right] & (6.37) \\ &\le \sum_{j=1}^{n} N(d,j)^{1/2}\left(\mathbb{E}_\mu^T\left[\|Q_j^{\boldsymbol{\rho}}\|_2^2\right]\right)^{1/2}. & (6.38) \end{aligned}$$

Recalling Proposition 22 yields

$$\mathbb{E}_\mu^T\left[\|Q_j^{\boldsymbol{\rho}}\|_2^2\right] = \begin{cases} \left(\frac{M(d,j)}{N(d,j)}\right)^T \|Q_j\|_2^2 & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases} \tag{6.39}$$

and by applying the Cauchy-Schwartz inequality to the right-hand side of 6.38 we obtain

$$\sum_{j=1}^{n} \mathbb{E}_\mu^T\left[\|Q_{2j}^{\boldsymbol{\rho}}\|_\infty\right] \le \left(\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} N(d,2j)\left(\frac{M(d,2j)}{N(d,2j)}\right)^T\right)^{1/2} \left(\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \|Q_{2j}\|_2^2\right)^{1/2}. \tag{6.40}$$

We now estimate

$$\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} N(d,2j)\left(\frac{M(d,2j)}{N(d,2j)}\right)^T \tag{6.41}$$

by breaking it up into two pieces and estimating each piece separately.

Lemma 8. *The following holds:*

$$\sum_{(d-2)\le 2j \le n} N(d,2j)\left(\frac{M(d,2j)}{N(d,2j)}\right)^T < \left(\frac{3}{4}\right)^{T(d-1)} e^d(1+1/\epsilon')^{d-1}. \tag{6.42}$$

Proof. By Proposition 23

$$\frac{M(d,2j)}{N(d,2j)} = \left(\frac{d+2j-2}{d+4j-2}\right)\frac{\begin{pmatrix} d+j-2 \\ d-2 \end{pmatrix}}{\begin{pmatrix} d+2j-2 \\ d-2 \end{pmatrix}}. \tag{6.43}$$

We have

$$\frac{\begin{pmatrix} d+j-2 \\ d-2 \end{pmatrix}}{\begin{pmatrix} d+2j-2 \\ d-2 \end{pmatrix}} = \prod_{i=1}^{d-2}\frac{j+i}{2j+i} < \left(\frac{d-2+j}{d-2+2j}\right)^{d-2}. \tag{6.44}$$

However, we also have that

$$\left(\frac{d-2+j}{d-2+2j}\right)$$

is increasing in d for every fixed j and since by assumption $2j \geq d-2$ then

$$\frac{d-2+j}{d-2+2j} < \frac{3}{4}$$

and

$$\frac{M(d,2j)}{N(d,2j)} < \left(\frac{3}{4}\right)^{d-1} \tag{6.45}$$

for all $2j \geq d-2$. In particular,

$$\sum_{(d-2)\leq 2j\leq n} N(d,2j)\left(\frac{M(d,2j)}{N(d,2j)}\right)^{T} < \left(\frac{3}{4}\right)^{T(d-1)}\sum_{0\leq 2j\leq n} N(d,2j). \tag{6.46}$$

We have the combinatorial identity

$$\sum_{0\leq 2j\leq n} N(d,2j) = \begin{pmatrix} d+n-1 \\ d-1 \end{pmatrix} \tag{6.47}$$

and consequently we will need upper bounds on binomial coefficients to estimate the right-hand side of 6.46. Recalling Stirling's formula

$$n! = n^n e^{-n}\sqrt{2\pi n}e^{\theta(n)/12n} \quad (0 < \theta(n) < 1) \tag{6.48}$$

yields

$$\binom{n}{m} < \left(e \cdot \frac{n}{m}\right)^m$$

and in particular

$$\binom{d+n-1}{d-1} < \left(e \cdot \frac{d+n-1}{d-1}\right)^{d-1}. \tag{6.49}$$

Combining 6.46,6.47 and 6.49 and recalling that $n = \lceil d/\epsilon' \rceil$ finally gives

$$\begin{aligned} \sum_{(d-2)\leq 2j \leq n} N(d,2j)\left(\frac{M(d,2j)}{N(d,2j)}\right)^T &< \left(\frac{3}{4}\right)^{T(d-1)} e^{d-1}\left(\frac{d}{d-1}\right)^{d-1}(1+1/\epsilon')^{d-1} \\ &< \left(\frac{3}{4}\right)^{T(d-1)} e^{d}(1+1/\epsilon')^{d-1}. \end{aligned}$$

□

We now estimate the other piece.

Lemma 9. *The following holds:*

$$\sum_{2j<(d-2)} N(d,2j)\left(\frac{M(d,2j)}{N(d,2j)}\right)^T \leq \frac{e^2\left(\frac{1}{2}\right)^{T-2}}{1-e^2\left(\frac{1}{2}\right)^{T-2}}. \tag{6.50}$$

Proof. We first use the combinatorial identity

$$\frac{\binom{d+j-2}{d-2}}{\binom{d+2j-2}{d-2}} = \frac{\binom{d+j-2}{j}}{\binom{d+2j-2}{2j}} = \prod_{i=1}^{j} \frac{j+i}{d-2+j+i} < \left(\frac{2j}{d-2+2j}\right)^{j}$$

which gives (by recalling Proposition 23)

$$N(d,2j)\left(\frac{M(d,2j)}{N(d,2j)}\right)^{T} < \binom{d+2j-2}{d-2}\left(\frac{2j}{d-2+2j}\right)^{jT}. \tag{6.51}$$

However, since $\frac{2j}{d-2+2j}$ is increasing in j and $2j < d-2$ then

$$\frac{2j}{d-2+2j} < \frac{1}{2}. \tag{6.52}$$

Recalling the inequality 6.49 for binomial coefficients and using 6.51 and 6.52 finally gives

$$\begin{aligned}
\sum_{2j<(d-2)} N(d,2j)\left(\frac{M(d,2j)}{N(d,2j)}\right)^{T} &< \sum_{2j<(d-2)} \left(e\cdot\frac{d+2j-2}{2j}\right)^{2j}\left(\frac{2j}{d+2j-2}\right)^{jT} \\
&< \sum_{2j<(d-2)} \left[e\left(\frac{1}{2}\right)^{\frac{T-2}{2}}\right]^{2j} \\
&< \sum_{j=1}^{\infty} \left[e\left(\frac{1}{2}\right)^{\frac{T-2}{2}}\right]^{2j} \\
&= \frac{e^2\left(\frac{1}{2}\right)^{T-2}}{1-e^2\left(\frac{1}{2}\right)^{T-2}}.
\end{aligned} \tag{6.53}$$

□

6.2.5 Final Results

We see from 6.40 and Lemmas 8 and 9 that $\mathbb{E}_\mu^T\left[\delta\left(M_{\boldsymbol{\rho}}(K), \frac{b(K)}{2}B^d\right)\right]$ is bounded by:

$$\left(\frac{e^2\left(\frac{1}{2}\right)^{T-2}}{1-e^2\left(\frac{1}{2}\right)^{T-2}} + \left(\frac{3}{4}\right)^{T(d-1)} e^d(1+1/\epsilon')^{d-1}\right)^{1/2}\left(\sum_{j=1}^{\lfloor\frac{n}{2}\rfloor}\|Q_{2j}\|_2^2\right)^{1/2}. \tag{6.54}$$

However, we also have the basic inequality

$$\left(\sum_{j=1}^{\lfloor\frac{n}{2}\rfloor}\|Q_{2j}\|_2^2\right)^{1/2} \le \|P_{\epsilon'}(K,u)\|_2 \le \|P_{\epsilon'}(K,u)\|_\infty \le cc_3\epsilon' + c. \tag{6.55}$$

If we let

$$\alpha(T,\epsilon',d) = \left(\frac{e^2\left(\frac{1}{2}\right)^{T-2}}{1-e^2\left(\frac{1}{2}\right)^{T-2}} + \left(\frac{3}{4}\right)^{T(d-1)} e^d(1+1/\epsilon')^{d-1}\right)^{1/2}(cc_3\epsilon'+c) \tag{6.56}$$

then by using 6.54 and 6.55, we have

$$\mathbb{E}_\mu^T\left[\delta\left(M_{\boldsymbol{\rho}}(K), \frac{b(K)}{2}B^d\right)\right] < 2cc_3\epsilon' + \alpha(T,\epsilon',d) \tag{6.57}$$

for all $T \ge 1$. We let $T(\epsilon, d)$ denote the minimal number T such that

$$2cc_3\epsilon' + \alpha(\epsilon', T, d) \le \epsilon \tag{6.58}$$

for some $\epsilon' > 0$. If we put

$$\epsilon' = \frac{\epsilon}{4cc_3}, \epsilon'' = \left(\frac{2\epsilon}{\epsilon+4c}\right)^2$$

then it follows from 6.57 and 6.58 that if T is large enough to satisfy

$$\frac{e^2\left(\frac{1}{2}\right)^{T-2}}{1-e^2\left(\frac{1}{2}\right)^{T-2}} \le \epsilon''/2, \tag{6.59}$$

$$e^d\left(1+1/\epsilon'\right)^{d-1}\left(\frac{3}{4}\right)^{T(d-1)} \le \epsilon''/2 \tag{6.60}$$

then $T(\epsilon, d) \leq T$. Solving the inequalities 6.59 and 6.60 yields

$$T(\epsilon, d) \leq T_1(\epsilon) \vee T_2(\epsilon, d) \tag{6.61}$$

with

$$\begin{aligned} T_1(\epsilon) &= 2 + \frac{1}{2}\left[\log\left(\frac{1+\epsilon/2}{\epsilon/2}\right) - 2\right], \\ T_2(\epsilon, d) &= \frac{1}{\log(4/3)(d-1)}\left[d + (d-1)\log\left(1 + \frac{4cc_3}{\epsilon}\right) + \log(2/\epsilon'')\right]. \end{aligned}$$

We immediately see that there exists a numerical constant c_4 such that

$$T(\epsilon, d) \leq cc_4 \log(1/\epsilon). \tag{6.62}$$

We thus have:

Theorem 11. *[9, p.1333]There exists a numerical constant c_4 such that for all $\epsilon > 0$*

$$\mathbb{E}_{\mu}^{\lceil cc_4 d \log(1/\epsilon)\rceil}\left[\delta\left(M_{\boldsymbol{\rho}}(K), \frac{b(K)}{2}B^d\right)\right] \leq \epsilon$$

for every convex body K satisfying $R(K) \leq c$.

Corollary 8. *[9, Theorem 1.3]There exists a numerical constant c_4 such that for all $\epsilon > 0$ there exists at most $\lceil cc_4 d \log(1/\epsilon)\rceil$ Minkowski symmetrizations that transform any convex body K with $R(K) \leq c$ into a new convex body K' with the property*

$$\delta\left(K', \frac{b(K)}{2}B^d\right) \leq \epsilon.$$

Proposition 24 (Urysohn's Inequality)**.** $\frac{b(K)}{2} \geq 1$ *for every convex body K with volume κ_d.*

Proof. By Corollary 8, there exists a sequence of unit vectors $\{u_i\}_{i=1}^{\infty}$ such that

$$\lim_{n\to\infty} \delta\left(M_{u_1,\ldots,u_n}(K), \frac{b(K)}{2}B^d\right) = 0. \tag{6.63}$$

However, Minkowski symmetrization increases volume (since it contains Steiner symmetrization and the latter preserves volume) and thus by 6.63

$$\kappa_d \le \lim_{n\to\infty} V_d(M_{u_1,\ldots,u_n}(K)) = (b(K)/2)^d \kappa_d. \tag{6.64}$$

□

6.3 Steiner and Minkowski Symmetrization: Making the Link.

6.3.1 Preliminaries

Notation and Preliminary Facts

If K is a convex body, we will denote by $p(K,x)$ the closest point to x in K and we will denote by $u(K,x)$ the unit vector $(x - p(K,x))d(x,K)^{-1}$ with $d(x,K)$ the distance between x and K. The *support set* of K with outer normal u will be denoted by

$$F(K,u) := \{x \in \partial K : \langle x, u\rangle = h(K,u)\}$$

and the *normal cone* of $x \in \partial K$ will be denoted by

$$N(K,x) = \{y : p(K,y) = x\}.$$

Two polytopes P_1, P_2 are said to be strongly isomorphic if for every $u \in S^{d-1}$

$$\dim\left(F(P_1,u)\right) = \dim\left(F(P_2,u)\right).$$

We say that a convex body is regular if its boundary does not contain any singularities i.e., every point of the boundary is contained in a single support set. Lastly, we say that a convex body is strictly convex if all of its support sets are single points.

Section Outline

Let K be a convex body with volume κ_d. It is clear that $\frac{b(K)}{2} < R(K)$ unless K is a ball. As explained in section 6.1.3 "Outline of the Strategy ", we will need to quantify how small $\frac{b(K)}{2}$ is relative to $R(K)$ in order to derive rates of convergence for Steiner symmetrization from those for Minkowski symmetrizations derived in the previous section. The purpose of this section is to give a self-contained derivation of the following result:

Proposition 25. *[9, Theorem 6.1]Let $\epsilon > 0$ and suppose $K \subset B(0, 1+\epsilon)$ is a convex body with $V_d(K) = \kappa_d$ then $\frac{b(K)}{2} < 1 + (1 - 1/d^2)\,\epsilon$. If $\epsilon < 1/d$ then $\frac{b(K)}{2} < 1 + (1 - 1/2d)\epsilon$.*

6.3.2 Differential Geometry of Convex Bodies

We will now review some definitions and results from the differential geometry of convex bodies. We must warn the reader that some proofs will be omitted. For a complete treatment, the interested reader should consult chapters 2,4 and 5 of [15] from which the presentation is largely based on.

Curvature Measures and Quermassintegrals

Let K be a d dimensional convex body then for every $\xi > 0$, we can define a measure $\mu_\xi(K,\cdot)$ defined on the collection $\mathcal{B}$ of Borel sets contained in the product space $S^{d-1} \times \mathbb{R}^d$

by letting

$$\mu_\xi(A) = V_d\left(\{x : 0 < \delta(x,K) \le \xi, (u(K,x), p(K,x)) \in A\}\right)$$

for every $A \in \mathcal{B}$. The measures $\mu_\xi(K,\cdot)$ have the nice property that they are weakly continuous with respect to the Hausdorff metric on the space of convex bodies $\mathcal{K}^d$. In other words, if K_n is a sequence of convex bodies converging in Hausdorff metric to a convex body K then for any open set U in $S^{d-1} \times \mathbb{R}^d$

$$\liminf_{n\to\infty} \mu_\xi(K_n, U) \le \mu_\xi(K, U).$$

Let P be a fixed polytope and $A \in \mathcal{B}$, we will now show that $\mu_\xi(P,A)$ is a degree $d-1$ polynomial in ξ. We denote by $\mathcal{F}_m$ the faces of P of dimension $m = 0, \dots, d-1$. If $F \in \mathcal{F}_m$ then

$$\begin{aligned}
\mu_\xi(P,A) &= \sum_{m=0}^{d-1} \sum_{F\in\mathcal{F}_m} \int_F V_{d-m}\{t\cdot u : 0 \le t \le \xi, u \in N(P,F), (u,y) \in A\} dV_m(y) \\
&= \sum_{m=0}^{d-1} \sum_{F\in\mathcal{F}_m} \frac{\xi^{d-m}}{d-m} \cdot \int_F V_{d-m-1}\{u : (u,y) \in A\} dV_m(y) && (6.65) \\
&= \sum_{m=0}^{d-1} \xi^{d-m} \begin{pmatrix} d-1 \\ m \end{pmatrix} \Theta_m(P,A) && (6.66)
\end{aligned}$$

with the measures $\Theta_m(P,A)$ implicitly defined by 6.65 and 6.66. Setting $\xi = 1, \dots, d$ in 6.66, we obtain

$$\begin{pmatrix} \Theta_0(P,A) \\ \vdots \\ \Theta_{d-1}(P,A) \end{pmatrix} = B^{-1} \begin{pmatrix} \mu_1(P,A) \\ \vdots \\ \mu_d(p,A) \end{pmatrix}, \quad B^j_m = j^{d-m} \begin{pmatrix} d-1 \\ m \end{pmatrix}.$$

To extend the measures Θ_m to arbitrary convex bodies, we let

$$\begin{pmatrix} \Theta_0(K,\cdot) \\ \vdots \\ \Theta_{d-1}(K,\cdot) \end{pmatrix} = B^{-1} \begin{pmatrix} \mu_1(K,\cdot) \\ \vdots \\ \mu_d(K,\cdot) \end{pmatrix}.$$

From this definition and weak continuity of the measures $\mu_i(K,\cdot)$ $i = 1,\ldots,d$, we deduce that the measures $\Theta_m(K,\cdot)$ are also weakly continuous. Lastly, note that equation can be extended to general convex bodies via approximation by polytopes and weak continuity of the measures $\mu_\xi(\cdot,A)$ and $\Theta_m(\cdot,A)$:

$$\mu_\xi(K,A) = \sum_{m=0}^{d-1} \xi^{d-m} \begin{pmatrix} d-1 \\ m \end{pmatrix} \Theta_m(K,A). \tag{6.67}$$

We can now define the so called *curvature measures* by restricting the measures Θ_m to specific sets in $\mathcal{B}$. More precisely, for $0 \le m \le d-1$ and for any Borel subset β of ∂K and Borel subset ω of S^{d-1} we let

$$C_m(K,\beta) = \Theta_m(K, S^{d-1} \times \beta), S_m(K,\omega) = \Theta_m(K, \omega \times \partial K).$$

We also let

$$\sigma(K,\beta) = \bigcup_{x\in\beta} N(K,x) \cap S^{d-1}, \tau(K,\omega) = \bigcup_{u\in\omega} F(K,u)$$

denote the *spherical image map* and *reverse spherical image map* respectively. For regular and strictly convex bodies, one has

$$C_m(K,\tau(K,\omega)) = S_m(K,\omega), C_m(K,\beta) = S_m(K,\sigma(K,\beta)). \tag{6.68}$$

The Quermassintegrals $W_m(K)$ for $m = 1,\ldots,d$ are defined in terms of the total curvature measures for K:

$$W_m(K) = \frac{1}{d} \cdot C_{d-m}(K, \partial K) = \frac{1}{d} \cdot S_{d-m}(K, S^{d-1}). \tag{6.69}$$

We also let $W_0 = V_d(K)$. With this definition, we have by 6.67

$$V_d(K + \xi B^d) = \sum_{m=0}^{d} \xi^m \begin{pmatrix} d \\ m \end{pmatrix} W_m(K). \tag{6.70}$$

Mixed Volumes and Inequalities

We saw in the previous subsection that $V_d(K + \xi B^d)$ is a polynomial in ξ. In this subsection, we will give an overview of how we can express $V_d(K + \xi B^d)$ in a different way using *mixed volumes*.

Let P be a d dimensional polytope with facets $F_1, \dots, F_m$ and corresponding outward unit normal vectors $u_1, \dots, u_m$. Suppose first that 0 is in the interior of the convex body then $V_d(P)$ is the sum of the volume of m pyramids with base F_i and height $h(P, u_i)$ i.e.,

$$V_d(P) = 1/d \cdot \sum_{i=1}^{m} h(P, u_i) V_{d-1}(F_i). \tag{6.71}$$

It turns out that 6.71 holds even if 0 is not in the interior of the polytope. Indeed, one can show the weighted average unit normal $\sum_{i=1}^m u_i \cdot V_{d-1}(F_i) = 0$ and thus if $x \in K^\circ$

$$\begin{aligned} V_d(P) &= V_d(P - x) \\ &= \frac{1}{d} \cdot \sum_{i=1}^{d} h(P - x, u_i) V_{d-1}(F_i) \\ &= \frac{1}{d} \cdot \sum_{i=1}^{m} h(P, u_i) V_{d-1}(F_i) - \left\langle x, \sum_{i=1}^{m} u_i V_{d-1}(F_i) \right\rangle \\ &= \frac{1}{d} \cdot \sum_{i=1}^{m} h(P, u_i) V_{d-1}(F_i). \end{aligned} \tag{6.72}$$

We now introduce some useful definitions and notation. We let $(i,j) \in I$ if and only if $F_i \cap F_j = F_{ij} \neq \varnothing$. Fix a facet F_i and consider it as a $d-1$ polytope lying in its $d-1$ dimensional ambient space. The facets of F_i are F_{ij} with $(i,j) \in I$ and have corresponding unit normals v_{ij}. Lastly, we let $h_i = h(F_i, u_i) = h(P, u_i)$ and $h_{ij} = h(F_i, v_{ij})$. Clearly h_i is orthogonal to v_{ij} and $\langle u_j, v_{ij}\rangle > 0$. A straightforward computation yields for any $v \in F_i$:

$$\begin{aligned}
\langle v, v_{ij}\rangle &= \frac{\langle v, u_j\rangle - \langle u_i, u_j\rangle \cdot \langle v, u_i\rangle}{\langle v_{ij}, u_j\rangle} \\
&= \frac{\langle v, u_j\rangle - \langle u_i, u_j\rangle \cdot h_i}{\langle v_{ij}, u_j\rangle}. \qquad (6.73)
\end{aligned}$$

If we maximize the left hand side of 6.73 with respect to v, we obtain

$$h_{ij} = \frac{h_j - \langle u_i, u_j\rangle \cdot h_i}{\langle v_{ij}, u_j\rangle}. \qquad (6.74)$$

Using 6.72 and equation 6.74, it is a straightforward inductive argument to show that for any d dimensional polytope P, $V_d(P)$ can be written as a degree d homogeneous polynomial in the variables $h(P, u_i)$ whose coefficients are invariant under permutation of indices and equal to the coefficients of the polynomial $V_d(P')$ in the variables $h(P', u_i)$ whenever P' is strongly isomorphic to P. If $P_1, \ldots, P_n$ are strongly isomorphic d dimensional polytopes then $\lambda_1 K_1 + \cdots + \lambda_n K_n$ is strongly isomorphic to P_i for all $1 \leq i \leq n$ and thus

$$\begin{aligned}
V_d(\lambda_1 P_1 + \cdots + \lambda_n K_n) &= \sum_{i_1,\ldots,i_m=1}^{d} a_{i_1,\ldots,i_d} h_{i_1} \cdots h_{i_d} \\
&= \sum_{i_1,\ldots,i_m=1}^{d} a_{i_1,\ldots,i_d} \prod_{j=1}^{d} \left(\sum_{r=1}^{m} \lambda_r h_{i_j}^{r} \right) \\
&= \sum_{i_1,\ldots,i_m=1}^{d} a_{i_1,\ldots,i_d} \left(\sum_{r_1,\ldots,r_d=1}^{n} \lambda_{r_1} \cdots \lambda_{r_d} \cdot h_{i_1}^{r_1} \cdots h_{i_d}^{r_d} \right) \\
&= \sum_{r_1,\ldots,r_d=1}^{n} \lambda_{r_1} \cdots \lambda_{r_d} \left(\sum_{i_1,\ldots,i_m=1}^{d} a_{i_1,\ldots,i_d} \cdot h_{i_1}^{r_1} \cdots h_{i_d}^{r_d} \right).
\end{aligned}$$

If we define

$$V(P_1,\ldots,P_d)=\sum_{i_1,\ldots,i_m=1}^{d} a_{i_1,\ldots,i_d}\cdot h_{i_1}^1\cdots h_{i_d}^d \tag{6.75}$$

as the *mixed volume* of the strongly isomorphic polytopes $P_1,\ldots,P_m$ then we obtain the formula

$$V_d\left(\lambda_1 P_1+\cdots+\lambda_m P_m\right)=\sum_{r_1,\ldots,r_d=1}^{m}\lambda_{r_1}\cdots\lambda_{r_d}V(P_{r_1},\ldots,P_{r_d}). \tag{6.76}$$

In analogy with the previous subsection, one will generalise the notion of a mixed volume to general convex bodies by first writing the mixed volume for strongly isomorphic polytopes $P_1,\ldots,P_d$ as a linear expression in the variables $V_d(P_{i_1}+\cdots+P_{i_j})$ with $1\le j\le d$ and then use an approximation result. In fact, one can check that

$$V(P_1,\ldots,P_d)=\frac{1}{d!}\sum_{k=1}^{d}(-1)^{k+d}\cdot\sum_{i_1<\cdots<i_k}V_d(P_{i_1}+\cdots+P_{i_d}). \tag{6.77}$$

If $K_1,\ldots,K_d$ are convex bodies, we let

$$V(K_1,\ldots,K_d)=\frac{1}{d!}\sum_{k=1}^{d}(-1)^{k+d}\cdot\sum_{i_1<\cdots<i_k}V(K_{i_1}+\cdots+K_{i_d}) \tag{6.78}$$

define the mixed volume of $K_1,\ldots,K_d$. Note that 6.78 shows mixed volumes are always nonnegative. According to [15, p.102], for every sequence $K_1,\ldots,K_d$ of convex bodies, we can choose a sequence of d dimensional polytopes $\{P_i^r\}_{i=1}^{\infty}$ such that P_i^r are strongly isomorphic for all $i\ge 1$ and P_i^r converges to K_r for all $1\le r\le d$. From this approximation result, one easily extends 6.76 to general sequences $K_1,\ldots,K_d$:

$$V\left(\lambda_1 K_1+\cdots+\lambda_m K_m\right)=\sum_{r_1,\ldots,r_d=1}^{m}\lambda_{r_1}\cdots\lambda_{r_d}V(K_{r_1},\ldots,K_{r_d}). \tag{6.79}$$

We can simplify 6.79 by introducing some useful notation. We let

$$V(K_1[j_1],\ldots,K_m[j_m])$$

denote the mixed volume of the sequence of d convex bodies with K_1 appearing j_1 times, K_2 appearing j_2 times and so forth and let

$$\begin{pmatrix} d \\ j_1, \ldots, j_m \end{pmatrix} = \frac{n!}{j_1! \cdots j_m!}$$

denote the number of ways of choosing d elements from m different classes of d elements with j_1 chosen from the first class, j_2 from the second and so forth. With this notation, the right-hand side of 6.79 simplifies to

$$\sum_{j_1 + \cdots + j_m = d} \lambda_1^{j_1} \cdots \lambda_m^{j_m} \begin{pmatrix} d \\ j_1, \ldots, j_m \end{pmatrix} V(K_1[j_1], \ldots, K_m[j_m]). \tag{6.80}$$

To conclude the discussion of mixed volumes, we present a famous inequality due to Minkowski that will be used in the final section of this chapter.

Mixed Area Measures

Let $P_1, \ldots, P_d$ be a sequence of strongly isomorphic d dimensional polytopes then using the notation of the previous subsection, we know from equation 6.76 that

$$\begin{aligned} S_{d-1}(\lambda_1 P_1 + \cdots + \lambda_m P_m, \omega) &= \sum_{u_i \in \omega} V_{d-1}(F(\lambda_1 P_1 + \cdots + \lambda_m P_m, u_i)) \\ &= \sum_{u_i \in \omega} \sum_{j_1, \ldots, j_{d-1} = 1}^{m} \lambda_{j_1} \cdots \lambda_{j_{d-1}} v(F_i^{j_1}, \ldots, F_i^{j_{d-1}}) \\ &= \sum_{i_1, \ldots, i_{d-1} = 1}^{m} \lambda_{i_1} \cdots \lambda_{i_{d-1}} \sum_{u_i \in \omega} v(F_i^{j_1}, \ldots, F_i^{j_{d-1}}). \end{aligned}$$

with $v(\cdot, \ldots, \cdot)$ the mixed volume in $d-1$ dimensions. Hence if $P_1, \ldots, P_{d-1}$ is a sequence of strongly isomorphic d dimesonal polytopes and we let

$$S(P_1, \ldots, P_{d-1}, \cdot) = \sum_{u_i \in \cdot} v(F_i^1, \ldots, F_i^{d-1}) \tag{6.81}$$

denote the *mixed area measure* for $P_1, \dots, P_{d-1}$ then we have the formula

$$S_{d-1}(\lambda_1 P_1 + \cdots + \lambda_m P_m, \cdot) = \sum_{j_1,\dots,j_{d-1}=1}^{m} \lambda_{i_1} \cdots \lambda_{i_{d-1}} S(P_{j_1}, \dots, P_{j_{d-1}}, \cdot). \tag{6.82}$$

One can show using induction on the dimension d that 6.72 extends to mixed volumes in the sense that

$$\begin{aligned} v(P_1, \dots, P_d) &= \frac{1}{d} \sum_{i=1}^{n} h_i^1 v(F_i^2, \dots, F_i^d) \\ &= \frac{1}{d} \int_{S^{d-1}} h(P_1, u) dS(P_2, \dots, P_d, u). \end{aligned} \tag{6.83}$$

We wish to extend area measures to general sequences $K_1, \dots, K_{d-1}$. To do so, we copy the strategy used to extend mixed volumes to general sequences of convex bodies. Firstly, one checks that

$$S(P_1, \dots, P_{d-1}, \cdot) = \frac{1}{(d-1)!} \sum_{k=1}^{d-1} (-1)^{k+d-1} \cdot \sum_{i_1 < \cdots < i_k} S_{d-1}(P_{i_1} + \cdots + P_{i_{d-1}}, \cdot) \tag{6.84}$$

and secondly we define

$$S(K_1, \dots, K_{d-1}, \cdot) = \frac{1}{(d-1)!} \sum_{k=1}^{d-1} (-1)^{k+d-1} \cdot \sum_{i_1 < \cdots < i_k} S_{d-1}(K_{i_1} + \cdots + K_{i_{d-1}}, \cdot) \tag{6.85}$$

to be the mixed area measure of the convex bodies $K_1, \dots, K_{d-1}$.

We deduce from 6.85 that area measures are positive measures. Furthermore, from the weak continuity of curvature measures we deduce that the mixed area measures are weakly continuous. Approximating by strongly isomorphic polytopes and appealing to the weak continuity of both S_{d-1} and of S, we can extend equation 6.82:

$$S_{d-1}(\lambda_1 K_1 + \cdots + \lambda_m K_m, \cdot) = \sum_{j_1,\dots,j_{d-1}=1}^{m} \lambda_{i_1} \cdots \lambda_{i_{d-1}} S(K_{j_1}, \dots, K_{j_{d-1}}, \cdot). \tag{6.86}$$

As for the mixed volumes, the right-hand side of equation 6.86 can be simplified to

$$\sum_{j_1+\cdots+j_m=d-1} \lambda_1^{j_1}\cdots\lambda_m^{j_m} \binom{d-1}{j_1,\ldots,j_m} S(K_1[j_1],\ldots,K_m[j_m],\cdot). \tag{6.87}$$

Lastly, we wish to extend 6.83 to general sequences of convex bodies $K_1,\ldots,K_d$. To achieve this extension, we can choose a sequence of polytopes $\{P_i^r\}_{i=1}^{\infty}$ such that P_i^r are strongly isomorphic for all $i \geq 1$ and P_i^r converges to K_r for all $1 \leq r \leq d$. This implies that $h(P_i^1, u)$ converges uniformly to $h(K_1, u)$ as i tends to infinity and by the weak continuity of area measures one has

$$\lim_{i\to\infty} \int_{S^{d-1}} f(u)dS(P_i^2,\ldots,P_i^d,u) = \int_{S^{d-1}} f(u)dS(K_2,\ldots,K_d,u) \tag{6.88}$$

for every continuous function $f(u)$. This implies

$$\begin{aligned}
d\cdot V(K_1,\ldots,K_d) &= \lim_{i\to\infty} \int_{S^{d-1}} h(P_i^1,u)dS(P_i^2,\ldots,P_i^d,u) \\
&= \lim_{i\to\infty} \int_{S^{d-1}} h(P_i^1,u) - h(K_1,u)dS(P_i^2,\ldots,P_i^d,u) \\
&+ \lim_{i\to\infty} \int_{S^{d-1}} h(K_1,u)dS(P_i^2,\ldots,P_i^d,u) \\
&= \lim_{i\to\infty} \int_{S^{d-1}} h(P_i^1,u) - h(K_1,u)dS(P_i^2,\ldots,P_i^d,u) \\
&+ \int_{S^{d-1}} h(K_1,u)dS(K_2,\ldots,K_d).
\end{aligned}$$

Note that $S(P_i^2,\ldots,P_i^d,S^{d-1})$ is uniformly bounded since it converges to $S(K_2,\ldots,K_d,S^{d-1})$ and since $h(P_i^1,u)$ converges uniformly to $h(K_1,u)$ then

$$\lim_{i\to\infty} \int_{S^{d-1}} h(P_i^1,u) - h(K_1,u)dS(P_i^2,\ldots,P_i^d,u) = 0$$

and finally

$$V(K_1,\ldots,K_d) = \frac{1}{d}\cdot \int_{S^{d-1}} h(K_1,u)dS(K_2,\ldots,K_d). \tag{6.89}$$

Integral Representations of Quermassintegrals

In this subsection, we will consider the quermassintegrals of convex bodies with nice boundaries. Suppose K has a C^k boundary then we can consider the bijective map $v : \partial K \to S^{d-1}$ which maps x to its unique outward unit normal. If the inverse exists v^{-1} and is C^1 then we say that K is a C^k_+ body. Let K be a C^2_+ body and temporarily let $h(K,u) = h(u)$. If $N(y)$ is any C^2 local parametrization of S^{d-1} then $v^{-1}(x/|x|) = \nabla h(x)$ for all $x \in \mathbb{R}^d$ (see [15, p.40]) implies

$$\nabla(v^{-1} \circ N) = \nabla^2 h \circ \nabla N \tag{6.90}$$

with $\nabla^2 h$ the Hessian of h. Note that the Hessian exists since we are assuming that v^{-1} is C^1. From $\nabla h(x) = v^{-1}(x/|x|)$ we deduce that

$$\nabla^2 h(u)(u) = 0 \quad \forall u \in S^{d-1}. \tag{6.91}$$

We let T_u denote the tangent space of u spanned by $\{N_i(y)\}_{i=1}^{d-1}$ with $N_i = \partial_i N$. Note that $X = v^{-1} \circ N$ is a local C^2 parametrization of ∂K and

$$\langle \partial_i X(y), u \rangle = \left\langle \nabla^2 h(u)(N_i(y)), u \right\rangle = \left\langle N_i(y), \nabla^2 h(u)(u) \right\rangle = 0. \tag{6.92}$$

This implies that

$$X_i(y) = \partial_i X(y) = \sum_{j=1}^{d-1} (\nabla v^{-1}(u))_i^j N_j(y) \quad (i = 1, \ldots, d-1) \tag{6.93}$$

for some real valued matrix $(\nabla v^{-1}(u))_i^j$. If $\eta = \eta_i^j(y)$ denotes the matrix $\langle N_i(y), N_j(y) \rangle$ acting as a linear transformation of T_u then

$$X_i = \sum_{j=1}^{d-1} \langle X_i, N_k \rangle \, \eta^{-1} \circ N_j$$

and consequently

$$\nabla v^{-1} = b \circ \eta^{-1} \tag{6.94}$$

with b the matrix

$$b_i^j = \langle X_i, N_j \rangle = \left\langle \nabla^2 h \circ N_i, N_j \right\rangle = \left\langle N_i, \nabla^2 h \circ N_j \right\rangle = b_j^i. \tag{6.95}$$

In particular, we see from 6.94 and 6.95 that ∇v^{-1} is symmetric. It is well known [6, pp.102-103] that for any integrable $f : \partial K \to \mathbb{R}$ and integrable $g : S^{d-1} \to \mathbb{R}$

$$\int_{X(M)} f(x) dV_{d-1}(x) = \int_M f(X(y)) \sqrt{\det(\gamma(y))} dV_{d-1}(y) \tag{6.96}$$

$$\int_{N(M)} g(u) dV_{d-1}(u) = \int_M g(N(y)) \sqrt{\det(\eta(y))} dV_{d-1}(y) \tag{6.97}$$

with $\gamma(y)$ the matrix $\gamma(y)_i^j = \langle X_i(y), X_j(y) \rangle$. We compute

$$\begin{aligned} \langle X_i(y), X_j(y) \rangle &= \sum_{k=1}^{d-1} \sum_{r=1}^{d-1} \langle N_k, N_r \rangle \nabla v^{-1}(u)_i^k \cdot \nabla v^{-1}(u)_j^r \\ &= (\nabla v^{-1}(u) \circ \eta(y) \circ \nabla v^{-1}(u))_i^j. \end{aligned}$$

In particular, we have

$$\det(\gamma(y)) = \det(\nabla v^{-1}(u))^2 \det(\eta(y)). \tag{6.98}$$

Combining 6.96, 6.97 with 6.98 yields

$$\int_{X(M)} f(x) dV_{d-1}(x) = \int_{N(M)} f(v^{-1}(u)) \det(\nabla v^{-1}(u)) dV_{d-1}(u) \tag{6.99}$$

$$\int_{N(M)} g(u) dV_{d-1}(u) = \int_{X(M)} g(v(x)) \det(\nabla v(x)) dV_{d-1}(x). \tag{6.100}$$

If A denotes a symmetric $d \times d$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_d$ we let

$$s_q(A) = \begin{cases} \sum_{i_1 < \cdots < i_q} \lambda_{i_1} \cdots \lambda_{i_q} & \text{if } 1 \le q \le d, \\ 1 & \text{if } q = 0 \end{cases}$$

denote the q^{th} elementary symmetric function of A. A simple calculation shows that

$$s_j(A) = \frac{s_{d-j}}{s_d}\left(A^{-1}\right). \tag{6.101}$$

We consider the Hessian $\nabla^2 h(u)$ with respect to the basis

$$\{N_1(y), \ldots, N_{d-1}(y), u\}$$

for $\mathbb{R}^d$. By 6.90 and 6.93, we see that the restriction of $\nabla^2 h(u)$ to T_u is $\nabla v^{-1}(u)$ and since u belongs to the kernel of $\nabla^2 h(u)$ then clearly

$$s_j\left(\nabla v^{-1}(u)\right) = s_j\left(\nabla^2 h(u)\right) \quad (j = 1, \ldots, d-1). \tag{6.102}$$

As a result, 6.99 and 6.100 transform to

$$\int_{X(M)} f(x) dV_{d-1}(x) = \int_{N(M)} f(\nabla h(u)) s_{d-1}(\nabla^2 h(u)) dV_{d-1}(u) \tag{6.103}$$

$$\int_{N(M)} g(u) dV_{d-1}(u) = \int_{X(M)} g(v(x)) s_{d-1}^{-1}(\nabla^2 h(v(x)) dV_{d-1}(x). \tag{6.104}$$

Let X be a C^2 local parametrization of ∂K and let $N = v \circ X$ define a local parametrization of S^{d-1}. We will compute $\mu_\xi(K, S^{d-1} \times X(M))$. We have

$$\mu_\xi(K, S^{d-1} \times X(M)) = V\left(\{X(y) + N(y) : y \in M, 0 < t \le \xi\}\right). \tag{6.105}$$

In other words, we want to find the volume of the image of $M \times (0,\xi)$ under the map $\phi(y,t) = X(y) + tN(y)$. The Jacobian $J\phi$ evaluated at (y,t) is

$$\det\left((X_1(y) + tN_1(y), \ldots, X_{d-1}(y) + tN_{d-1}(y), N(y))\right)$$

and since $X_i(y) = \nabla^2 h(u)(N_i(y))$, we obtain

$$J\phi(y,t) = \det(\nabla^2 h + tI) \cdot \det((N_1, \ldots, N_{d-1}, N)). \tag{6.106}$$

Since $N(y)$ is orthogonal to $N_1(y), \ldots, N_{d-1}(y)$, it is well known [1, pp.193-194] that

$$\det((N_1, \ldots, N_{d-1}, N)) = \sqrt{\det(\eta)} \tag{6.107}$$

and by computing in an orthonormal basis we have

$$\det((N(y)) + tI) = \sum_{m=0}^{d-1} t^{d-1-m} s_m(\nabla^2 h(N(y))). \tag{6.108}$$

Using 6.106,6.107 and 6.108, we deduce that $\mu_\xi(K, S^{d-1} \times X(M))$ equals

$$\int_M \int_0^\xi \sum_{m=0}^{d-1} t^{d-m-1} s_m(\nabla^2 h(N(y)))\sqrt{\det(e_{ij}(y))}dtdy$$

which in turn equals

$$\sum_{m=0}^{d-1} \frac{\xi^{d-m}}{d-m} \int_M s_m(\nabla^2 h(N(y)))\sqrt{\det(e_{ij}(y))}dy \tag{6.109}$$

By 6.97, 6.109 transforms to

$$\sum_{m=0}^{d-1} \frac{\xi^{d-m}}{d-m} \int_{\sigma(X(M))} s_m(\nabla^2 h(u))dV_{d-1}(u) \tag{6.110}$$

and by comparing with 6.67 and using 6.68,6.97 and 6.101 we get

$$\binom{d-1}{m} C_m(K, X(M)) = \frac{1}{d-m} \int_{\sigma(X(M))} s_m(\nabla^2 h(u)) dV_{d-1}(u) \tag{6.111}$$

and

$$\binom{d-1}{m} S_m(K, \sigma(X(M))) = \frac{1}{d-m} \int_{X(M)} s_{d-1-m}(Dv(x)) dV_{d-1}(x). \tag{6.112}$$

The open neighborhoods of ∂K and S^{d-1} generate the Borel sigma algebras of ∂K and S^{d-1} respectively and thus by the monotone class theorem, 6.111 and 6.112 extend to arbitrary Borel sets $\beta \subset \partial K$ and $\omega \subset S^{d-1}$. In particular

$$i \cdot \binom{d}{i} \cdot W_i(K) = \int_{S^{d-1}} s_{d-i}(\nabla^2 h(u)) dV_{d-1}(u) \tag{6.113}$$

$$= \int_{\partial K} s_{i-1}(Dv(x)) dV_{d-1}(x). \tag{6.114}$$

We will now derive another useful integral representation for the Quermassintegrals by using mixed volumes and mixed areas. Recalling 6.80 and 6.87,

$$V(K + \xi B^d) = \sum_{m=1}^{d} \xi^{d-m} \binom{d}{m} V(K[m], B^d[d-m]), \tag{6.115}$$

$$S_{d-1}(K + \xi B^d, \cdot) = \sum_{m=0}^{d-1} \xi^{d-1-m} \binom{d-1}{m} S(K[m], B^d[d-1-m], \cdot) \tag{6.116}$$

and comparing with 6.70 and 6.67 (in dimensions $d-1$), we obtain

$$W_i(K) = V(K[d-i], B^d[i]) \tag{6.117}$$

and

$$S(K[m], B^d[d-1-m], \cdot) = \binom{d-1}{m}^{-1} \frac{1}{d-m} \int_{\cdot} s_m(\nabla^2 h(u)) dV_{d-1}(u). \tag{6.118}$$

Combining 6.117, 6.118 as well as 6.89 yields

$$\begin{aligned} W_i(K) &= V(K[d-i], B^d[i]) \\ &= \frac{1}{d} \int_{S_{d-1}} h(K,u) dS(K[d-i-1], B^d[i], u) \\ &= \frac{1}{d(i+1)} \binom{d-1}{d-1-i}^{-1} \int_{S_{d-1}} h(K,u) s_{d-i-1}(\nabla^2 h(u)) dV_{d-1}(u) \\ &= \frac{1}{i+1} \binom{d}{i+1}^{-1} \int_{S_{d-1}} h(K,u) s_{d-i-1}(\nabla^2 h(u)) dV_{d-1}(u). \end{aligned} \tag{6.119}$$

6.3.3 Inequalities between Quermassintegrals

We are now ready to give a complete presentation of a paper of Heil and Bukowski [2] that yields inequalities between the quermassintegrals W_i. These inequalities serve as the main tool to prove Proposition 25. We begin with a simple lemma:

Lemma 10. *Suppose $F(x)$ and $f(x)$ are C^1 and homogeneous of degree p and q respectively then*

$$\int_{S^{d-1}} f(u) \cdot divF(u) d\theta(u)$$

equals

$$(p+q+d-1) \int_{S^{d-1}} \left[\langle f(u)F(u), u \rangle - \langle \nabla f(u), F(u) \rangle \right] d\theta(u).$$

Proof. By the divergence theorem, we have

$$\gamma_d \int_{S^{d-1}} \langle f(u)F(u), u \rangle \, d\theta(u) = \int_{B(0,1)} \left[f(x) \mathrm{div} F(x) + \langle \nabla f(x), F(x) \rangle \right] dx. \tag{6.120}$$

By switching to polar coordinates and by using the homogeneity of f and F 6.120 equals

$$\gamma_d \int_{S^{d-1}} \int_0^1 [r^p f(u) r^{q-1} \mathrm{div} F(u) + \langle r^{p-1} \nabla f(u), r^q F(u) \rangle] r^{d-1} dr d\theta(u). \tag{6.121}$$

A computation shows that 6.121 equals

$$\gamma_d \cdot (p+q+d-1) \cdot \int_{S^{d-1}} f(u) \mathrm{div} F(u) + \langle \nabla f(u), F(u) \rangle \, d\theta(u).$$

□

If A denotes a symmetric $d \times d$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_d$, we let

$$T_q(A) = \sum_{k=0}^{q} (-1)^k \cdot s_{q-k}(A) \cdot A^k \tag{6.122}$$

denote the q^{th} Newtonian transformation of A. In the following lemma, the symbol $\mathrm{sgn}\begin{pmatrix} i_1 & \cdots & i_q \\ j_1 & \cdots & j_q \end{pmatrix}$ will denote the sign of the permutation $\sigma(i_r) = j_r$ and will be 0 otherwise.

Lemma 11. *The following holds [14, pp.373-375]:*

i) $s_q(A) = \frac{1}{q!} \sum sgn \begin{pmatrix} i_1 & \cdots & i_q \\ j_1 & \cdots & j_q \end{pmatrix} A_{i_1}^{j_1} \cdots A_{i_q}^{j_q}$

ii) $T_q(A)_i^j = \frac{1}{q!} \sum sgn \begin{pmatrix} i_1 & \cdots & i_q & i \\ j_1 & \cdots & j_q & j \end{pmatrix} A_{i_1}^{j_1} \cdots A_{i_q}^{j_q}$

iii) $Tr(T_q(A) \cdot A) = (q+1) S_{q+1}(A)$

iv) $T_q(A) = s_q I - T_{q-1}(A) \circ A$

v) $Tr\,(T_q(A)) = (d-q) \cdot s_q(A)$

vi) If f *is* C^3 *then* $\sum_{i=1}^d \partial_i T_q(\nabla^2 f)_i^j = 0, \forall j = 1, \ldots, d.$

vii) If f is C^3 then $q \cdot s_q(\nabla^2 f) = div(T_{q-1}(\nabla^2 f)(\nabla f))$.

Proof. i) Follows from the well-known representation of $s_q(A)$ in terms of principal minors. ii) follows by first computing in an orthonormal basis of eigenvectors and noting that both sides of ii) transform similarily. To prove iii), we have

$$\begin{aligned}
Tr(T_q(A) \circ A) &= \sum_{i=1}^{d} \sum_{j=1}^{d} T_q(A)_i^j \cdot A_j^i \\
&= \frac{1}{q!} \sum_{i=1}^{d} \sum_{j=1}^{d} \sum \delta \begin{pmatrix} i_1 & \cdots & i_q & i \\ j_1 & \cdots & j_q & j \end{pmatrix} A_{i_1}^{j_1} \cdots A_{i_q}^{j_q} A_i^j \\
&= \frac{1}{q!} \cdot (q+1)! \cdot s_{q+1}(A) \\
&= (q+1) \cdot s_{q+1}(A).
\end{aligned}$$

iv) is a trivial computation and v) follows from iii) and iv). To prove vi), we compute

$$\begin{aligned}
\sum_{i=1}^{d} \partial_i T_q(\nabla^2 f)_i^j &= \frac{1}{q!} \sum_{i=1}^{d} \sum \delta \begin{pmatrix} i_1 & \cdots & i_q & i \\ j_1 & \cdots & j_q & j \end{pmatrix} \sum_{k=1}^{q} f_{i_1 j_1} \cdots f_{i_k j_k i} \cdots f_{i_q j_q} \\
&= \frac{1}{q!} \sum_{i=1}^{d} \sum_{k=1}^{q} \sum \delta \begin{pmatrix} i_1 & \cdots & i_q & i \\ j_1 & \cdots & j_q & j \end{pmatrix} f_{i_1 j_1} \cdots f_{i_k j_k i} \cdots f_{i_q j_q} \\
&= \frac{1}{(q-1)!} \sum_{i=1}^{d} \sum \delta \begin{pmatrix} i_1 & \cdots & i_q & i \\ j_1 & \cdots & j_q & j \end{pmatrix} f_{i_1 j_1} \cdots f_{i_q j_q i} \\
&= \frac{-1}{(q-1)!} \sum_{i=1}^{d} \sum \delta \begin{pmatrix} i_1 & \cdots & i & i_q \\ j_1 & \cdots & j_q & j \end{pmatrix} f_{i_1 j_1} \cdots f_{i j_q i_q} \\
&= -\sum_{i=1}^{d} \partial_i T_q(\nabla^2 f)_i^j \\
&= 0.
\end{aligned}$$

To prove vii), we have by vi) and iii)

$$\begin{aligned}
\operatorname{div}\left(T_{q-1}(\nabla^2 f)(\nabla f)\right) &= \sum_{i=1}^{d}\sum_{j=1}^{d}\partial_i\left(T_{q-1}(\nabla^2(f))_i^j\partial_j f\right)\\
&= \sum_{i=1}^{d}\sum_{j=1}^{d}\left[\partial_i\left(T_{q-1}(\nabla^2 f)_i^j\right)\partial_j f + T_{q-1}(\nabla^2 f)_i^j\cdot\partial_i\partial_j f\right]\\
&= \sum_{i=1}^{d}\sum_{j=1}^{d}T_{q-1}(\nabla^2 f)_i^j\cdot\partial_i\partial_j f\\
&= Tr\left(T_{q-1}(\nabla^2 f)\circ\nabla^2 f\right)\\
&= q\cdot s_q(\nabla^2 f).
\end{aligned}$$

□

Proposition 26. *[2, Theorem 1]Let K be a convex body whose support function is C^3 then*

$$\begin{aligned}
i\binom{d}{i}W_{i-2}(K) &= \gamma_d\cdot\frac{1}{i-1}\cdot\int_{S^{d-1}}\left[s_{d-i}(\nabla^2 h(u))(ih^2(u)-|\nabla h(u)|^2)\right.\\
&\quad \left.+\left\langle T_{d-i-1}(\nabla^2 h(u))\left(\nabla h(u)\right),\nabla h(u)\right\rangle\right]d\theta(u)
\end{aligned}$$

Proof. We have by 6.113

$$(i-1)\binom{d}{i-1}W_{i-2} = \gamma_d\int_{S^{d-1}}h(u)s_{d-i+1}(u)d\theta(u) \tag{6.123}$$

and by part *vii* of Lemma 11 the right-hand side of 6.123 equals

$$\frac{1}{d-i+1}\cdot\gamma_d\int_{S^{d-1}}h(u)\cdot\operatorname{div}\left(T_{d-i}(\nabla^2 h(u))(\nabla h(u))\right)d\theta(u). \tag{6.124}$$

In order to use lemma 10, we need to consider the homogeneity of h and $T_{d-i}(\nabla^2 h(u))(\nabla h(u))$. Clearly h has homogeneity 1 and as a consequence $\nabla(h), \nabla^2 h$ and $s_q(\nabla^2 h)$ have homogeneity $0, -1$ and $-q$ respectively. This implies that $T_{d-i}(\nabla^2 h(u))(\nabla h(u))$ has homogeneity $i - d$ and thus 6.124 transforms to

$$\gamma_d \cdot \frac{1}{d-i+1} \cdot \int_{S^{d-1}} \Big[i \left\langle h(u) T_{d-i}(\nabla^2 h(u))(\nabla h(u)), u \right\rangle \\ - \left\langle \nabla h(u), T_{d-i}(\nabla^2 h)(\nabla h(u)) \right\rangle \Big] d\theta(u). \tag{6.125}$$

Recalling 6.91, we obtain the relations

$$\begin{aligned} T_q(\nabla^2 h(u))(u) &= u \cdot s_q(\nabla^2 h(u)), \\ \left\langle h(u) T_{d-i}(\nabla^2 h(u))(\nabla h(u)), u \right\rangle &= h(u) \left\langle \nabla h(u), T_{d-i}(\nabla^2 h(u))(u) \right\rangle \\ &= h^2(u) \cdot s_{d-i}(\nabla^2 h(u)). \end{aligned}$$

Using the above relations as well as part $iv)$ of Lemma 11, 6.125 equals

$$\gamma_d \cdot \frac{1}{d-i+1} \cdot \int_{S^{d-1}} \Big[i \cdot H^2(u) \cdot s_{d-i}(\nabla^2 h(u)) - \Big\langle \nabla h(u), \nabla h(u) \cdot \\ s_{d-i}(\nabla^2 h(u)) - T_{d-i-1}(\nabla^2 h) \circ \nabla^2 h(u)(\nabla h(u)) \Big\rangle \Big] d\theta(u)$$

which in turn equals

$$\gamma_d \cdot \frac{1}{d-i+1} \cdot \int_{S^{d-1}} \Big[s_{d-i}(\nabla^2 h(u))(i \cdot H^2(u) - |\nabla h(u)|^2) \\ + \left\langle \nabla h(u), T_{d-i-1}(\nabla^2 h) \circ \nabla^2 h(u)(\nabla h(u)) \right\rangle \Big] d\theta(u).$$

To complete the proof, note that

$$i \binom{d}{i} = (d-i+1) \binom{d}{i-1}.$$

□

It will now be shown how one can use these special integral representation to obtain inequalities between the W_i. Before doing so, we consider the notion of a *concave sequence.* We say that a sequence $\{a_s\}_{s=1}^m$ is concave if the value of every term in the sequence is larger than the average of the values preceeding and proceeding it i.e., $\frac{a_{i-1}+a_{i+1}}{2} \le a_i$ for every $i = 2, \ldots, m$. From this definition, one easily deduces that for every $1 \le i < j < k \le m$

$$\frac{a_j - a_i}{j - i} \ge \frac{a_k - a_j}{k - j} \Leftrightarrow a_i(j - k) + a_j(k - i) + a_k(i - j) \ge 0. \tag{6.126}$$

Proposition 27. *[2, Theorem 2]For every convex body K and for every $0 \le i < j < k \le d$, the sequence $\{(s+1)R^s(K)W_s(K)\}_{s=0}^d$ is concave and $R^i(K)W_i(K)(i+1)(j-k) + R^j(K)W_j(K)(j+1)(k-i) + R^k(K)(k+1)W_k(K)(i-j) \ge 0$.*

Proof. Since every convex body K can be approximated by C^∞ bodies one may assume that K is C^∞ and by scaling one may set $R = 1$. It suffices to show that

$$i\begin{pmatrix} d \\ i \end{pmatrix}[(i+1)W_i - 2(i)W_{i-1} + (i-1)W_{i-2}] \tag{6.127}$$

is nonnegative. By 6.114,6.119 and Proposition 26, 6.127 equals

$$\gamma_d \cdot \int_{S^{d-1}} \Big[s_{d-i}(\nabla^2 h(u)) \left((i+1) - 2ih(u) + ih^2(u) - |\nabla h(u)|^2 \right) \\ + \left\langle \nabla h(u), T_{d-i-1}(\nabla^2 h) \circ \nabla^2 h(u)(\nabla h(u)) \right\rangle \Big] d\theta(u).$$

Since $\nabla(h)(u)$ belongs to the boundary of K and K is contained in the unit ball then $|\nabla(h)(u)|^2 \le 1$ and in particular

$$(i+1) - 2ih(u) + ih^2(u) - |\nabla h(u)|^2 = i(h(u) - 1)^2 + (1 - |\nabla h(u)|^2) \ge 0.$$

It follows Lemma 11 that there exists an orthonormal basis whose elements are eigenvectors (with nonnegative eigenvalues) for both $T_{d-i-1}(\nabla^2 h(u))$ and $\nabla^2 h(u)$. This implies that $T_{d-i-1}(\nabla^2 h) \circ \nabla^2 h(u)$ is positive definite and in particular

$$\left\langle \nabla h(u), T_{d-i-1}(\nabla^2 h) \circ \nabla^2 h(u)(\nabla h(u)) \right\rangle \geq 0. \tag{6.128}$$

The second statement in the proposition follows from 6.126. □

6.3.4 The Key Estimates

Proof of Proposition 25. By considering $i = 0, j = d-1, k = d$ in Proposition 27, we have the following inequality

$$W_0 - d^2 \cdot (1+\epsilon)^{d-1} \cdot W_{d-1} + (d^2-1)(1+\epsilon)^d \cdot W_d \geq 0. \tag{6.129}$$

Since $W_0(K) = \kappa_d, W_{d-1}(K) = \kappa_d \cdot \frac{b(K)}{2}$ and $W_d(K) = \kappa_d$ then inequality 6.129 yields

$$\begin{aligned} \frac{b(K)}{2} &\leq 1/(d^2 \cdot (1+\epsilon)^{d-1}) + (1 - 1/d^2) \cdot (1+\epsilon) \qquad (6.130) \\ &< 1/d^2 + (1-1/d^2) \cdot (1+\epsilon) \\ &= 1 + \epsilon \cdot (1 - 1/d^2). \end{aligned}$$

If $\epsilon < 1/d$ then the inequality can be improved by estimating $(1+\epsilon)^{1-d}$ via its Taylor expansion and using Leibniz's theorem. More precisely, we have for $\epsilon < 1/d$

$$\begin{aligned} (1+\epsilon)^{1-d} &= 1 + \sum_{k=1}^{\infty} (-1)^k (d-1) \cdots (d-1+k) \cdot \frac{1}{k!} \cdot \epsilon^k \qquad (6.131) \\ &= \sum_{k=0}^{\infty} (-1)^k \cdot a_k \cdot \epsilon^k \qquad (6.132) \end{aligned}$$

with $\{a_k\}_{k=1}^{\infty}$ implicitly defined by 6.131 and 6.132. We note that $\{a_k \cdot \epsilon^k\}_{k=0}^{\infty}$ is a decreasing sequence if and only if $\epsilon < 1/d \leq \frac{1+k}{d+k}$ for all k. Hence for $\epsilon < 1/d$, $(1+\epsilon)^{1-d} < 1 - (d-1) \cdot$

$\epsilon + \frac{d(d-1)}{2} \cdot \epsilon^2$ and by 6.130

$$\begin{aligned} \frac{b(K)}{2} &< \frac{1}{d^2}\left[1-(d-1)\epsilon + \frac{d(d-1)}{2}\epsilon^2\right] + (1+\epsilon)(1-1/d^2) \\ &= 1+\epsilon - \frac{\epsilon}{d} + \frac{d(d-1)\cdot\epsilon^2}{2d^2} \\ &< 1+\epsilon-\frac{\epsilon}{d}+\frac{\epsilon^2}{2} \\ &< 1+\epsilon-\frac{\epsilon}{2d}. \end{aligned}$$

□

6.4 Steiner Symmetrizations

6.4.1 Circumradius

In order to use the results from section 11, we first use Theorem 9 and apply $3d$ Steiner symmetrizations to transform any convex body K into an isomorphic ball satisfying the property

$$c_1 B^d \subset K' \subset c_2 B^d \tag{6.133}$$

for some numerical constants c_1 and c_2. In other words, by initially applying $3d$ Steiner symmetrizations, we may assume from now on that K satisfies property 6.133.

We recall that $S_u(K) \subset M_u(K)$ for all $u \in S^{d-1}$ and thus if we let

$$S_\rho(K) = S_{\rho^1,\dots,\rho^d}(K) \tag{6.134}$$

for all $\rho = (\rho^1, \dots, \rho^d) \in \mathcal{O}$ and

$$S_{\boldsymbol{\rho}}(K) = \bigcirc_{i=1}^{T} S_{\rho_i}(K) \tag{6.135}$$

for all $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_T) \in \mathcal{O} \times \cdots \times \mathcal{O}$ (T times) then

$$\mathbb{E}_{\mu}^{T}\left[R(S_{\boldsymbol{\rho}}(K))\right] \leq \mathbb{E}_{\mu}^{T}\left[R(M_{\boldsymbol{\rho}}(K))\right] \tag{6.136}$$

for all $T \geq 1$. By Theorem 11 and 6.136, we have

$$\mathbb{E}_{\mu}^{T}\left[R(S_{\boldsymbol{\rho}}(K))\right] \leq \epsilon + \frac{b(K)}{2} \tag{6.137}$$

for $T \geq c_2 c_4 d \log(1/\epsilon)$. However, by Proposition 25, we also have

$$\frac{b(K)}{2} < \begin{cases} 1 + (1 - 1/d^2)(R(K) - 1) & \text{if } R(K) - 1 \geq 1/d \\ 1 + (1 - 1/2d)(R(K) - 1) & \text{if } R(K) - 1 < 1/d \end{cases} \tag{6.138}$$

and thus by 6.137 and 6.138 if

$$T \geq \begin{cases} c_2 c_4 d \log\left(\frac{2d^2}{(R(K)-1)}\right), & \text{for } R(K) \geq 1 + \frac{1}{d} \\ c_2 c_4 d \log\left(\frac{4d}{(R(K)-1)}\right), & \text{for } R(K) < 1 + \frac{1}{d} \end{cases} \tag{6.139}$$

then

$$\mathbb{E}_{\mu}^{TN}\left[R(S_{\boldsymbol{\rho}}(K))\right] < \begin{cases} \left(1 - \frac{1}{2d^2}\right)^N (R(K) - 1) + 1 & \text{for } R(K) \geq 1 + \frac{1}{d} \\ \left(1 - \frac{1}{4d}\right)^N (R(K) - 1) + 1 & \text{for } R(K) < 1 + \frac{1}{d} \end{cases} \tag{6.140}$$

for all $N \in \mathbb{N}$. We easily deduce from 6.139 and 6.140 that if $R(K) \geq 1 + \epsilon \geq 1 + 1/d$ then

$$\mathbb{E}_{\mu}^{TN}\left[R(S_{\boldsymbol{\rho}}(K))\right] \leq 1 + \epsilon \tag{6.141}$$

whenever

$$N \geq \frac{\log\left(\frac{c_2 - 1}{\epsilon}\right)}{\log\left(\frac{2d^2}{2d^2 - 1}\right)} \geq \frac{\log\left(\frac{R(K) - 1}{\epsilon}\right)}{\log\left(\frac{2d^2}{2d^2 - 1}\right)} \tag{6.142}$$

and

$$T \geq c_2 c_4 d \log\left(2d^3\right). \tag{6.143}$$

Similarily, if $1+\epsilon \leq R(K) < 1+1/d$ then

$$\mathbb{E}_{\mu}^{TN}\left[R(S_{\boldsymbol{\rho}}(K))\right] \leq 1+\epsilon \tag{6.144}$$

whenever

$$N \geq \frac{\log\left(\frac{d}{\epsilon}\right)}{\log\left(\frac{4d}{4d-1}\right)} \geq \frac{\log\left(\frac{d}{R(K)-1}\right)}{\log\left(\frac{4d}{4d-1}\right)} \tag{6.145}$$

and

$$T \geq c_2 c_4 d \log\left(\frac{4d}{\epsilon}\right). \tag{6.146}$$

It is clear that there exists a numerical contant c_5 such that

$$\frac{\log\left(\frac{c_2-1}{\epsilon}\right)}{\log\left(\frac{2d^2}{2d^2-1}\right)} \leq c_5 d^2 \log(1/\epsilon) \tag{6.147}$$

and

$$\frac{\log\left(d/\epsilon\right)}{\log\left(\frac{4d}{4d-1}\right)} \leq c_5 d \log(1/\epsilon). \tag{6.148}$$

If we account for the first $3d$ Steiner symmetrizations needed to transform K into an isomorphic ball see 6.133 then by using 6.141,6.144 as well as 6.147 and 6.148, we have the following result:

Theorem 12. *[9, Proposition 6.4]There exists a numerical constant c_6 such that we need at most*

$$\begin{cases} \left\lceil c_6 d^3 \log(d) \log(1/\epsilon) \right\rceil & \text{if } 1/d \leq \epsilon \\ \left\lceil c_6 \left(d^3 \log^2(d) + d^2 \log^2(1/\epsilon)\right) \right\rceil & \text{if } \epsilon < 1/d \end{cases} \tag{6.149}$$

Steiner symmetrizations to transform any convex body K with volume κ_d into a body K' with the property $R(K') \leq 1+\epsilon$.

6.4.2 Inradius

It is clear that relating Minkowski symmetrizations to Steiner symmetrizations will not yield any information about the behaviour of the inner radius under consecutive Steiner symmetrizations. We do, however, know that Steiner symmetrizations preserve volume and we have a good handle on how many Steiner symmetrizations we need to reduce the circumradius to an appropriate level. The fact that the convex body has volume equal to that of the unit ball does not automatically yield any information about the inner radius $r(K)$ since we can have convex bodies that have very large widths in some directions balanced with short widths in other directions. If we know beforehand that $R(K)$ is close to 1 then we cannot have very long widths and consequently we cannot have very short widths either. This intuition is embodied in the following lemma.

Lemma 12. *Let K be a convex body with volume κ_d. If $R(K) \leq 1+\left(\frac{\epsilon}{24}\right)^d$ then $r(K) \geq 1-\epsilon$.*

Proof. Temporarily let $\epsilon' = \left(\frac{\epsilon}{24}\right)^d$. The first step is to use Urysohn's inequality (Proposition 24):

$$\frac{b(K)}{2} \geq 1. \tag{6.150}$$

Now suppose there exists some u_0 such that $h(K,u_0) < 1-\epsilon$ then since $h(K,u)$ has Lipschitz constant at most $1+\epsilon'$ we have $h(K,w) < (1+\epsilon')(\epsilon/4)+1-\epsilon$ whenever $|w-u_0| \leq \epsilon/4$. Let $A = B(u_0,\epsilon/4) \cap S^{d-1}$ then

$$\theta(A) = \gamma_d^{-1} \int_{proj(A)} \left(1-|x|^2\right)^{-1/2} dx > \gamma_d^{-1} V_{d-1}(\text{proj}(A)) \tag{6.151}$$

with $\text{proj}(A)$ the projection of A onto $u_0^{\perp}$. We have

$$\sqrt{|x|^2 + (1-\sqrt{1-|x|^2})^2} = \sqrt{2}|x| \leq \epsilon/4 \tag{6.152}$$

whenever $x \in \mathbb{R}^{d-1}$ and $|x| \le \frac{\epsilon}{4\sqrt{2}}$. This implies

$$V_{d-1}(\text{proj}(A)) \ge V_{d-1}\left(\frac{\epsilon}{4\sqrt{2}}B^{d-1}\right) = \left(\frac{\epsilon}{4\sqrt{2}}\right)^{d-1}\kappa_{d-1}. \tag{6.153}$$

Combining 6.151 and 6.153 gives

$$\theta(A) > \frac{\kappa_{d-1}}{\gamma_d}\left(\frac{\epsilon}{4\sqrt{2}}\right)^{d-1} = \frac{\Gamma(d/2+1)}{d\sqrt{\pi}\Gamma((d-1)/2+1)}\left(\frac{\epsilon}{4\sqrt{2}}\right)^{d-1} > \frac{1}{d\sqrt{\pi}}\left(\frac{\epsilon}{4\sqrt{2}}\right)^{d-1}$$

and thus

$$\theta(A) > \left(\frac{\epsilon}{(d\sqrt{\pi})^{1/(d-1)}4\sqrt{2}}\right)^{d-1} \ge \left(\frac{\epsilon}{2\sqrt{\pi}4\sqrt{2}}\right)^{d-1} > \left(\frac{\epsilon}{24}\right)^{d-1}. \tag{6.154}$$

We will now obtain a contradiction by showing that the mean width is smaller than 2. We have

$$\begin{aligned}
\int_{S^{d-1}} h(K,u)d\theta(u) &< \theta(A)((1+\epsilon')(\epsilon/4)+1-\epsilon)+(1-\theta(A))(1+\epsilon') \\
&< \theta(A)\left(1-\frac{\epsilon}{2}\right)+(1-\theta(A))(1+\epsilon') \\
&= 1+\epsilon'-\theta(A)\left(\frac{\epsilon}{2}+\epsilon'\right) \\
&< 1+\epsilon'-\theta(A)\frac{\epsilon}{2} \\
&< 1.
\end{aligned}$$

□

6.4.3 Final Results

Proof of Theorem 6. If $0 < \epsilon < 1$ then $\left(\frac{\epsilon}{24}\right)^d < 1/d$. By Theorem 12, there exists a numerical contant c_6 such that we need at most

$$\left\lceil c_6\left(d^3\log^2(d)+d^4\log^2(24/\epsilon)\right)\right\rceil$$

Steiner symmetrizations to transform an arbitrary convex body K with volume κ_d into a new convex body K' with the property $R(K') \leq 1 + \left(\frac{\epsilon}{24}\right)^d$. By Lemma 12, $r(K') \geq 1 - \epsilon$. $\square$

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