

AN EULERIAN APPROACH TO OPTIMAL TRANSPORT WITH APPLICATIONS TO THE  
OTTO CALCULUS

by

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# Abstract

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This thesis studies the optimal transport problem with costs induced by Tonelli Lagrangians. The main result is an extension of the Otto calculus to higher order functionals, approached via the Eulerian formulation of the optimal transport problem. Open problems 15.11 and 15.12 from Villani's *Optimal Transport: Old and New* are resolved. A new class of displacement convex functionals is discovered that includes, as a special case, the functionals considered by Carrillo-Slepčev. Improved and simplified proofs of the relationships between the various formulations of the optimal transport problem, first seen in Bernard-Buffoni and Fathi-Figalli, are given. Progress is made towards developing a rigorous Otto calculus via the DiPerna-Lions theory of renormalized solutions. As well, progress is made towards understanding general Lagrangian analogues of various Riemannian structures.

*In memory of my mother, Donna*

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# Chapter 1

## Introduction

This thesis is focused on the theory of optimal transport. In the optimal transport problem, one seeks to find the most efficient way to transfer mass from one place to another. More formally, the problem is to choose, among all maps which push forward a specified initial probability measure to a specified final probability measure, the map that minimizes a given cost function.

In this thesis, several improvements are made to the *Otto calculus*, first developed by Otto ([40], [41]). The Otto calculus is a formal calculus on Wasserstein space, the space of probability measures on a manifold endowed with a notion of distance derived from the optimal transport problem. Although Wasserstein space lacks the structure needed to define derivatives—for instance, it is not a Hilbert manifold—the Otto calculus allows one to formally compute tangent vectors to curves and Hessians of functionals along these curves, where the notion of a curve in Wasserstein space needs to be interpreted correctly.

The project which resulted in this thesis was first presented to me by Wilfrid Gangbo (who eventually became my co-advisor) while he was visiting the Fields Institute for a semester in the fall of 2014. The problem originally posed to me was to understand the relationships between the various formulations of the optimal transport problem ([19], [5]) and to try to use this understanding to develop improved versions of the inequalities established in [22] and [14]. Over the course of working on this problem, as I learned the optimal transport literature, I became interested in McCann’s *displacement convexity* ([34]) and understanding it via an Eulerian approach to the Otto calculus.

The notion of displacement convexity of a functional on Wasserstein space is the analogue of geodesic convexity of a function on a Riemannian manifold. A functional on Wasserstein space is called displacement convex if it is convex along every Wasserstein geodesic. The correct analogue of geodesic curves in Wasserstein space are McCann’s *displacement interpolants*. The Otto calculus provides a natural way to test displacement convexity of functionals on Wasserstein space, by taking second derivatives—computing the *displacement Hessian* of the functional—and checking for positivity, analogous to checking whether a smooth function is geodesically convex on a Riemannian manifold.

The definition of a displacement Hessian of a functional is motivated by the definition of the

Hessian of a smooth function on a Riemannian manifold. Let  $\phi : M \rightarrow \mathbb{R}$  be a smooth function on a Riemannian manifold  $M$ . Its Hessian at a point  $x$  is the bilinear form defined by

$$\nabla\nabla\phi(v, v) = \frac{d^2}{ds^2}\phi(\gamma(s))\Big|_{s=0}$$

where  $\gamma$  is a geodesic curve with  $\gamma(0) = x$  and  $\gamma'(0) = v$ . If  $\nabla\nabla\phi$  is positive definite, then  $\phi$  is geodesically convex. The situation in Wasserstein space is analogous: given a functional  $\mathcal{F}$ , the Otto calculus allows for the computation of its displacement Hessian:

$$\frac{d^2}{ds^2}\mathcal{F}(\rho(s))$$

where  $\rho$  is a Wasserstein geodesic. The positivity of the displacement Hessian then corresponds to the convexity of the functional.

This concept has yielded several results; most notably, the Ricci curvature non-negativity condition associated to the displacement convexity of the entropy functional was first predicted via the Otto calculus in [41]. In learning the literature, it became clear to me that with a careful understanding of the different forms of the optimal transport problem, especially of the regularity of optimal trajectories in the Lagrangian formulation of the problem, and the regularity of the corresponding optimal vector field in the Eulerian formulation of the problem, improvements could be made to the Otto calculus.

The main contributions of this thesis are:

- I compute, with an Eulerian calculus extending the Otto calculus, a canonical form for the displacement Hessians of functionals involving arbitrary derivatives of densities. The problem of computing the displacement Hessian of a functional involving only the first derivative of the density was posed by Villani as Open Problem 15.11 in [46]; this is now resolved.
- I use this Eulerian calculus to find a new class of displacement convex functionals involving derivatives on  $S^1$ . This class contains, as a special case, the functionals considered by Carrillo-Slepčev ([11]), and thus provides an alternate proof of their displacement convexity.
- I extend the Otto calculus to the setting of the optimal transport problem with costs induced by Tonelli Lagrangians, which resolves Open Problem 15.12 in [46]. (Paul Lee has worked in a similar direction in [30]).

As well, this thesis contains a careful analysis of the regularity of the optimal trajectories in the Lagrangian formulation of the optimal transport problem. I believe the proofs of these results are simpler than what is currently in the literature ([5], [19], [46]), but are known to experts.

There are two ongoing avenues of investigation. Partial results in these directions are in this thesis; more substantial results are forthcoming.



- A long-standing open problem is that of developing a rigorous Otto calculus, which has so far been a purely formal method of computation. There are partial results in this direction, limited to the 2-Wasserstein space (Wasserstein space with cost given by squared Riemannian distance). In [42], Otto and Westdickenberg, using the Eulerian formulation of the classical optimal transport problem, show that a certain gradient flow being a contraction implies the displacement convexity of a corresponding functional. This idea is expanded upon in [17], where Daneri and Savaré provide a new proof of the displacement convexity of functionals satisfying the *McCann conditions* on manifolds with non-negative Ricci curvature. Daneri and Savaré work with the Otto calculus in a smooth setting and note the difficulty of extending their Otto calculus computations to the general non-smooth setting. The problem of developing a rigorous Otto calculus is discussed in [46], where Villani espouses a pessimistic view on the matter.

With an improved understanding of the regularity of the optimal vector field in the Eulerian formulation of the problem, together with results from the DiPerna-Lions theory of renormalized solutions ([18], [1], [13], [16]), the Eulerian calculus studied in this thesis can be made rigorous; in particular, this yields a rigorous Otto calculus.

- I am working towards understanding the geometry induced by a Tonelli Lagrangian. The formulation of the Otto calculus in this thesis suggests that there should be general Lagrangian analogues of Riemannian structures, such as the Riemannian volume form, Ricci curvature, Christoffel symbols, etc.

## 1.1 The Optimal Transport Problem from Lagrangian and Eulerian Perspectives

My description of the historical background on the optimal transport problem draws heavily on chapter 3 from [46].

The optimal transport problem was first described in the late eighteenth century by Monge. In [38], he posed the problem of the finding the least costly way to transport material extracted from the earth to the construction of a building. In Monge’s original formulation, the cost of transporting a mass  $m$  from position  $x$  to position  $y$  is the mass multiplied by distance:  $m|x - y|$ . There is a natural Lagrangian “particle trajectory” interpretation of this cost, that of a particle of mass  $m$  moving in a straight line from the point  $x$  to the point  $y$ . Little progress was made on the problem until the 20th century when Kantorovich ([28], [27]) reformulated and solved the optimal transport problem in a more general setting, viewing it as an infinite dimensional linear programming problem.

In 1991, Brenier characterized the solutions to the optimal transportation problem on  $\mathbb{R}^d$  with cost given by the square of Euclidean distance; he showed that optimal transport maps are given by the gradient of convex functions ([6]). In 1994, McCann developed this work fur-

ther, realizing that solutions to the Monge problem yielded *displacement interpolants*: geodesic curves in the space of probability measures ([34]). He further discovered the phenomenon of *displacement convexity*: several important functionals are convex along displacement interpolants ([34]). The interpretation of solutions to the optimal transport problem as corresponding to particle trajectories was present in McCann's work at this time ([33]). He later, in 2001, extended Brenier's theorem to Riemannian manifolds, characterizing optimal transport maps in this setting as particle trajectories defined by a potential function ([36]).

The particle trajectory view of optimal transport was made more explicit by Benamou and Brenier in [4], where they interpret the Monge optimal transport problem from an explicitly Lagrangian perspective, and, motivated by fluid mechanics, transform the problem into a corresponding Eulerian problem. The Lagrangian formulation was developed further by Figalli, Fathi-Figalli, and Bernard-Buffoni ([21], [19], [5]), all of whom consider the optimal transport problem with costs induced by Tonelli Lagrangians. In [19] and [5], they make explicit the relationships between the optimizers of the Lagrangian and Eulerian formulations of the optimal transport problem.

## 1.2 Outline of this thesis

In the second chapter, Tonelli Lagrangians are defined and the regularity properties of their induced costs are deduced. The Hamiltonian dual to the Tonelli Lagrangian is introduced as its Legendre transform and the properties of the corresponding flows on the cotangent and tangent bundles are stated.

In the first half of chapter three (section 3.2), background on the optimal transport problem is presented. The Monge, Kantorovich, and Kantorovich dual optimal transportation problems are defined. Several standard results are recalled.

In the second half of chapter three, the Lagrangian optimal transport problem is defined (section 3.3). A careful analysis of the regularity of the optimal trajectories in the Lagrangian formulation of the optimal transportation problem is performed (subsection 3.3.2). The Eulerian formulation of the optimal transport problem is defined (section 3.4). The equivalence of the five formulations of the optimal transport problem and the relationships between their minimizers is proven (theorems 3.5.1 and 3.5.2). The results here are, I believe, presented more simply than what is currently in the literature, and are perhaps slightly stronger, but will not be a surprise to experts.

In the fourth chapter, an Eulerian calculus extending the Otto calculus is introduced within the framework of the Eulerian optimal transport problem. Here, new results are proven.

In theorem 4.3.2, a canonical computation of the displacement Hessian of functionals involving derivatives of densities is presented. This resolves Open Problem 15.11 in Villani [46], which asks whether one can compute the displacement Hessian of a functional involving only a single derivative of the density.

In theorem 4.2.3, a new class of displacement convex functionals is discovered. This class includes, as a special case, the functionals considered by Carrillo-Slepčev ([11]); the proof of theorem 4.2.3 provides a new proof of their displacement convexity. A counterexample shows that certain aspects of the sufficient condition used to establish the displacement convexity of this new class of functionals are close to necessary.

The formulation of the Eulerian calculus in the fourth chapter resolves Open Problem 15.12 in Villani, where he asks whether it is possible to extend the Otto calculus to the setting of general Lagrangian induced costs. In appendix A, the displacement Hessian of the entropy functional is computed in coordinates; Villani's formulation of the Otto calculus is realized as a special case, when the cost function is Riemannian distance squared.

I am investigating, in ongoing work, a regularization scheme to make the Otto calculus rigorous, via the theory of renormalized solutions introduced by DiPerna-Lions ([18]). As well, the Otto calculus developed in chapter 4 suggests general Lagrangian analogues to various Riemannian structures (e.g. volume forms, Ricci curvature, etc.); I am working to make these notions precise. Discussion of these issues and partial results appear in the final chapter of this thesis and the appendix.

## Chapter 2

# Lagrangian Induced Cost Functions

### 2.1 Introduction

As discussed in the introduction, there is a natural “particle trajectory” interpretation of the optimal transport problem. A good approach to the optimal transport problem from a particle trajectory perspective is achieved by working with costs induced by Tonelli Lagrangians. This is the most general framework where the cost of sending a particle from a point  $x$  to a point  $y$  corresponds to a smooth curve starting at  $x$  and ending at  $y$ . Both the classical optimal transport problem and the optimal transport problem with costs given by squared Riemannian distance occur as special cases. The approach via Tonelli Lagrangians is more general and—perhaps more importantly—strips away all but the essential structure of the problem. These cost functions have been considered by, among others, Fathi-Figalli [19], Bernard-Buffoni [5] and are presented in chapter 7 of Villani’s book [46].

In the first half of this chapter, Tonelli Lagrangians will be defined and a careful application of the direct method in the calculus of variations will yield the existence of action minimizing curves. Cost functions will be defined by the least action of Tonelli Lagrangians. Regularity properties of action minimizing curves and cost functions will be proven. A large amount of information about Tonelli Lagrangian induced cost functions can be understood directly from this framework, without invoking the additional structure of the optimal transport problem.

In the latter half of this chapter, the Hamiltonian corresponding to a Tonelli Lagrangian will be introduced as the Legendre transform the Lagrangian, from the perspective of convex analysis. The relationships between Lagrangians and Hamiltonians, as presented in classical mechanics, will be shown. Properties of the Lagrangian flow on the tangent bundle of a manifold and of the Hamiltonian flow on the cotangent bundle will be proven.

Instead of treating the Lagrangian flow as a flow of a vector field on the tangent bundle of a manifold, it will be more fruitful to view the flow as families of curves on the manifold itself, each of which satisfy the Euler-Lagrange equation. This will be the perspective taken through most of this thesis. A technical issue that will arise (in a later chapter) will be to find a preferred family of these curves which themselves are the flow of a vector field on the

base manifold. These preferred curves will be interpreted as a solution to the optimal transport problem, and will be an object of interest, rather than the entire Lagrangian flow on the tangent bundle of a manifold.

The material in this chapter is classical, but is also crucial to the understanding of the optimal transport problem.

## 2.2 Tonelli Lagrangians

For notational convenience,  $W^{1,1}((0, 1); \mathbb{R}^d) = W^{1,1}$ . The tangent bundle of a manifold  $M$  is denoted by  $TM$ . When  $M = \mathbb{R}^d$ , which will almost always be the case in this document, then  $T\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d$ . The cotangent bundle is denoted  $T^*M$  and  $T^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d$ .

**Definition 2.2.1.** A function  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  is called a **Tonelli Lagrangian** if it satisfies the following properties.

- (a) The function  $L$  is of class  $C^2$ .
- (b) For every  $x \in \mathbb{R}^d$ , the function  $L(x, \cdot) : T_x\mathbb{R}^d \cong \mathbb{R}^d \rightarrow \mathbb{R}$  is strictly convex.
- (c) There exists a constant  $c_0$  and a non-negative function  $\theta : \mathbb{R}^d \rightarrow \mathbb{R}$  with superlinear growth, i.e.

$$\lim_{|v| \rightarrow +\infty} \frac{\theta(v)}{|v|} = +\infty,$$

such that

$$L(x, v) \geq c_0 + \theta(v).$$

The first variable in a Lagrangian will be referred to as **position** and the second variable will be referred to as **velocity**. For example, if  $L$  is a Tonelli Lagrangian, then  $L(x, v)$  is the Lagrangian evaluated at position  $x$  and velocity  $v$ . The derivatives and gradients of a Tonelli Lagrangian  $L$  in the position and velocity variables will be denoted by  $D_x L$ ,  $D_v L$ ,  $\nabla_x L$ , and  $\nabla_v L$ .

**Definition 2.2.2.** A function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  of class  $C^2$  is called **strongly convex** if  $\nabla^2 f(x)$  is positive definite for all  $x \in \mathbb{R}^d$ . Denote a point in  $\mathbb{R}^d \times \mathbb{R}^n$  by  $(x, y)$ . A function  $f : \mathbb{R}^d \times \mathbb{R}^n \rightarrow \mathbb{R}$  of class  $C^2$  is called **strongly convex in  $y$**  if  $D^2 f_x(y) = D_y^2 f(x, y)$  is positive definite for every  $(x, y) \in \mathbb{R}^d \times \mathbb{R}^n$ , where  $D^2 g(y)$  denotes the matrix of second derivatives of  $g$  and  $D_y^2 g(x, y)$  denotes the matrix of second derivatives of  $g$  in  $y$ .

Tonelli Lagrangians  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  will often be assumed to be strongly convex in the second argument. The reason strong convexity is not built into the definition of Tonelli Lagrangians is

that the definition is chosen to have the minimal assumptions needed to guarantee the existence of action minimizing curves.

**Definition 2.2.3.** Let  $\sigma$  be a curve in  $W^{1,1}((0,1); \mathbb{R}^d) = W^{1,1}$ . The **velocity** of the curve  $\sigma$  at  $t \in (0,1)$  is

$$\dot{\sigma}(t) := \frac{d}{ds}\sigma(s)\Big|_{s=t}.$$

The Tonelli Lagrangian naturally induced a functional on the space of weakly differentiable curves. This functional, the action of the Lagrangian, will be interpreted as a cost function.

**Definition 2.2.4.** Let  $L$  be a Tonelli Lagrangian. Let  $\sigma \in W^{1,1}$ . The **action of  $L$  on  $\sigma$**  is

$$A_L(\sigma) = \int_0^1 L(\sigma(t), \dot{\sigma}(t)) dt.$$

**Definition 2.2.5.** Fix  $x, y \in \mathbb{R}^d$ . The **cost of transport from  $x$  to  $y$  induced by  $L$**  is

$$c(x, y) = \inf\{A_L(\sigma) : \sigma \in W^{1,1}, \sigma(0) = x, \sigma(1) = y\}.$$

Since the Tonelli Lagrangian is bounded below, this infimum is guaranteed to be finite. A curve  $\sigma \in W^{1,1}$  from  $x$  to  $y$  which minimizes the action of  $L$  is called an **action minimizing curve** or an **optimal trajectory** from  $x$  to  $y$ . It is not obvious that action minimizing curves exist for general Tonelli Lagrangians. For example, the minimal growth assumptions on  $L$  (for instance,  $L(x, \cdot)$  may grow slower than  $|\cdot|^p$  for any  $p > 1$ ) present technical challenges to demonstrating the existence of action minimizers (the potentially slow asymptotic growth prevents using the Banach-Aloaglu theorem when trying to prove existence of action minimizers). However, Tonelli Lagrangians do indeed have action minimizing curves.

**Theorem 2.2.6** (Existence of Action Minimizers). *Let  $L$  be a Tonelli Lagrangian. Fix  $x, y \in \mathbb{R}^d$ . Let  $c(x, y) = \inf\{A_L(\sigma) : \sigma \in W^{1,1}, \sigma(0) = x, \sigma(1) = y\}$ . Then, the infimum is achieved. That is, there exists some curve  $\bar{\sigma} \in W^{1,1}$  such that  $c(x, y) = A_L(\bar{\sigma})$ .*

The following characterization of weakly convergent sequences in  $L^1$  from Buttazzo, Giacquinta, and Hildebrandt [8] will be used in the proof of the above theorem.

**Theorem 2.2.7** (Theorem 2.11 from Buttazzo, Giacquinta, and Hildebrandt [8]). *Suppose that  $\Omega$  is a bounded open set in  $\mathbb{R}^d$ . Suppose that  $(u_k)_{k=1}^\infty$  is a sequence in  $L^1(\Omega)$  with the following properties.*

(a) *The sequence  $(u_k)_{k=1}^\infty$  is uniformly bounded in  $L^1(\Omega)$ . That is,*

$$\sup_k \|u_k\|_1 < +\infty.$$

(b) The sequence of measures  $(u_k dx)_{k=1}^\infty$  is equiabsolutely continuous with respect to Lebesgue measure. That is, for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for every  $u_k$ , and for every  $E \subset \Omega$  with  $m(E) < \delta$ ,

$$\int_E |u_k(x)| dx < \epsilon.$$

Then, the sequence  $(u_k)_{k=1}^\infty$  has a subsequence which converges weakly in  $L^1(\Omega)$ . Further, if  $(u_k)_{k=1}^\infty$  converges weakly in  $L^1(\Omega)$ , then properties (a) and (b) hold.

The proof of the existence of action minimizers will proceed by showing the following results. Let  $(\sigma_n)_{n=1}^\infty$  be a minimizing sequence for  $c(x, y)$ .

**Step 1:** The sequence  $(\dot{\sigma}_n)_{n=1}^\infty$  is weakly compact in  $L^1((0, 1); \mathbb{R}^d)$ .

**Step 2:** A subsequence  $(\sigma_k)_{k=1}^\infty$  of the sequence  $(\sigma_n)_{n=1}^\infty$  is uniformly bounded and equicontinuous.

**Step 3:** The subsequence  $(\sigma_k)_{k=1}^\infty$  converges to a limit  $\sigma$  in  $L^1$  (in fact, the convergence will be uniform) and  $\dot{\sigma}_k \rightharpoonup \dot{\sigma}$  in  $L^1$ .

**Step 4:** This limit is a minimizer of the action of  $L$ .

*Proof.* **Step 1:** It suffices to show that  $(\dot{\sigma}_n)_{n=1}^\infty$  is bounded in  $L^1$  and is equiabsolutely continuous. Since  $(\sigma_n)_{n=1}^\infty$  is a minimizing sequence, by passing to a subsequence, the sequence  $(A_L(\sigma_n))_{n=1}^\infty$  may be taken to be non-increasing. Therefore, for every  $n$ ,

$$A_L(\sigma_1) \geq A_L(\sigma_n) \geq \int_0^1 c_0 + \theta(\dot{\sigma}_n(t)) dt = c_0 + \int_0^1 \theta(\dot{\sigma}_n(t)) dt.$$

Since  $\lim_{|v| \rightarrow \infty} \theta(v)/|v| = +\infty$  and for every  $n$ ,

$$\int_0^1 \theta(\dot{\sigma}_n(t)) dt \leq A_L(\sigma_1) - c_0 = K_0,$$

it follows that there exists a constant  $K_1$  such that for every  $n$ ,

$$\int_0^1 |\dot{\sigma}_n(t)| dt < K_1 \quad \implies \quad \sup_n \|\dot{\sigma}_n\|_1 < +\infty. \quad (2.1)$$

Fix  $\epsilon > 0$ . Let  $M$  be sufficiently large that

$$\frac{\theta(v)}{|v|} > \frac{K_0}{\epsilon} \quad \text{for all } |v| > M.$$

Then,

$$|\dot{\sigma}_n(t)| < \frac{\epsilon}{K_0} \theta(\dot{\sigma}_n(t)) \quad \text{for all } t \in F_M^n = \{t \in (0, 1) : |\dot{\sigma}_n(t)| > M\}.$$

From this, it follows that

$$\int_{F_M^n} |\dot{\sigma}_n(t)| dt < \frac{\epsilon}{K_0} \int_0^1 \theta(\dot{\sigma}_n(t)) dt = \epsilon.$$

In particular,  $m(F_M^n) < \frac{\epsilon}{M}$  for every  $n$ , for otherwise, if  $m(F_M^n) \geq \frac{\epsilon}{M}$  for some  $n$ , then

$$\int_{F_M^n} |\dot{\sigma}_n(t)| dt \geq m(F_M^n)M \geq \epsilon,$$

contradicting

$$\int_{F_M^n} |\dot{\sigma}_n(t)| dt < \epsilon.$$

Therefore, if  $m(E) < \frac{\epsilon}{M}$ , then for every  $n$ ,

$$\begin{aligned} \int_E |\dot{\sigma}_n(t)| dt &= \int_{E \cap F_M^n} |\dot{\sigma}_n(t)| dt + \int_{E \setminus F_M^n} |\dot{\sigma}_n(t)| dt \\ &\leq \int_{F_M^n} |\dot{\sigma}_n(t)| dt + m(E)M \\ &< \epsilon + \frac{\epsilon}{M}M = 2\epsilon. \end{aligned}$$

Since  $\epsilon > 0$  was arbitrary, the sequence of measures  $(\dot{\sigma}_n dx)_{n=1}^\infty$  is equiabsolutely continuous with respect to Lebesgue measure. By 2.2.7, the sequence  $(\dot{\sigma}_n)_{n=1}^\infty$  has a weakly convergent subsequence.

**Step 2:** Fix  $s \in (0, 1)$ . Then,

$$\begin{aligned} |\sigma_n(s)| &= \left| \sigma(0) + \int_0^s \dot{\sigma}_n(t) dt \right| \\ &= \left| x + \int_0^s \dot{\sigma}_n(t) dt \right| \\ &\leq |x| + \|\dot{\sigma}_n\|_1 \\ &\leq |x| + K_1. \end{aligned}$$

Therefore,

$$\sup_n \|\sigma_n\|_u < +\infty, \tag{2.2}$$

where  $\|\cdot\|_u$  denotes the uniform norm.

Fix  $\epsilon > 0$ . By the previous part, there exists  $\delta > 0$  such that for every set  $E$  with  $m(E) < \delta$ ,



for all  $n$ ,

$$\int_E |\dot{\sigma}_n(t)| dt < \epsilon.$$

If  $|r - s| < \delta$  (without loss of generality, assume  $r \geq s$ ), then,

$$\begin{aligned} |\sigma_n(r) - \sigma_n(s)| &= \left| x + \int_0^r \dot{\sigma}_n(t) dt - x - \int_0^s \dot{\sigma}_n(t) dt \right| \\ &= \left| \int_s^r \dot{\sigma}_n(t) dt \right| \\ &< \epsilon. \end{aligned}$$

Since this is independent of  $n$ , the sequence  $(\sigma_n)_{n=1}^\infty$  is equicontinuous on  $[0, 1]$ .

**Step 3:** By step 2, the sequence  $(\sigma_n)_{n=1}^\infty$  is uniformly bounded and equicontinuous, so satisfies the hypotheses of the Arzelà-Ascoli theorem. Therefore the sequence has a uniformly convergent subsequence. After possibly passing to a further subsequence, by the results from step one, the uniformly convergent subsequence  $(\sigma_{n_k})_{k=1}^\infty$  has derivatives  $(\dot{\sigma}_{n_k})_{k=1}^\infty$  which converge weakly in  $L^1$ .

The sequence  $\dot{\sigma}_n \rightharpoonup \gamma$  weakly, and  $\sigma_n \rightarrow \sigma$  uniformly. The function  $\gamma$  is in fact the velocity of  $\sigma$ . Fix  $s \in (0, 1)$ . Then,

$$\begin{aligned} \left| x + \int_0^s \gamma(t) dt - \sigma(s) \right| &\leq \left| x + \int_0^s \gamma(t) dt - \sigma_n(s) \right| + |\sigma_n(s) - \sigma(s)| \\ &= \left| \int_0^s \gamma(t) dt - \int_0^s \dot{\sigma}_n(t) dt \right| + |\sigma_n(s) - \sigma(s)|. \end{aligned}$$

Since  $\dot{\sigma}_n \rightharpoonup \gamma$  weakly and  $\sigma_n \rightarrow \sigma$  uniformly, both summands approach zero as  $n \rightarrow \infty$ . Therefore,

$$\left| x + \int_0^s \gamma(t) dt - \sigma(s) \right| = 0.$$

Hence,  $\gamma = \dot{\sigma}$ .

**Step 4:** Let  $F_M = \{t \in (0, 1) : |\dot{\sigma}(t)| > M\}$ . Let

$$A = \sup_n \left\{ \|\dot{\sigma}_n - \dot{\sigma}\|_{L^1(0,1)} \right\}.$$

By equation (2.1), together with the fact that  $\dot{\sigma} \in L^1$ , it follows that  $A < +\infty$ . Set  $M > A$ . Then, as  $\nabla_v L$  is of class  $C^1$ , it is Lipschitz on

$$\overline{B}_M \times \overline{B}_M,$$

where  $\overline{B}_M$  is the closed ball of radius  $M$  centred at the origin. Let  $\ell_M$  be the Lipschitz constant.

Then,

$$\begin{aligned}
& \int_{F_M^c} |(\nabla_v L(\sigma_n, \dot{\sigma}) - \nabla_v L(\sigma, \dot{\sigma}))(\dot{\sigma}_n - \dot{\sigma})| dt \\
& \leq \|\nabla_v L(\sigma_n, \dot{\sigma}) - \nabla_v L(\sigma, \dot{\sigma})\|_{L^\infty(F_M^c)} \|\dot{\sigma}_n - \dot{\sigma}\|_{L^1(0,1)} \\
& \leq \ell_M \|\sigma_n - \sigma\|_u A.
\end{aligned} \tag{2.3}$$

Since  $\sigma_n \rightarrow \sigma$  uniformly as  $n \rightarrow \infty$ ,

$$A \ell_M \|\sigma_n - \sigma\|_u \rightarrow 0$$

as  $n \rightarrow \infty$ . For  $t \in F_M^c$ ,

$$\begin{aligned}
L(\sigma_n, \dot{\sigma}_n) & \geq L(\sigma_n, \dot{\sigma}) + \nabla_v L(\sigma_n, \dot{\sigma})(\dot{\sigma}_n - \dot{\sigma}) \\
& = L(\sigma_n, \dot{\sigma}) + (\nabla_v L(\sigma_n, \dot{\sigma}) - \nabla_v L(\sigma, \dot{\sigma}))(\dot{\sigma}_n - \dot{\sigma}) + \nabla_v L(\sigma, \dot{\sigma})(\dot{\sigma}_n - \dot{\sigma}).
\end{aligned}$$

Then, using the estimate from equation (2.3),

$$\int_{F_M^c} L(\sigma_n, \dot{\sigma}_n) dt \geq \int_{F_M^c} L(\sigma_n, \dot{\sigma}) dt - A \ell_M \|\sigma_n - \sigma\|_u + \int_{F_M^c} \nabla_v L(\sigma, \dot{\sigma})(\dot{\sigma}_n - \dot{\sigma}) dt. \tag{2.4}$$

Without loss of generality, it may be assumed that  $L \geq 0$ . By Fatou's lemma,

$$\liminf_{n \rightarrow \infty} \int_{F_M^c} L(\sigma_n, \dot{\sigma}) dt \geq \int_{F_M^c} L(\sigma, \dot{\sigma}) dt. \tag{2.5}$$

Since  $\dot{\sigma}_n \rightarrow \dot{\sigma}$  and

$$\nabla_v L(\sigma, \dot{\sigma}) \chi_{F_M^c} \in L^\infty(0, 1),$$

where  $\chi_E$  denotes the characteristic function of the set  $E$ , it follows that for each  $M$ ,

$$\lim_{n \rightarrow \infty} \int_0^1 \nabla_v L(\sigma, \dot{\sigma}) \chi_{F_M^c}(\dot{\sigma}_n - \dot{\sigma}) dt = 0. \tag{2.6}$$

Combining the estimates from (2.4), (2.5), and (2.6), and using that  $L \geq 0$ ,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_0^1 L(\sigma_n, \dot{\sigma}_n) dt &\geq \liminf_{n \rightarrow \infty} \int_{F_M^c} L(\sigma_n, \dot{\sigma}) dt \\ &\quad + \lim_{n \rightarrow \infty} \left( -A\ell_M \|\sigma_n - \sigma\|_u + \int_{F_M^c} \nabla_v L(\sigma, \dot{\sigma})(\dot{\sigma}_n - \dot{\sigma}) dt \right) \\ &= \liminf_{n \rightarrow \infty} \int_{F_M^c} L(\sigma_n, \dot{\sigma}) dt \\ &\geq \liminf_{n \rightarrow \infty} \int_{F_M^c} L(\sigma, \dot{\sigma}) dt. \end{aligned}$$

Taking the limit as  $M \rightarrow \infty$  yields

$$\liminf_{n \rightarrow \infty} \int_0^1 L(\sigma_n, \dot{\sigma}_n) dt \geq \int_0^1 L(\sigma, \dot{\sigma}) dt.$$

Since  $(A_L(\sigma_n))_{n=1}^\infty$  is a minimizing sequence, it follows that  $\sigma$  minimizes the action of  $L$ . ■

### 2.2.1 Lagrangian Systems and Action Minimizing Curves

Let  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  be a Tonelli Lagrangian and let  $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the induced cost function. Let  $x, y \in \mathbb{R}^d$ . By theorem 2.2.6, there exists an optimal trajectory  $\sigma : [0, 1] \rightarrow \mathbb{R}^d$  with  $\sigma(0) = x$  and  $\sigma(1) = y$  which minimizes the action of  $L$ , so

$$c(x, y) = A_L(\sigma).$$

A priori, the curve  $\sigma$  is merely an element of  $W^{1,1}$ ; it is not obvious that it possesses better regularity properties than this. The main result of this subsection is that  $\sigma$  does, in fact, possess good regularity properties: curves which minimize the action of  $L$  are of class  $C^2$  and satisfy the Euler-Lagrange equation. This subsection consists mostly of results from chapters 15 and 16 of Clarke's *Functional Analysis, Calculus of Variations, and Optimal Control* [12].

First, it will be shown that optimal trajectories are Lipschitz continuous. This will be done by verifying that  $L$  satisfies a sufficient growth condition:

**Definition 2.2.8.** A Lagrangian  $L$  has **Nagumo growth** along the curve  $\phi$  if there exists a function  $\tilde{\theta} : [0, +\infty) \rightarrow \mathbb{R}$  with the properties

- (a)  $\lim_{t \rightarrow +\infty} \tilde{\theta}(t)/t = +\infty$
- (b)  $L(\phi(t), v) \geq \tilde{\theta}(|v|)$  for all  $t \in [0, 1]$  and for all  $v \in \mathbb{R}^d$ .

**Theorem 2.2.9** (Theorem 16.18 from Clarke). *Let  $\sigma : [0, 1] \times \mathbb{R}^d$  be an absolutely continuous curve which minimizes the action of  $L$ , where the Lagrangian  $L$  is continuous, autonomous (no explicit dependence on  $t$ ), convex in  $v$ , and has Nagumo growth along  $\sigma$ . Then,  $\sigma$  is Lipschitz continuous.*

Upon showing that the Tonelli Lagrangians have Nagumo growth along every optimal trajectory, it will follow that optimal trajectories are Lipschitz continuous.

**Proposition 2.2.10.** *Let  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  be a Tonelli Lagrangian. Then  $L$  has Nagumo growth along every curve.*

*Proof.* By definition 2.2.1, there exists a function  $\theta : \mathbb{R}^d \rightarrow \mathbb{R}$  and a constant  $c \in \mathbb{R}$  such that

$$L(x, v) \geq c + \theta(v) \quad \text{for all } x, v \in \mathbb{R}^d \quad \text{and} \quad \lim_{|v| \rightarrow +\infty} \frac{\theta(v)}{|v|} = +\infty.$$

Therefore, the function

$$\tilde{\theta}(t) = \inf_{|v| \geq t} \{\theta(v) + c\}$$

satisfies

$$\lim_{t \rightarrow +\infty} \frac{\tilde{\theta}(t)}{t} = +\infty,$$

and for all  $x, v \in \mathbb{R}^d$ ,  $L(x, v) \geq \tilde{\theta}(|v|)$ , so  $L$  has Nagumo growth along every curve.  $\blacksquare$

**Corollary 2.2.11.** *Let  $\sigma : [0, 1] \rightarrow \mathbb{R}^d$  be an action minimizing curve for the Tonelli Lagrangian  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$ . Then  $\sigma$  is Lipschitz continuous.*

Having established that optimal trajectories are locally Lipschitz, it will be seen that they satisfy the Euler-Lagrange equation in a weak sense. This will be “bootstrapped” to the desired regularity.

**Definition 2.2.12.** A curve  $\sigma : [0, 1] \rightarrow \mathbb{R}^d$  satisfies the **integral Euler equation** if there exists a constant  $c \in \mathbb{R}^d$  such that for almost every  $t \in [0, 1]$ ,

$$\nabla_v L(\sigma(t), \dot{\sigma}(t)) = c + \int_0^t \nabla_x L(\sigma(s), \dot{\sigma}(s)) ds. \quad (2.7)$$

**Theorem 2.2.13** (Theorem 15.2 from Clarke). *Suppose that  $\sigma$  minimizes  $A_L$  and  $\sigma$  is Lipschitz continuous. Then,  $\sigma$  satisfies the integral Euler equation.*

**Theorem 2.2.14** (Theorem 15.5 from Clarke). *Suppose that  $\sigma$  is Lipschitz continuous, satisfies the integral Euler equation, and for almost every  $t \in [0, 1]$ , the function  $v \mapsto L(\sigma(t), v)$  is strictly convex. Then,  $\sigma$  is of class  $C^1$ .*

**Theorem 2.2.15** (Theorem 15.7 from Clarke). *Suppose that  $\sigma$  is Lipschitz continuous and satisfies the integral Euler equation. Suppose the Lagrangian  $L$  is of class  $C^m$  for  $m \geq 2$  and for all  $t \in [0, 1]$  and all  $v \in \mathbb{R}^d$ , the Hessian  $\nabla_v^2 L(\sigma(t), v)$  is positive definite. Then,  $\sigma$  is of class  $C^m$ .*

Combining corollary 2.2.11 and theorems 2.2.13, 2.2.14, and 2.2.15 yields the desired regularity of optimal trajectories, which is recorded in the following proposition.

**Proposition 2.2.16.** *Let  $L$  be a Tonelli Lagrangian which is strongly convex in the second argument. Let  $\sigma$  be a minimizer for  $A_L$ . Then,  $\sigma$  is of class  $C^2$  and satisfies the Euler equation.*

*Proof.* By corollary 2.2.11, the optimal trajectory  $\sigma$  is Lipschitz continuous.

By theorem 2.2.13, the curve  $\sigma$  satisfies the integral Euler equation. Then, since  $L(x, \cdot)$  is strictly convex, by theorem 2.2.14, the curve  $\sigma$  is of class  $C^1$ . Finally, since  $\nabla_v^2 L(x, v)$  is positive definite for all  $x, v \in \mathbb{R}^d$  and  $L$  is of class  $C^2$ , by theorem 2.2.15, the curve  $\sigma$  is also of class  $C^2$ .

Since the curve  $\sigma$  satisfies the integral Euler equation (2.7) and is of class  $C^2$ , differentiating the integral Euler equation with respect to  $t$  shows that  $\sigma$  satisfies the Euler-Lagrange equation. ■

## 2.2.2 Properties of Cost Functions Induced by Lagrangians

Let  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  be a Tonelli Lagrangian which is strongly convex in velocity. By the results thus far, for every  $x, y \in \mathbb{R}^d$ , there is an action minimizing curve  $\sigma_x^y$  with  $\sigma_x^y(0) = x$  and  $\sigma_x^y(1) = y$  which is of class  $C^2$  and satisfies the Euler equation; the cost induced by  $L$  is

$$c(x, y) = A_L(\sigma_x^y).$$

The goal of this subsection is to establish regularity properties for cost functions induced by Tonelli Lagrangians: they are locally Lipschitz and superdifferentiable. Of particular interest will be the differentiability of the cost function: when the cost function  $c$  is differentiable at a point  $(x, y)$ , there will be a unique optimal trajectory from  $x$  to  $y$ . There are additional important properties of cost functions induced by Tonelli Lagrangians, but these will need to wait until the next chapter when more structure from the optimal transport problem has been introduced.

To proceed, some facts about sub- and superdifferentiability are stated.

**Definition 2.2.17.** Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ . The **subdifferential** of  $f$  at  $x$  is the set

$$\partial f(x) = \{v \in \mathbb{R}^d : f(y) - f(x) \geq v \cdot (y - x) - o(x - y) \forall y\}.$$

If the subdifferential of a function  $f$  is non-empty at a point  $x$ , the function  $f$  is said to be **subdifferentiable** at  $x$ . The **superdifferential** of a function  $f$  is minus the subdifferential of  $-f$ . If  $-f$  is subdifferentiable at  $x$ , then  $f$  is said to be **superdifferentiable** at  $x$ .

A function is subdifferentiable at a point if it locally has a supporting hyperplane at that point. The following proposition (stated without proof) records a standard fact from convex analysis about the relationship between subdifferentiability, superdifferentiability, and differentiability.

**Proposition 2.2.18.** *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ . The function  $f$  is differentiable at a point  $x$  if and only if it is both subdifferentiable and superdifferentiable at  $x$ . If  $f$  is differentiable at  $x$ , then both its sub- and superdifferentials at  $x$  are singletons and the unique element in its sub- and superdifferential is the derivative of  $f$  at  $x$ .*

With this in place, the Lipschitz regularity of cost functions induced by Tonelli Lagrangians can be proven.

**Proposition 2.2.19.** *The cost  $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  induced by the Tonelli Lagrangian  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  is locally Lipschitz.*

*Proof.* Fix  $x, y \in \mathbb{R}^d$ . Let  $\sigma$  be an optimal path from  $x$  to  $y$ . Fix  $u, v \in \mathbb{R}^d$ . For  $s > 0$ , the curve

$$[0, 1] \ni t \mapsto \sigma(t) + tsv + (1-t)su \in \mathbb{R}^d \quad (2.8)$$

is a (potentially non-optimal) curve from  $x + su$  to  $y + sv$ . Then,

$$\lim_{s \rightarrow 0} \frac{c(x + su, y + sv) - c(x, y)}{s} \quad (2.9)$$

$$\leq \lim_{s \rightarrow 0} \frac{1}{s} \int_0^1 [L(\sigma(t) + tsv + (1-t)su, \dot{\sigma}(t) + sv - su) - L(\sigma(t), \dot{\sigma}(t))] dt \quad (2.10)$$

$$= \int_0^1 \lim_{s \rightarrow 0} \frac{L(\sigma(t) + tsv + (1-t)su, \dot{\sigma}(t) + sv - su) - L(\sigma(t), \dot{\sigma}(t))}{s} dt. \quad (2.11)$$

The inequality in equation (2.10) holds because  $t \mapsto \sigma(t) + tsv + (1-t)su$  is a (potentially non-optimal) curve from  $x + su$  to  $y + sv$ . In equation (2.11) the limit may be brought inside the integral because the integrand is of class  $C^1$ . Further, the integrand in (2.11) is a derivative:

$$\begin{aligned} & \lim_{s \rightarrow 0} \frac{L(\sigma(t) + tsv + (1-t)su, \dot{\sigma}(t) + sv - su) - L(\sigma(t), \dot{\sigma}(t))}{s} \\ &= \left. \frac{d}{ds} L(\sigma(t) + tsv + (1-t)su, \dot{\sigma}(t) + sv - su) \right|_{s=0}. \end{aligned} \quad (2.12)$$

Substituting equation (2.12) into (2.11) yields

$$\begin{aligned} & \lim_{s \rightarrow 0} \frac{c(x + su, y + sv) - c(x, y)}{s} \\ & \leq \int_0^1 \left. \frac{d}{ds} L(\sigma(t) + tsv + (1-t)su, \dot{\sigma}(t) + sv - su) \right|_{s=0} dt \end{aligned} \quad (2.13)$$

$$= \int_0^1 [\nabla_x L(\sigma(t), \dot{\sigma}(t)) \cdot [tv + (1-t)u] + \nabla_v L(\sigma(t), \dot{\sigma}(t)) \cdot (v - u)] dt. \quad (2.14)$$

Since  $\sigma$  solves the Euler-Lagrange equation,  $\nabla_x L(\sigma(t), \dot{\sigma}(t)) = \frac{d}{dt} \nabla_v L(\sigma(t), \dot{\sigma}(t))$ . Substituting

this into (2.14) produces

$$\begin{aligned} & \lim_{s \rightarrow 0} \frac{c(x + su, y + sv) - c(x, y)}{s} \\ &= \int_0^1 \left[ \left( \frac{d}{dt} \nabla_v L(\sigma(t), \dot{\sigma}(t)) \right) \cdot [tv + (1-t)u] + \nabla_v L(\sigma(t), \dot{\sigma}(t)) \cdot (v - u) \right] dt \end{aligned} \quad (2.15)$$

$$= \int_0^1 \frac{d}{dt} [\nabla_v L(\sigma(t), \dot{\sigma}(t)) \cdot [tv + (1-t)u]] dt \quad (2.16)$$

$$= \nabla_v L(y, \dot{\sigma}(1)) \cdot v - \nabla_v L(x, \dot{\sigma}(0)) \cdot u. \quad (2.17)$$

By fixing  $u$  and  $v$  to have norm 1 and allowing  $x$  and  $y$  to vary over a large compact set  $B$ , it follows that for every vector  $v$ ,

$$\lim_{s \rightarrow 0} \frac{c(x + su, y + sv) - c(x, y)}{s} \leq 2 \|\nabla_v L(w, z)\|_{(w,z) \in B} =: K.$$

Then,

$$\begin{aligned} -K &\leq \lim_{s \rightarrow 0} \frac{c(x, y) - c(x + su, y + sv) -}{s} \\ &= \lim_{t \rightarrow 0} \frac{c(x, y + t(-v)) - c(x, y)}{t} \quad (t = -s) \\ &\leq K. \end{aligned}$$

Therefore,  $c$  is locally Lipschitz. ■

It follows that cost functions induced by Tonelli Lagrangians are superdifferentiable in each argument.

**Corollary 2.2.20.** *The cost  $c$  induced by the Tonelli Lagrangian  $L$  is superdifferentiable in each argument.*

*Proof.* From equation (2.17), if  $\sigma$  is an action minimizing curve from  $x$  to  $y$ , then  $\nabla_v L(y, \dot{\sigma}(1))$  is a superderivative of  $c(x, \cdot)$  at  $y$  and  $-\nabla_v L(x, \dot{\sigma}(0))$  is a superderivative for  $c(\cdot, y)$  at  $x$ . ■

Since cost functions induced by Tonelli Lagrangians are locally Lipschitz, by applying Rademacher's theorem, it immediately follows that the cost function  $c$  is differentiable everywhere.

**Corollary 2.2.21.** *The cost  $c$  induced by the Tonelli Lagrangian  $L$  is differentiable almost everywhere.*

*Proof.* By the previous proposition, the cost function is a locally Lipschitz function from  $\mathbb{R}^d \times \mathbb{R}^d$  to  $\mathbb{R}$ . By Rademacher's theorem, a locally Lipschitz function from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is differentiable almost everywhere. ■

**Corollary 2.2.22.** *If the cost function  $c$  induced by the Tonelli Lagrangian  $L$  is differentiable in the second argument, its derivative is*

$$\frac{d}{dy}c(x, y) = \nabla_v L(y, \dot{\sigma}(1)),$$

where  $\sigma$  is an optimal curve from  $x$  to  $y$ . If  $c$  is differentiable in the first argument, its derivative is

$$\frac{d}{dx}c(x, y) = -\nabla_v L(x, \dot{\sigma}(0)).$$

*Proof.* Let  $\sigma$  be an optimal curve from  $x$  to  $y$ . By corollary 2.2.20,  $\nabla_v L(y, \dot{\sigma}(1))$  is a superderivative for  $c(x, \cdot)$  at  $y$  and  $-\nabla_v L(x, \dot{\sigma}(0))$  is a superderivative for  $c(\cdot, y)$  at  $x$ . By 2.2.21,  $c$  is differentiable almost everywhere. ■

The differentiability of the cost function  $c$  induced by the Tonelli Lagrangian  $L$  leads to a uniqueness result about action minimizing curves of  $L$ .

**Proposition 2.2.23.** *For any pair  $x, y$  such that either  $\frac{d}{dx}c(x, y)$  or  $\frac{d}{dy}c(x, y)$  exists, there exists a unique action minimizing curve from  $x$  to  $y$ .*

In particular, for each  $x \in \mathbb{R}^d$ ,  $c(x, \cdot)$  is differentiable for almost every  $y \in \mathbb{R}^d$ , and for each  $y \in \mathbb{R}^d$ ,  $c(\cdot, y)$  is differentiable for almost every  $x \in \mathbb{R}^d$ .

*Proof.* Let  $(x, y)$  be a point at which  $\frac{d}{dy}c(x, y)$  exists. Suppose that  $\sigma$  and  $\gamma$  are both action minimizing curves from  $x$  to  $y$ . Then, by proposition 2.2.22,

$$\nabla_y c(x, y) = \nabla_v L(y, \dot{\sigma}(1)) = \nabla_v L(y, \dot{\gamma}(1)).$$

By lemma 2.3.5 and proposition 2.4.7, the function  $\nabla_v L(y, \cdot)$  is injective, so  $\dot{\sigma}(1) = \dot{\gamma}(1)$ . Hence, the curves  $\gamma$  and  $\sigma$  both satisfy the Euler-Lagrange equation with final condition

$$(\sigma(1), \dot{\sigma}(1)) = (\gamma(1), \dot{\gamma}(1)).$$

By uniqueness of solutions to the Lagrangian flow,  $\sigma = \gamma$ . An analogous argument holds for the case where  $\frac{d}{dx}c(x, y)$  exists. ■

## 2.3 Hamiltonians

The dual object to a Lagrangian on the tangent bundle of a manifold is a function on the cotangent bundle, called a Hamiltonian. The Hamiltonian corresponding to a Tonelli Lagrangian will be defined via the Legendre transform, and the usual properties (for instance, those seen in a standard classical mechanics course) will be derived.

Let  $E$  be a normed vector space and let  $E^*$  be its dual.



**Definition 2.3.1.** Let  $f : E \rightarrow \mathbb{R} \cup \{\infty\}$  which is not identically  $+\infty$ . The **Legendre transform** or **convex conjugate** of  $f$  is the map  $f^* : E^* \rightarrow \mathbb{R} \cup \{\infty\}$  given by

$$f^*(x) = \sup_{y \in E} \{\langle y, x \rangle - f(y)\}.$$

An important fact about Legendre transforms is given by the Fenchel-Moreau theorem, quoted from [7].

**Theorem 2.3.2** (Fenchel-Moreau). *Let  $E$  be a normed vector space and let  $f : \mathbb{R} \cup \{\infty\}$ . Suppose that  $f$  is convex, lower semi-continuous, and  $f \not\equiv +\infty$ . Then,  $f^{**}|_E = f$ .*

Let  $L : T\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a Tonelli Lagrangian. Assume that  $L$  is strongly convex in the second argument, so  $\nabla_v^2 L(x, v)$  is positive definite at every point  $(x, v) \in T\mathbb{R}^d$ .

**Definition 2.3.3.** The **Hamiltonian** corresponding to  $L$  is the function  $H : T^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  given by

$$H(q, p) := \sup_{v \in \mathbb{R}^d} \{p \cdot v - L(q, v)\}.$$

That is,  $H$  is the Legendre transform of  $L$  with respect to velocity.

The first argument in a Hamiltonian will often be referred to as **position** and the second argument will often be referred to as **momentum**. For example, if  $H$  is a Hamiltonian, then  $H(q, p)$  is the Hamiltonian evaluated at position  $q$  and momentum  $p$ . The derivatives and gradients of a Hamiltonian in the position and momentum variables will be denoted  $D_x H$ ,  $D_p H$ ,  $\nabla_x H$ , and  $\nabla_p H$ .

**Proposition 2.3.4.** *Let  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  be a Tonelli Lagrangian. Let  $H : T^*\mathbb{R}^d \rightarrow \mathbb{R}$  be the corresponding Hamiltonian. Then,*

$$L(x, v) = \sup_{p \in \mathbb{R}^d} \{v \cdot p - H(x, p)\}.$$

*Proof.* Tonelli Lagrangians are convex with respect to velocity, are continuous, and are not identically infinite. For each  $x$ , the Hamiltonian  $H(x, \cdot)$  is the Legendre transform of  $L(x, \cdot)$ . By the Fenchel-Moreau theorem, the Legendre transform of  $H$  in the second variable is therefore  $L$ . ■

Since  $L$  is of class  $C^2$ , is strictly convex, and has superlinear growth, the maximization problem  $\sup_{v \in \mathbb{R}^d} \{p \cdot v - L(q, v)\}$  can be solved with calculus:

$$H(q, p) = p \cdot r - L(q, r) \quad \text{where} \quad p = \nabla_v L(q, r). \quad (2.18)$$

An application of the implicit function theorem will show that the variable  $r$  in equation (2.18) is of class  $C^1$ , which will be the first step in deriving the classical structure of the Hamiltonian corresponding to a Lagrangian; this will require a useful lemma about the injectivity of gradients of convex functions.

**Lemma 2.3.5.** *Let  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  be function of class  $C^2$  with a positive definite Hessian. Then, the gradient  $\nabla\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is injective.*

*Proof.* Fix  $x \neq y \in \mathbb{R}^d$ . Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by  $f(t) = \varphi((1-t)x + ty)$ . The function  $f$  is strictly convex:

$$f''(t) = (x - y)^T \nabla^2 \varphi((1-t)x + ty)(x - y),$$

and since  $\nabla^2 \varphi$  is positive definite, the second derivative  $f''(t) > 0$ . Then,  $f' : \mathbb{R} \rightarrow \mathbb{R}$  is strictly increasing, so  $f'(0) < f'(1)$ , and

$$f'(0) = \nabla\varphi(x)(x - y) < \nabla\varphi(y)(x - y) = f'(1).$$

Hence,  $\nabla\varphi(x) \neq \nabla\varphi(y)$ . ■

**Proposition 2.3.6.** *The variable  $r$  in the formula for  $H(q, p)$  in equation (2.18) is a  $C^1$  function of  $q$  and  $p$ .*

*Proof.* Let  $F(q, p, r) = p - \nabla_v L(q, r)$ . The function  $F$  is of class  $C^1$ , since  $L$  is of class  $C^2$ . Then,

$$\nabla_r F(q, p, r) = -\nabla_v^2 L(q, r).$$

By assumption,  $\nabla_v^2 L(q, r)$  is positive definite. Hence, it is invertible. The hypotheses of the implicit function theorem are therefore satisfied, so  $r$  can be written locally as a  $C^1$  function of  $q$  and  $p$ :  $r = \phi(q, p)$ .

By lemma 2.3.5, for every  $x \in \mathbb{R}^d$ , the function  $\nabla_v L(x, \cdot)$  is injective. It follows that, given  $p$  and  $q$ , there is at most one vector  $r$  satisfying  $p = \nabla_v L(q, r)$ .

Taken together, it follows that  $r$  can be written as a  $C^1$  function of  $q$  and  $p$ . ■

**Proposition 2.3.7.** *The Hamiltonian  $H(q, p)$  is of class  $C^2$ .*

*Proof.* The Hamiltonian can be written as

$$H(q, p) = p \cdot r - L(q, r) \quad \text{where} \quad r = \phi(q, p)$$

and  $\phi$  is of class  $C^1$ . Then,

$$\nabla_x H(q, p) = p \cdot \nabla_x \phi(q, p) - \nabla_x L(q, \phi(q, p)) - \nabla_v L(q, \phi(q, p)) \cdot \nabla_x \phi(q, p).$$

Since  $p = \nabla_v L(q, r) = \nabla_v L(q, \phi(q, p))$ ,

$$\nabla_x H(q, p) = p \cdot \nabla_x \phi(q, p) - \nabla_x L(q, \phi(q, p)) - p \cdot \nabla_x \phi(q, p) = -\nabla_x L(q, \phi(q, p)).$$

Similarly,

$$\nabla_p H(q, p) = \phi(q, p) + p \cdot \nabla_x \phi(q, p) - \nabla_x L(q, \phi(q, p)) \cdot \nabla_x \phi(q, p) = \phi(q, p). \quad (2.19)$$

Since  $\phi$  is of class  $C^1$  and  $L$  is of class  $C^2$ , it follows that  $\nabla_x H$  and  $\nabla_p H$  are both of class  $C^1$ , so  $H$  is of class  $C^2$ . ■

**Corollary 2.3.8.** *Let  $q, r \in \mathbb{R}^d$ . Let  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  be a Tonelli Lagrangian and let  $H$  be the corresponding Hamiltonian. Then*

$$\nabla_p H(q, \nabla_v L(x, r)) = r.$$

That is,  $\nabla_p H(q, \cdot) \circ \nabla_v L(q, \cdot) = id$ .

*Proof.* From the proof of proposition 2.3.7, in equation (2.19), the variables  $p$  and  $r$  satisfy  $p = \nabla_v L(q, \phi(q, p))$  and  $r = \phi(q, p)$ . Substituting these into equation (2.19) yields  $\nabla_p H(q, \nabla_v L(x, r)) = r$ . ■

By the Fenchel-Moreau theorem, the function  $L(x, \cdot)$  is the Legendre transform of  $H(x \cdot)$ . It can be checked that  $\nabla_v L(x, \cdot)$  is also a left inverse to  $\nabla_v L(x, \cdot)$ , in addition to a right inverse. With this, the Legendre transform may be recast as a  $C^1$  diffeomorphism:

**Definition 2.3.9.** Let  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  be a Tonelli Lagrangian. The **global Legendre transform** for  $L$  is the map  $\mathcal{L} : T\mathbb{R}^d \rightarrow T^*\mathbb{R}^d$  given by

$$\mathcal{L}(x, v) = (x, \nabla_v L(x, v)).$$

**Proposition 2.3.10.** *The global Legendre transform is a  $C^1$ -diffeomorphism from  $T\mathbb{R}^d$  to  $T^*\mathbb{R}^d$ .*

*Proof.* The map  $\mathcal{L}^{-1} : T^*\mathbb{R}^d \rightarrow T\mathbb{R}^d$  given by

$$(x, p) \mapsto (x, \nabla_p H(x, p))$$

satisfies  $\mathcal{L} \circ \mathcal{L}^{-1} = id$  and  $\mathcal{L}^{-1} \circ \mathcal{L} = id$ . From corollary 2.3.8 and the discussion following corollary 2.3.8, the Hamiltonian  $H$  is of class  $C^2$ . Tonelli Lagrangians are, by definition, of class  $C^2$ . Therefore, both  $\mathcal{L}$  and  $\mathcal{L}^{-1}$  are of class  $C^1$ . ■

## 2.4 Lagrangian and Hamiltonian Systems

Let  $L : T\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a Tonelli Lagrangian which is strongly convex in the second argument. Let  $H : T^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be the corresponding Hamiltonian.

**Definition 2.4.1.** The **Lagrangian System** is the system of differential equations

$$\begin{cases} \dot{x} = v \\ \frac{d}{dt} \nabla_v L(x, v) = \nabla_x L(x, v). \end{cases} \quad (2.20)$$

The differential equation  $\frac{d}{dt} \nabla_v L(x, \dot{x}) = \nabla_x L(x, \dot{x})$  is called the **Euler-Lagrange Equation**.

**Definition 2.4.2.** The **Lagrangian flow from time  $s$  to time  $t$**  is the map  $\Phi_{s,t} : T\mathbb{R}^d \rightarrow T\mathbb{R}^d$  given by

$$\Phi_{s,t}(x, v) = (\sigma(t), \dot{\sigma}(t)),$$

where  $\sigma$  solves the Lagrangian system with time  $s$  initial conditions

$$(\sigma(s), \dot{\sigma}(s)) = (x, v).$$

**Definition 2.4.3.** The **Hamiltonian System** is the system of differential equations

$$\begin{cases} \dot{q} = \nabla_p H(q, p) \\ \dot{p} = -\nabla_x H(q, p) \end{cases}$$

**Definition 2.4.4.** The **Hamiltonian flow from time  $s$  to time  $t$**  is the map  $\Psi_{s,t} : T^*\mathbb{R}^d \rightarrow T^*\mathbb{R}^d$  given by

$$\Psi_{s,t}(x, r) = (q(t), p(t)),$$

where  $(q, p)$  solves the Hamiltonian system with time  $s$  initial conditions

$$(q(s), p(s)) = (x, r).$$

The main goal of this section is to show that these systems are equivalent in a precise sense and they both have solutions of class  $C^2$ .

### 2.4.1 Hamiltonian Systems Have Unique Local Solutions of Class $C^2$

The Hamiltonian system

$$\begin{cases} \dot{q} = \nabla_p H(q, p) \\ \dot{p} = -\nabla_x H(q, p) \\ (q, p)(0) = (q_0, p_0) \end{cases}$$

is a system of first order ordinary differential equations. Write  $y = \begin{pmatrix} q \\ p \end{pmatrix}$  and  $F(y) = \begin{pmatrix} \nabla_p H(q,p) \\ -\nabla_x H(q,p) \end{pmatrix}$ . Since  $H$  is  $C^2$ , the first order ordinary differential equation  $\dot{y} = F(y)$  has  $F$  locally Lipschitz. Therefore, this system satisfies the hypotheses of the Picard-Lindelöf theorem, so there exists a unique local solution  $(q, p)$ , which is of class  $C^1$  (that the functions are of class  $C^1$  is given by the theorem). It immediately follows that the solution  $(q, p)$  is in fact of class  $C^2$ , since

$$\dot{q} = \nabla_p H(q, p) \quad \text{and} \quad \dot{p} = -\nabla_x H(q, p)$$

are both of class  $C^1$ .

## 2.4.2 Equivalence of Hamiltonian and Lagrangian Systems

A standard fact from convex analysis is stated.

**Proposition 2.4.5.** *Suppose that  $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$  is a convex function on a normed vector space  $X$ . Let  $\varphi^* : X^* \rightarrow \mathbb{R} \cup \{+\infty\}$  denote its Legendre-Fenchel transform. Then,*

$$\varphi(a) + \varphi^*(b) = \langle a, b \rangle \iff b \in \partial\varphi(a) \iff a \in \partial\varphi^*(b),$$

where  $\partial f(x)$  denotes the subdifferential of  $f$  at  $x$ .

**Proposition 2.4.6.** *Suppose that  $(q, p)$  is a  $C^2$  solution to the Hamiltonian system. Then,  $q$  is a  $C^2$  solution to the Lagrangian system (with compatible initial conditions).*

*Proof.* Suppose that  $(q, p)$  is a  $C^2$  solution to the Hamiltonian system. (Solutions are necessarily of class  $C^2$  by the previous subsection). Then, by the results of the section on Hamiltonians,

$$\begin{aligned} H(q, p) &= p \cdot \nabla_p H(q, p) - L(q, \nabla_p H(q, p)) \\ &= p \cdot \dot{q} - L(q, \dot{q}). \end{aligned}$$

The second equality follows by the assumption that  $(q, p)$  is a solution to the Hamiltonian system, so  $\dot{q} = \nabla_p H(q, p)$ . Then,

$$H(q, p) + L(q, \dot{q}) = p \cdot \dot{q}.$$

By proposition 2.4.5, it follows that

$$p = \nabla_v L(q, \dot{q}) \implies \dot{p} = \frac{d}{dt} \nabla_v L(q, \dot{q}).$$

And, from the computation of the gradients of  $H$  in the proof of proposition 2.3.7,

$$\nabla_x H(q, p) = -\nabla_x L(q, \dot{q}).$$

Then, since  $(q, p)$  is a solution to the Hamiltonian system,

$$\dot{p} = -\nabla_x H(q, p) = \nabla_x L(q, \dot{q}).$$

Therefore,  $\nabla_x L(q, \dot{q}) = \frac{d}{dt} \nabla_v L(q, \dot{q})$ , so  $q$  is a solution to the Lagrangian system. ■

**Proposition 2.4.7.** *Suppose  $q$  is of class  $C^2$  and is a solution to the Lagrangian system. Then,  $(q, p) = (q, \nabla_v L(q, \dot{q}))$  is a solution to the Hamiltonian system with corresponding initial conditions.*

*Proof.* Suppose that  $q$  is of class  $C^2$  and satisfies  $\nabla_x L(q, \dot{q}) = \frac{d}{dt} \nabla_v L(q, \dot{q})$ . Since  $q$  is of class  $C^2$ , it ensures that the right hand side is defined, as

$$\frac{d}{dt} \nabla_v L(q, \dot{q}) = \nabla_x \nabla_v L(q, \dot{q}) \cdot \dot{q} + \nabla_v^2 L(q, \dot{q}) \cdot \ddot{q}.$$

The Hamiltonian  $H$  satisfies

$$H(q, p) = p \cdot r - L(q, r) \quad \text{where } p = \nabla_v L(q, r) \text{ and } r = \nabla_p H(q, p).$$

Set  $p = \nabla_v L(q, \dot{q})$ . We wish to show that  $(q, \nabla_v L(q, \dot{q}))$  is a solution to the Hamiltonian system. Then,

$$\nabla_v L(q, r) = \nabla_v L(q, \dot{q}).$$

Since  $L$  is of class  $C^2$  and  $\nabla_v^2 L$  is positive definite, by lemma 2.3.5,  $\nabla_v L(q, \cdot)$  is injective, and it follows that  $\dot{q} = r$ . From proposition 2.4.6 and the proof of proposition 2.3.7,

$$\nabla_x H(q, p) = -\nabla_x L(q, r = \dot{q}).$$

Therefore,

$$\begin{aligned} \dot{p} &= \frac{d}{dt} \nabla_v L(q, \dot{q}) && \text{(Set } p = \nabla_v L(q, \dot{q})) \\ &= \nabla_x L(q, \dot{q}) && \text{(} q \text{ solves the Lagrangian system)} \\ &= -\nabla_x H(q, p). \end{aligned}$$

And,  $\dot{q} = r = \nabla_p H(q, p)$ . Therefore,  $(q, \nabla_v L(q, \dot{q}))$  is a solution to the Hamiltonian system. ■

### 2.4.3 Conservation of Hamiltonians

**Proposition 2.4.8.** *Let  $(q, p)$  be a solution to the Hamiltonian system. Then, for all  $t$  where the solution is defined,*

$$H(q(t), p(t)) = H(q(0), p(0)).$$

*Proof.* For all time  $t$  where the solution  $(q, p)$  is defined,

$$\begin{aligned} \frac{d}{dt}H(q, p) &= \nabla_x H(q, p) \cdot \dot{q} + \nabla_p H(q, p) \cdot \dot{p} \\ &= \nabla_x H(q, p) \cdot \nabla_p H(q, p) + \nabla_p H(q, p) \cdot (-\nabla_x H(q, p)) \\ &= 0. \end{aligned}$$

Therefore,  $H(q(t), p(t))$  is constant.  $\blacksquare$

Assume that there exists a constant  $c_1$  and a function  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ , where  $\lim_{|p| \rightarrow +\infty} \psi(p) = +\infty$ , and

$$H(q, p) \geq c_1 + \psi(p).$$

If  $(q, p)$  is a solution to the Hamiltonian system, then  $H$  is constant along the solution, so, for all time  $t$  where a solution is defined,  $p(t)$  is bounded. Otherwise, the above assumption would lead to a contradiction.

#### 2.4.4 Dependence on Initial Conditions

**Proposition 2.4.9.** *The Lagrangian flow  $\Phi_{s,t} : T\mathbb{R}^d \rightarrow T\mathbb{R}^d$  is of class  $C^1$ .*

This does not follow from the standard ODE theory; Tonelli Lagrangians are only of class  $C^2$ , so the left hand side of the Euler-Lagrange equation is, a priori, only continuous.

*Proof.* By standard results from the theory of ordinary differential equations, (see, for instance, theorem 17.9 in [29]), the Hamiltonian flow  $\Psi_{s,t}$  is of class  $C^1$ . Since the Lagrangian flow satisfies

$$\Phi_{s,t} = \mathcal{L}^{-1} \circ \Psi_{s,t} \circ \mathcal{L} : T\mathbb{R}^d \rightarrow T\mathbb{R}^d,$$

where  $\mathcal{L}$  is the global Legendre transform, which is a  $C^1$  diffeomorphism by proposition 2.3.10, it follows that the Lagrangian flow is of class  $C^1$ .  $\blacksquare$

A computation of the spatial derivative of the Hamiltonian flow will later prove useful.

**Proposition 2.4.10.** *Let  $\pi_q : T^*\mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\pi_p : T^*\mathbb{R}^d \rightarrow \mathbb{R}^d$  be projections given by  $\pi_q(q, p) = q$  and  $\pi_p(q, p) = p$ . Then,*

$$\pi_q D_2 \Psi_{s,t}(q, p) = (t - s) \nabla_p^2 H(q, p) + o(t - s).$$

*Proof.* The Hamiltonian flow  $\Psi_{s,t}$  solves the differential equation

$$\dot{\Psi}_{s,t}(q, p) = \begin{pmatrix} \nabla_p H(\Psi_{s,t}(q, p)) \\ -\nabla_x H(\Psi_{s,t}(q, p)) \end{pmatrix}.$$

By standard results from ODE theory (see, for instance, [26]), the derivative  $D\Psi_{s,t}$  satisfies the differential equation

$$\dot{z} = \begin{pmatrix} \nabla_p \nabla_x H(\Psi_{s,t}(q,p)) & \nabla_p^2 H(\Psi_{s,t}(q,p)) \\ -\nabla_x^2 H(\Psi_{s,t}(q,p)) & -\nabla_p \nabla_x H(\Psi_{s,t}(q,p)) \end{pmatrix} z, \quad (2.21)$$

where  $z$  is a  $2d \times 2d$  matrix. Let  $(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d$ . Then, the directional derivative of  $\Psi_{s,t}$  in the direction  $(\xi, \eta)$  is given by

$$D\Psi_{s,t} \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

and these directional derivatives continue to solve the linearized differential equation (2.21).

Note that  $\Psi_{s,s} = id$ , so  $D\Psi_{s,s} = I$ . Substituting  $z = D\Psi_{s,s}$  into (2.21) yields

$$D\dot{\Psi}_{s,s} = \begin{pmatrix} \nabla_p \nabla_x H(\Psi_{s,t}) & \nabla_p^2 H(\Psi_{s,t}) \\ -\nabla_x^2 H(\Psi_{s,t}) & -\nabla_p \nabla_x H(\Psi_{s,t}) \end{pmatrix} D\Psi_{s,s} \quad (2.22)$$

$$= \begin{pmatrix} \nabla_p \nabla_x H(\Psi_{s,t}) & \nabla_p^2 H(\Psi_{s,t}) \\ -\nabla_x^2 H(\Psi_{s,t}) & -\nabla_p \nabla_x H(\Psi_{s,t}) \end{pmatrix}. \quad (2.23)$$

Taylor expanding  $D\Psi_{s,t}$  in  $t$  around  $t = s$  yields,

$$D\Psi_{s,t} = D\Psi_{s,s} + (t-s)D\dot{\Psi}_{s,s} + o(t-s). \quad (2.24)$$

Substituting equation (2.23) into equation (2.24):

$$D\Psi_{s,t} = D\Psi_{s,s} + (t-s) \begin{pmatrix} \nabla_p \nabla_x H(\Psi_{s,t}) & \nabla_p^2 H(\Psi_{s,t}) \\ -\nabla_x^2 H(\Psi_{s,t}) & -\nabla_p \nabla_x H(\Psi_{s,t}) \end{pmatrix} + o(t-s). \quad (2.25)$$

The above equation holds true when multiplied by any vector in  $\mathbb{R}^d \times \mathbb{R}^d$ . Let  $\pi_x(x,p) = x$  and  $\pi_p(x,p) = p$ . Multiplying equation (2.25) by a vector  $(0, \eta) \in \mathbb{R}^d \times \mathbb{R}^d$ , remembering that  $D\Psi_{s,s} = I$ , yields

$$\begin{pmatrix} \pi_q D_q \Psi_{s,t} & \pi_q D_p \Psi_{s,t} \\ \pi_p D_q \Psi_{s,t} & \pi_p D_p \Psi_{s,t} \end{pmatrix} \begin{pmatrix} 0 \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ \eta \end{pmatrix} + (t-s) \begin{pmatrix} \nabla_p^2 H(\Psi_{s,t}) \cdot \eta \\ -\nabla_p \nabla_x H(\Psi_{s,t}) \cdot \eta \end{pmatrix} + o(t-s).$$

Restricting to the first  $d$  coordinates, it follows that

$$\pi_q D_p \Psi_{s,t}(p, q) = (t-s) \nabla_p^2 H(\Psi_{s,t}(p, q)) + o(t-s). \quad \blacksquare$$



## Chapter 3

# Five Formulations of the Optimal Transport Problem

### 3.1 Introduction

In this chapter, the optimal transport problem is introduced. First, the Monge, Kantorovich, and Kantorovich dual problems are presented, together with criteria for when their infima are equal. These results are standard and are cited from various sources; some simple results are proven to give a feeling for the arguments involved in proving the equality of the infima of these three costs. In this initial section, many of the results are cited from [45], [37], and [46].

After recording these results, the Lagrangian formulation of the optimal transport problem is stated. A careful analysis of the relationships between the different formulations of the optimal transport problem relates the optimal trajectories in the Lagrangian problem to both the Monge problem and the Kantorovich dual problem. Following from the results of the previous chapter, optimal trajectories in the Lagrangian problem will be seen to be solutions of a boundary value problem with boundary conditions given by the solution to the Monge problem. Complementary to this, but more subtle, optimal trajectories in the Lagrangian problem will also be seen to be solutions of an initial value problem with initial conditions given by the Kantorovich dual problem.

A careful analysis of the relationships between the optimal trajectories and the corresponding potential functions from the Kantorovich dual problem establishes the best possible regularity results for optimal trajectories. The key tool in this analysis is Mather's shortening lemma, which gives a quantitative estimate on the maximum distance between optimal trajectories in terms of the distance between optimal trajectories at their midpoints.

Once the regularity of the optimal trajectories in the Lagrangian formulation of the problem is established, the Eulerian formulation of the optimal transport problem is introduced. The regularity of the optimizers in the Eulerian problem will follow directly from the regularity of the optimal trajectories from the previous section.

The chapter concludes with a proof of the equivalence of the five forms of the optimal transport problem. In particular, the relationships between the minimizers is made explicit. These results are known to experts ([5],[19]), but I believe this is the first time these relationships have been presented this simply.

### 3.2 Some General Optimal Transport Theory

Background on the optimal transport problem is presented in this section. The Monge, Kantorovich, and Kantorovich dual optimal transport problems are defined, together with criteria for their infima to agree. This section mostly follows McCann-Guillen's lecture notes [37], Villani's *Topics in Optimal Transportation* [45], and Fathi-Figalli [19].

Let  $\mathcal{P}(\mathbb{R}^d)$  denote the space of Borel probability measures on  $\mathbb{R}^d$ . Let  $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$  and let  $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, +\infty]$ . The function  $c$  is called a **cost function**.

**Definition 3.2.1.** A Borel measurable map  $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is said to **push forward  $\mu$  to  $\nu$** , denoted  $T_{\#}\mu = \nu$ , if for all Borel subsets  $B \subset \mathbb{R}^d$ ,

$$\mu(T^{-1}(B)) = \nu(B).$$

**Definition 3.2.2.** The **Monge formulation of the optimal transport problem from  $\mu$  to  $\nu$  for the cost  $c$**  is the minimization problem

$$\min_T \{\text{cost}(T)\} := \min_T \left\{ \int_{\mathbb{R}^d} c(x, T(x)) d\mu(x) : T_{\#}\mu = \nu \right\}.$$

**Definition 3.2.3.** A Borel measurable map  $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$  which minimizes  $\text{cost}(T)$  and pushes forward  $\mu$  to  $\nu$  is called an **optimal transport map from  $\mu$  to  $\nu$  for the cost  $c$**  or a **Monge map from  $\mu$  to  $\nu$  for the cost  $c$** .

**Definition 3.2.4.** Let  $\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ . The measure  $\gamma$  has **first marginal  $\mu$**  and **second marginal  $\nu$**  if for all Borel subsets  $B \subset \mathbb{R}^d$ ,

$$\gamma(B \times \mathbb{R}^d) = \mu(B) \quad \text{and} \quad \gamma(\mathbb{R}^d \times B) = \nu(B).$$

The set of Borel probability measures on  $\mathbb{R}^d \times \mathbb{R}^d$  with first marginal  $\mu$  and second marginal  $\nu$  is denoted  $\Gamma(\mu, \nu)$ . If  $\mu$  and  $\nu$  are absolutely continuous with respect to Lebesgue measure and are represented by densities  $\rho_0$  and  $\rho_1$ , respectively, then this set is denoted  $\Gamma(\rho_0, \rho_1)$ .

**Definition 3.2.5.** The **Kantorovich formulation of the optimal transport problem from  $\mu$  to  $\nu$  for the cost  $c$**  is the minimization problem

$$\min_{\gamma} \{\text{cost}(\gamma)\} := \min_{\gamma} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\gamma(x, y) : \gamma \in \Gamma(\mu, \nu) \right\}.$$

**Definition 3.2.6.** A measure  $\gamma \in \Gamma(\mu, \nu)$  which minimizes  $\text{cost}(\gamma)$  is called an **optimal transport plan from  $\mu$  to  $\nu$  for the cost  $c$** .

**Definition 3.2.7.** The (**Kantorovich**) **cost from  $\mu$  to  $\nu$  for the cost  $c$**  is

$$W_c(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\gamma(x, y) \right\}.$$

**Definition 3.2.8.** The **Monge cost from  $\mu$  to  $\nu$  for the cost  $c$**  is

$$W_c^M(\mu, \nu) = \inf_T \left\{ \int_{\mathbb{R}^d} c(x, T(x)) d\mu(x) : T_{\#}\mu = \nu \right\}.$$

Unless otherwise specified, the cost from  $\mu$  to  $\nu$  for the cost  $c$  refers to the Kantorovich cost.

**Proposition 3.2.9.** *Let  $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$  and  $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, +\infty]$ . Then, the Kantorovich cost from  $\mu$  to  $\nu$  for the cost  $c$  is no larger than the Monge cost. That is,*

$$W_c(\mu, \nu) \leq W_c^M(\mu, \nu).$$

*Proof.* Let  $T$  be a Borel map pushing forward  $\mu$  to  $\nu$ . The corresponding measure  $\gamma_T = (\text{id} \times T)_{\#}\mu$  has the following properties:

- (a)  $\gamma_T \in \Gamma(\mu, \nu)$ .
- (b)  $\text{cost}(\gamma_T) = \text{cost}(T)$ .

For (a), let  $B \subset \mathbb{R}^d$  be a Borel subset. Then,

$$\begin{aligned} \gamma_T(B \times \mathbb{R}^d) &= \mu((\text{id} \times T)^{-1}(B \times \mathbb{R}^d)) \\ &= \mu(\text{id}^{-1}(B) \cap T^{-1}(\mathbb{R}^d)) \\ &= \mu(B), \end{aligned}$$

and,

$$\begin{aligned} \gamma_T(\mathbb{R}^d \times B) &= \mu((\text{id} \times T)^{-1}(\mathbb{R}^d \times B)) \\ &= \mu(\text{id}^{-1}(\mathbb{R}^d) \cap T^{-1}(B)) \\ &= \mu(T^{-1}(B)) \\ &= \nu(B), \end{aligned}$$

where the last equality holds since  $T$  is assumed to push forward  $\mu$  to  $\nu$ .

For (b), note that for Borel subset  $A, B \subset \mathbb{R}^d$ ,

$$\begin{aligned}\gamma_T(A \times B) &= \mu((\text{id} \times T)^{-1}(A \times B)) \\ &= \mu(A \cap T^{-1}(B)) \\ &= \mu(\{x \in A \mid T(x) \in B\}).\end{aligned}$$

Therefore,

$$\begin{aligned}\text{cost}(\gamma_T) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\gamma_T(x, y) \\ &= \int_{\{(x, y) \mid y=T(x)\}} c(x, y) d\gamma_T(x, y) + \int_{\{(x, y) \mid y=T(x)\}^c} c(x, y) d\gamma_T(x, y).\end{aligned}$$

Since  $\{(x, y) \mid y = T(x)\}^c$  is  $\gamma_T$ -measure 0,

$$\begin{aligned}\text{cost}(\gamma_T) &= \int_{\{(x, y) \mid y=T(x)\}} c(x, y) d\gamma_T(x, y) \\ &= \int_{\mathbb{R}^d} c(x, T(x)) d\mu(x) \\ &= \text{cost}(T).\end{aligned}$$

Finally, let  $(T_n)_{n=1}^\infty$  be a minimizing sequence in the Monge problem. Then,

$$W_c(\mu, \nu) \leq \lim_{n \rightarrow \infty} \text{cost}(\gamma_{T_n}) = \lim_{n \rightarrow \infty} \text{cost}(T_n) = W_c^M(\mu, \nu),$$

concluding the proof.  $\blacksquare$

### 3.2.1 Equivalence of the Monge and Kantorovich Problems

Under appropriate hypotheses, the Monge and Kantorovich costs coincide.

**Definition 3.2.10.** A cost function  $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, +\infty)$  is said to be **left-twisted** or to satisfy the **left twist condition** if the map

$$(x, y) \mapsto (x, \nabla_x c(x, y))$$

defined on the domain  $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : \nabla_x c(x, y) \text{ exists}\}$  is injective.

When there is no ambiguity, the left twist condition will be referred to as the twist condition. The right twist condition refers to the injectivity of the map

$$(x, y) \mapsto (\nabla_y c(x, y), y).$$

If a cost function  $c$  satisfies the twist condition, then given  $(x, p) \in T^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d$ , there is at most one  $y \in \mathbb{R}^d$  satisfying  $\nabla_x c(x, y) + p = 0$ . If such a  $y$  exists, it is denoted  $Y(x, p)$ .

The cost functions considered in the first chapter, those induced by Tonelli Lagrangians, satisfy the twist condition, as recorded in the following theorem from Fathi-Figalli [19].

**Proposition 3.2.11** (Proposition 4.1 from Fathi-Figalli). *Let  $L$  be a Tonelli Lagrangian which is strongly convex in velocity. Then, the cost function induced by  $L$  is continuous, bounded below, satisfies the twist condition, and the family of functions  $x \mapsto c_y(x) = c(x, y)$  is locally semi-concave in  $x$  locally uniformly in  $y$ .*

The twist condition plays an important role in ensuring that optimizers to the Monge optimal transport problem exist and that the Monge and Kantorovich costs coincide. The following theorem from McCann-Guillen [37] states a sufficient condition for the Monge and Kantorovich costs and minimizers to coincide in the classical setting, which they cite this theorem from Gangbo's habilitation thesis.

**Theorem 3.2.12** (Theorem 2.9 from McCann-Guillen). *Let  $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ . Suppose that  $\mu$  is absolutely continuous with respect to Lebesgue measure. Let  $c \in C(\mathbb{R}^d \times \mathbb{R}^d)$  be differentiable in the first  $d$  coordinates and satisfy the twist condition. Suppose that  $\nabla_x c(x, y)$  is locally bounded in  $x$  and globally bounded in  $y$ . Then, there exists a Lipschitz function  $u : \mathbb{R}^d \rightarrow \mathbb{R}$  such that*

- (a) *The map  $G(x) = Y(x, Du(x))$  pushes forward  $\mu$  to  $\nu$ . (The map  $Du(x)$  is defined almost everywhere.)*
- (b) *The map  $G$  is the unique minimizer to Monge's formulation of the optimal transport problem.*
- (c) *The measure  $\gamma_G = (id \times G)_\# \mu$  is the unique minimizer to Kantorovich's formulation of the optimal transport problem.*

In the proof that  $W_c(\mu, \nu) \leq W_c^M(\mu, \nu)$ , it was shown that  $\text{cost}(G) = \text{cost}(\gamma_G)$ . The map  $Y$  is defined in the discussion following definition 3.2.10.

A more delicate sufficient condition is given in Fathi-Figalli [19].

**Theorem 3.2.13** (Theorem 3.2 from Fathi-Figalli). *Let  $\mu$  and  $\nu$  be probability measures on  $\mathbb{R}^d$ . Suppose that the cost function  $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is lower semicontinuous and bounded below. Suppose the following conditions hold:*

- (a) *The family of maps  $x \mapsto c(x, y) = c_y(x)$  is locally semi-concave in  $x$  locally uniformly in  $y$ ,*
- (b) *The cost  $c$  satisfies the twist condition,*
- (c) *The measure  $\mu$  is absolutely continuous with respect to Lebesgue measure,*
- (d) *The cost from  $\mu$  to  $\nu$  for the cost  $c$  is finite.*

Then, there exists a Borel optimal transport map  $T$  from  $\mu$  to  $\nu$  for the cost  $c$ , this map is unique  $\mu$ -a.e., and any optimal transport plan  $\gamma \in \Gamma(\mu, \nu)$  from  $\mu$  to  $\nu$  for the cost  $c$  is concentrated on the graph of  $T$ .

The definition for a function being locally semi-concave in  $x$  locally uniformly in  $y$  is given in appendix A of Fathi-Figalli (in particular, see definitions A.1, A.3, A.10, A.13, and A.15). Fathi-Figalli actually use a slightly more general hypothesis than (c), requiring only that the measure  $\mu$  does not give mass to “small sets”. Finally, because any optimal transport plan has its cost concentrated on the graph of the optimal transport map, it follows that the Monge cost and Kantorovich cost coincide.

### 3.2.2 Kantorovich Duality

Let  $\mu$  and  $\nu$  be absolutely continuous probability measures on  $\mathbb{R}^d$ . Let  $c$  be a cost function. There is a dual optimization problem to the Kantorovich optimal transport problem from  $\mu$  to  $\nu$  for the cost  $c$ .

**Definition 3.2.14.** The **Kantorovich dual formulation of the optimal transport problem from  $\mu$  to  $\nu$  for the cost  $c$**  (or the **Kantorovich dual problem**) is the optimization problem

$$\max_{u,v} \{\text{cost}(u, v)\} := \max_{u,v} \left\{ \int_{\mathbb{R}^d} v(y) d\nu(y) - \int_{\mathbb{R}^d} u(x) d\mu(x) : \right. \\ \left. v(y) - u(x) \leq c(x, y) \text{ for all } x, y \in \mathbb{R}^d \right\}. \quad (3.1)$$

The supremum in this optimization problem is called the **Kantorovich dual cost from  $\mu$  to  $\nu$  for the cost  $c$** .

Different authors have different conventions for the signs in front of the functions  $u$  and  $v$  in the Kantorovich dual problem. I am following the convention of Villani’s 2008 book [46]. (This disagrees with the convention in Villani’s 2003 book [45].)

**Definition 3.2.15.** A pair of functions  $u$  and  $v$  which are optimizers in the Kantorovich dual problem from  $\mu$  to  $\nu$  for the cost  $c$  are called **Kantorovich potential functions** or **Kantorovich potentials**.

The Kantorovich dual cost and the Kantorovich cost agree under weak hypotheses, given in the following theorem from Villani [45].

**Theorem 3.2.16** (Theorem 1.3 from [45]). *Let  $X$  and  $Y$  be Polish spaces (complete separable metric spaces). Let  $\mu \in \mathcal{P}(X)$  and  $\nu \in \mathcal{P}(Y)$ . Let  $c : X \times Y \rightarrow [0, +\infty]$  be a lower semicontinuous cost function. Then,*

$$\inf_{\gamma \in \Gamma(\mu, \nu)} \{\text{cost}(\gamma)\} = \sup_{\substack{(u,v) \in L^1(\mu) \times L^1(\nu) \\ v(y) - u(x) \leq c(x,y)}} \{\text{cost}(u, v)\} = \sup_{\substack{(u,v) \in C_b(X) \times C_b(Y) \\ v(y) - u(x) \leq c(x,y)}} \{\text{cost}(u, v)\},$$

where  $C_b(X)$  and  $C_b(Y)$  denote the spaces of bounded continuous functions on  $X$  and  $Y$ , respectively. Further, the infimum on the left hand side is achieved by a measure in  $\Gamma(\mu, \nu)$ .

More precise information about the structure of optimizers in the Kantorovich optimal transport problem and Kantorovich dual problem are given by Schachermayer-Teichmann [43] and Villani [46].

**Theorem 3.2.17** (Part of theorem 5.10 from [46]). *Let  $\mu$  and  $\nu$  be absolutely continuous probability measures on  $\mathbb{R}^d$  and let  $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, +\infty]$  be a lower semicontinuous cost function. Then, there exists an optimizer  $\bar{\gamma} \in \Gamma(\mu, \nu)$  for the Kantorovich optimal transport problem:*

$$\text{cost}(\bar{\gamma}) = \inf_{\gamma \in \Gamma(\mu, \nu)} \{\text{cost}(\gamma)\},$$

and there exists a pair of Kantorovich potentials  $u : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $v : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty\}$  satisfying

$$v(y) - u(x) \leq c(x, y) \text{ for all } x, y \in \mathbb{R}^d$$

with equality holding  $\bar{\gamma}$ -a.e. These functions are optimizers in the Kantorovich dual problem and satisfy

$$v(y) = \inf_{x \in \mathbb{R}^d} \{c(x, y) + u(x)\} \quad \text{and} \quad u(x) = \sup_{y \in \mathbb{R}^d} \{v(y) - c(x, y)\}.$$

If the functions  $u$ ,  $v$ , and  $c$  are all differentiable, then their derivatives are related to each other, as given in the following proposition.

**Proposition 3.2.18.** *Suppose that the cost function  $c$  as well as the Kantorovich potentials  $u$  and  $v$  are differentiable. Let  $\gamma$  be an optimal transport plan for the corresponding Kantorovich optimal transport problem. Then, at each point in the support of  $\gamma$ ,*

$$\frac{\partial c}{\partial x} = -\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial c}{\partial y} = \frac{\partial v}{\partial y}$$

*Proof.* Since  $c$ ,  $u$ , and  $v$  are differentiable, the function  $c(x, y) + u(x) - v(y)$  is differentiable, so in particular is differentiable on  $\text{supp}(\gamma)$ . By construction,  $c(x, y) + u(x) - v(y) \geq 0$  and, by proposition 3.2.17,  $c(x, y) + u(x) - v(y) = 0$  almost everywhere on  $\text{supp}(\gamma)$ . Therefore, the function  $c(x, y) + u(x) - v(y)$  has a minimum at almost every point of  $\text{supp}(\gamma)$ . Since a differentiable function has derivative 0 at its local extrema, the proposition holds. ■

### 3.3 The Lagrangian Optimal Transport Problem

Let  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  be a Tonelli Lagrangian which is strongly convex in velocity. In the previous chapter, cost functions were defined in terms of the action of Tonelli Lagrangians (definition 2.2.5) via

$$c(x, y) = \inf \left\{ \int_0^1 L(\sigma(t), \dot{\sigma}(t)) dt : \sigma \in W^{1,1}, \sigma(0) = x, \sigma(1) = y \right\}. \quad (3.2)$$

In theorem 2.2.6, it was shown that for any  $x, y \in \mathbb{R}^d$ , there is an optimal trajectory  $\sigma : [0, 1] \rightarrow \mathbb{R}^d$  with endpoints  $x$  and  $y$  which yields the cost.

In subsection 2.2.1, it was shown that optimal trajectories are of class  $C^2$  and satisfy the Euler-Lagrange equation. Furthermore, it was seen that the cost function  $c$  is locally Lipschitz (proposition 2.2.19), superdifferentiable (corollary 2.2.20), and differentiable almost everywhere (corollary 2.2.21); the derivatives of  $c$  were computed explicitly (corollary 2.2.22). Finally, proposition 2.2.23 showed that if either  $\partial_x c(x, y)$  or  $\partial_y c(x, y)$  exist, then there exists a unique optimal trajectory from  $x$  to  $y$ . This motivates structuring the optimal transport problem explicitly in terms of a Tonelli Lagrangian.

**Definition 3.3.1.** Let  $\rho_0$  and  $\rho_1$  be probability densities on  $\mathbb{R}^d$ . The **Lagrangian optimal transport problem from  $\rho_0$  to  $\rho_1$  for the cost  $c$  induced by the Tonelli Lagrangian  $L$**  is the minimization problem

$$\inf_{\sigma : \mathbb{R}^d \times [0,1] \rightarrow \mathbb{R}} \left\{ \int_0^1 \int_{\mathbb{R}^d} L(\sigma(t, x), \dot{\sigma}(t, x)) \rho_0(x) dx dt : \sigma(1, \cdot) \# \rho_0 = \rho_1 \right\},$$

where  $\sigma(\cdot, x) \in W^{1,1}$  for every  $x$  and  $\sigma(0, \cdot) = id$ .

With the results established in the first chapter, the admissible families of trajectories in the Lagrangian problem may be assumed to be of class  $C^2$  with respect to time.

Let  $\rho_0$  and  $\rho_1$  be probability densities on  $\mathbb{R}^d$ . By proposition 3.2.11 and theorem 3.2.13, the Monge optimal transport problem from  $\rho_0$  to  $\rho_1$  for the cost  $c$  induced by the Tonelli Lagrangian  $L$  has a solution  $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$  which pushes forward  $\rho_0$  to  $\rho_1$ . In this case, the solution to the Monge optimal transport problem satisfies

$$\int_{\mathbb{R}^d} c(x, T(x)) \rho_0(x) dx = \int_{\mathbb{R}^d} \int_0^1 L(\sigma(t, x), \dot{\sigma}(t, x)) dt \rho_0(x) dx \quad (3.3)$$

where  $\sigma(\cdot, x)$  is an optimal trajectory from  $x$  to  $T(x)$ .

More substantial relationships between the minimizers of the Monge, Kantorovich dual, and Lagrangian optimal transport problems will be established in the proceeding subsections.



### 3.3.1 Further Properties of Cost Functions Induced by Lagrangians

Let  $\mu$  and  $\nu$  be absolutely continuous probability measures on  $\mathbb{R}^d$ . Let  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  be a Tonelli Lagrangian which is strongly convex in velocity and let  $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, +\infty)$  be the induced cost function. Let  $T$  be the Monge map for the Monge optimal transport problem from  $\mu$  to  $\nu$  for the cost  $c$ . Let  $u$  and  $v$  be Kantorovich potentials for the corresponding Kantorovich dual problem.

This subsection will establish that for almost every  $x$ , there is a unique action minimizing curve from  $x$  to  $T(x)$ . This does not follow directly from proposition 2.2.23; it requires a more subtle argument. The first step will be to show that the Kantorovich potential  $u$  is differentiable almost everywhere.

The following theorem from Figalli-Gigli [23] establishes the differentiability of the Kantorovich potentials when the cost function is given by the square Riemannian distance on a non-compact Riemannian manifold. As noted by Figalli-Gigli, this theorem follows from a generalization of the arguments in Appendix C of Gangbo-McCann [25].

**Theorem 3.3.2** (Theorem 1 from Figalli-Gigli). *Let  $M$  be a Riemannian manifold of dimension  $n$  with distance function  $d : M \times M \rightarrow [0, +\infty)$ . Let  $c(x, y) = \frac{1}{2}d^2(x, y)$ . Let  $\mu$  and  $\nu$  be probability measures on  $M$ . Let  $u$  and  $v$  be Kantorovich potentials for the Kantorovich dual problem from  $\mu$  to  $\nu$  for the cost  $c$ . Let*

$$\partial^c u(x) = \{y \in M : u(x) = v(y) - c(x, y)\}$$

and

$$\partial^c u = \cup_{x \in M} (\{x\} \times \partial^c u(x)).$$

Let  $D = \{x \in M : u(x) < +\infty\}$ . Then, the potential  $u$  is locally semiconvex in  $\text{int}D$ , the set  $\partial^c u(x)$  is non-empty for all  $x \in \text{int}D$ , and  $\partial^c u$  is locally bounded in  $\text{int}D$ . Moreover, the set  $D \setminus \text{int}D$  is  $(n - 1)$ -rectifiable.

Under the hypotheses of this theorem, there is an immediate corollary about the differentiability of the Kantorovich potential  $u$ .

**Corollary 3.3.3.** *With the hypotheses of the previous theorem (that is, on a Riemannian manifold with the cost given by  $c = \frac{1}{2}d^2$ ), the Kantorovich potential  $u$  is differentiable almost everywhere.*

*Proof.* Locally semiconvex functions are differentiable almost everywhere. ■

As noted in Remark 3 of Figalli-Gigli [23], these results continue to hold when the cost function  $\frac{1}{2}d^2$  is replaced with a cost function induced by a Tonelli Lagrangian. This is recorded in the following proposition.

**Proposition 3.3.4.** *Let  $\mu$  and  $\nu$  be probability measures on  $\mathbb{R}^d$ . Let  $c$  be a cost function induced by a Tonelli Lagrangian and let  $u$  and  $v$  be Kantorovich potentials for the Kantorovich dual problem from  $\mu$  to  $\nu$  for the cost  $c$ . Then, the function  $u$  is semiconcave, differentiable almost everywhere, and its derivative  $\partial u$  is locally bounded.*

The following proposition and corollary establish the differentiability of the cost function  $c$  at almost every point of the form  $(x, T(x))$ . This proposition and its proof are a modification of Lemma 7 from McCann [36].

**Proposition 3.3.5.** *Suppose that the Kantorovich potential  $u$  is differentiable at  $x$ . If equality holds in*

$$c(x, y) + u(x) - v(y) \geq 0, \quad (3.4)$$

*then  $y$  is the endpoint after unit time of the flow of Lagrangian system*

$$\begin{cases} \dot{x} = v \\ \frac{d}{dt} \nabla_v L(x, v) = \nabla_x L(x, v) \\ (x_0, v_0) = (x, \nabla_p H(x, -\partial u(x))). \end{cases}$$

*Proof.* Suppose that  $u$  is differentiable at the point  $x$  and suppose that equality holds in (3.4). Then, for all  $z \in \mathbb{R}^d$ ,

$$\begin{aligned} c(z, y) + u(z) - v(y) &\geq 0 \\ &= c(x, y) + u(x) - v(y). \end{aligned}$$

Rearranging and eliminating  $v(y)$  yields

$$c(z, y) \geq c(x, y) + u(x) - u(z) \quad (3.5)$$

$$\geq c(x, y) + u(x) - u(x) - \partial u(x) \cdot (z - x) + o(z - x) \quad (3.6)$$

$$= c(x, y) - \partial u(x) \cdot (z - x) + o(z - x). \quad (3.7)$$

Equation (3.6) follows from (3.5) since  $\partial u(x)$  is a subdifferential of  $u$  at  $x$ . This proves  $-\partial u(x)$  is a subdifferential of  $c(\cdot, y)$  at  $x$ . And, by corollary 2.2.20, the function  $c(\cdot, y)$  is always superdifferentiable. Therefore,  $c(\cdot, y)$  is differentiable at  $x$ .

By a simple modification of the proof of proposition 2.2.23, whenever  $c(\cdot, y)$  is differentiable, there is a unique optimal curve from  $x$  to  $y$ . From proposition 3.2.18, when  $c(x, y) + u(x) - v(y) = 0$  and both  $c(\cdot, y)$  and  $u$  are differentiable at the point  $x$ , then

$$\nabla_x c(x, y) = -\partial u(x),$$

and from corollary 2.2.22,

$$\nabla_x c(x, y) = -\nabla_v L(x, \dot{\sigma}(0)),$$

where  $\sigma$  is an optimal curve from  $x$  to  $y$ . (In particular, since  $\nabla_x c(x, y)$  exists, a unique optimal curve is guaranteed to exist by proposition 2.2.23.) Since  $\nabla_p H(x, \nabla_v L(x, v)) = v$ ,

$$\nabla_p H(x, \partial u(x)) = \nabla_p H(x, \nabla_v L(x, \dot{\sigma}(0))) = \dot{\sigma}(0).$$

Therefore, the solution to

$$\begin{cases} \dot{x} = v \\ \frac{d}{dt} \nabla_v L(x, v) = \nabla_x L(x, v) \\ (x_0, v_0) = (x, \nabla_p H(x, -\partial u(x))) \end{cases}$$

is the unique optimal curve  $\sigma$  going from  $x$  to  $y$ . ■

**Corollary 3.3.6.** *For almost every  $x$ , the partial derivative*

$$\nabla_z c(z, T(x)) \Big|_{z=x} \tag{3.8}$$

*exists and there is a unique optimal curve from  $x$  to  $T(x)$ .*

*Proof.* By proposition 3.3.4, the Kantorovich potential  $u$  is differentiable at almost every  $x \in \mathbb{R}^d$ . By proposition 3.3.5 and its proof, when  $u$  is differentiable at  $x$  and  $c(x, y) + u(x) - v(y) = 0$ , then  $c(\cdot, y)$  is differentiable at  $x$ . And then, by proposition 2.2.23 there exists a unique optimal curve from  $x$  to  $y$ .

By theorem 3.2.13, every optimal transport plan is concentrated on the graph of the Monge map, and by theorem 3.2.17,  $c(x, y) + u(x) - v(y) = 0$  for almost every point  $(x, y)$  in the support of any optimal transport plan. So, for almost every  $x$ ,

$$c(x, T(x)) + u(x) - v(T(x)) = 0.$$

Therefore, for almost every  $x$ , the hypotheses of theorem 3.3.5 are satisfied at the point  $(x, T(x))$ , which proves that

$$\nabla_z c(z, T(x)) \Big|_{z=x}$$

exists and there is a unique optimal curve from  $x$  to  $T(x)$ . ■

### 3.3.2 Regularity of Optimal Trajectories

In section 2.2.1, optimal trajectories were found to be of class  $C^2$  in time. This section establishes the regularity of optimal trajectories with respect to position. This is more subtle, and involves considering optimal trajectories with respect to their position after a short time, rather than the true initial position. In this framework, optimal trajectories and their time derivatives will turn out to both be locally Lipschitz with respect to their (after a short time) “initial” positions. First, however, the injectivity of optimal trajectories will be established.

**Proposition 3.3.7.** *Let  $\sigma^{x_0} : [0, 1] \rightarrow \mathbb{R}^d$  be the solution of the Lagrangian system*

$$\begin{cases} \dot{x} = v \\ \frac{d}{dt} \nabla_v L(x, v) = \nabla_x L(x, v) \\ (x_0, v_0) = (x_0, \nabla_p H(x_0, -\partial u(x_0))) \end{cases}.$$

Let  $\sigma_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be given by

$$\sigma_t(x) = \sigma(t, x) = \sigma^x(t).$$

Then,  $\sigma_1 = T$  and, for every  $t \in [0, 1)$ , the map  $\sigma_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is injective.

*Proof.* By corollary 3.3.4, the Kantorovich potential  $u$  is differentiable almost everywhere, so the map  $\sigma$  is well defined in  $L^1([0, 1] \times \mathbb{R}^d; \mathbb{R}^d)$ .

By corollary 3.3.6, for almost every  $x$ , the partial derivative  $\nabla_z c(z, T(x))|_{z=x}$  exists and there is a unique action minimizing curve from  $x$  to  $T(x)$ . From the proofs of proposition 3.3.5 and corollary 3.3.6, this curve is the solution to the Lagrangian system

$$\begin{cases} \dot{x} = v \\ \frac{d}{dt} \nabla_v L(x, v) = \nabla_x L(x, v) \\ (x_0, v_0) = (x, \nabla_p H(x, -\partial u(x))) \end{cases}.$$

By the uniqueness of solutions to Lagrangian systems, it follows that for almost every  $x$ ,

$$\sigma^x(1) = T(x) \quad \implies \quad \sigma^1 = T.$$

And again by the uniqueness of solutions to Lagrangian systems, if  $t \in [0, 1)$  and

$$\sigma^{x_1}(t) = \sigma^{x_2}(t),$$

then  $x_1 = x_2$ , which proves that  $\sigma_t$  is injective for all  $t \in [0, 1)$ . ■

An important tool for understanding the regularity of optimal trajectories with respect to initial conditions is Mather’s shortening lemma.

**Theorem 3.3.8** (Mather's Shortening Lemma; Cor. 8.2 from [46]). *Let  $M$  be a smooth Riemannian manifold. Let  $d$  denote Riemannian distance on  $M$ . Let  $L : TM \rightarrow \mathbb{R}$  be a Tonelli Lagrangian which is strongly convex in velocity. Let  $c$  be the induced cost function. Then, for any compact set  $K \subset M$ , there exists a constant  $C_K$  such that, whenever  $x_1, y_1, x_2, y_2$  are four points in  $K$  which satisfy*

$$c(x_1, y_1) + c(x_2, y_2) \leq c(x_1, y_2) + c(x_2, y_1)$$

and  $\gamma_i$  is the action minimizing curve from  $x_i$  to  $y_i$ , then, for any  $t_0 \in (0, 1)$ ,

$$\sup_{t \in [0, 1]} d(\gamma_1(t), \gamma_2(t)) \leq \frac{C_K}{\min t_0, 1 - t_0} d(\gamma_1(t_0), \gamma_2(t_0)).$$

Regularity for optimal trajectories will proceed in four steps. First, a local boundedness lemma will be proven. Then, the family of trajectories  $\sigma_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$  will be shown to be Lipschitz continuous with respect to its time  $s$  initial conditions when  $s \in (0, 1)$ —regularity fails to hold with respect to initial conditions at times 0. The time derivative map  $\dot{\sigma}_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$  will be shown to be Hölder-1/2 continuous. Finally,  $\dot{\sigma}_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$  will be shown to be locally Lipschitz.

**Lemma 3.3.9.** *Let  $\rho_0, \rho_1 \in \mathcal{P}^{ac}(\mathbb{R}^d)$ . Let  $L$  be a Tonelli Lagrangian and let  $\sigma : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the family of trajectories solving the optimal transport problem from  $\rho_0$  to  $\rho_1$  for the cost induced by  $L$ . Let  $t \in (0, 1)$ . For each  $y \in \mathbb{R}^d$  such that  $y = \sigma_{t_0}(x)$ , there exists a neighbourhood  $N \ni y$  and a compact set  $K$  such that whenever  $\sigma_{t_0}(z) \in N$ , then the endpoints of the curve  $\sigma^z : [0, 1] \rightarrow \mathbb{R}^d$  are contained in  $K$ .*

*Proof.* Let  $\gamma : [0, t_0] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\eta : [0, 1 - t_0] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be given by

$$\begin{cases} \gamma(t, y) = \sigma(t_0 - t, x) & \text{where } \sigma(t_0, x) = y \\ \eta(t, y) = \sigma(t_0 + t, x) & \text{where } \sigma(t_0, x) = y. \end{cases}$$

The family of curves  $\eta$  solves the time  $1 - t_0$  optimal transport problem from  $\rho_{t_0}$  to  $\rho_1$ . The family of curves  $\gamma$  solves the time  $t_0$  optimal transport problem from  $\rho_{t_0} = (\sigma_{t_0})_{\#}\rho_0$  to  $\rho_0$  with the cost  $\tilde{C}$  induced by the Lagrangian  $\tilde{L}$  given by

$$\tilde{L}(x, v) = L(x, -v).$$

By proposition 3.3.7, the trajectories  $\gamma(\cdot, y)$  and  $\eta(\cdot, y)$  are solutions to the Lagrangian system where the initial velocities are  $C^1$  functions of Kantorovich potentials for the respective optimal transport problems. By proposition 3.3.4, the potentials are locally bounded, so there are neighbourhoods  $N_\gamma, N_\eta \ni y$  on which the families of curves  $\gamma$  and  $\eta$  are bounded.

Let  $N$  be the intersection of  $N_\gamma$  and  $N_\eta$ . The proof concludes by noting that the image of a curve  $\sigma^x : [0, 1] \rightarrow \mathbb{R}^d$  with  $\sigma^x(t_0) = y$  is the union of the images of  $\gamma(\cdot, y)$  and  $\eta(\cdot, y)$ .  $\blacksquare$

**Proposition 3.3.10.** *Let  $\rho_0, \rho_1 \in \mathcal{P}^{ac}(\mathbb{R}^d)$ . Let  $L$  be a Tonelli Lagrangian which is strongly convex in velocity. Let  $c$  be the induced cost function and let  $\sigma : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the family of trajectories solving the optimal transport problem from  $\rho_0$  to  $\rho_1$ . Let  $\epsilon > 0$ . Let  $\tilde{\sigma} : [\epsilon, 1 - \epsilon] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be given by*

$$\tilde{\sigma}(t, y) = \sigma(t, \sigma_t^{-1}(y)).$$

*Then, the map  $\tilde{\sigma}_t = \tilde{\sigma}(t, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is locally Lipschitz for all  $t \in [\epsilon, 1 - \epsilon]$ .*

*Proof.* Fix  $t \in [\epsilon, 1 - \epsilon]$ . By lemma 3.3.9, given  $\sigma_t x = y$ , there is a neighbourhood  $N \ni y$  such that all trajectories  $\tilde{\sigma}^z$  which intersect  $N$  at time  $t$  are contained in a compact set  $K$ .

Since  $\sigma$  solves the optimal transport problem, for almost every pair  $x, y \in \mathbb{R}^d$ ,

$$c(\sigma_0(x), \sigma_1(x)) + c(\sigma_0(y), \sigma_1(y)) \leq c(\sigma_0(x), \sigma_1(z)) + c(\sigma_0(y), \sigma_1(x)).$$

After possibly redefining  $\sigma$  on a set of measure zero, it satisfies the hypotheses of Mather's lemma (3.3.8). So, by Mather's lemma,

$$\sup_{s \in [0, 1]} \|\sigma_s(x) - \sigma_s(y)\| \leq \begin{cases} \frac{C_K}{\min t, 1 - t} \|\sigma_t(x) - \sigma_t(y)\| \\ \frac{C_K}{\min \epsilon, 1 - \epsilon} \|\sigma_\epsilon(x) - \sigma_\epsilon(y)\|. \end{cases}$$

In particular, there are constants  $C_1$  and  $C_2$  depending only on  $K$  and  $\epsilon$  such that

$$C_1 \|\sigma_\epsilon(x) - \sigma_\epsilon(y)\| \leq \|\sigma_t(x) - \sigma_t(y)\| \leq C_2 \|\sigma_\epsilon(x) - \sigma_\epsilon(y)\|.$$

Rewriting this in terms of  $\tilde{\sigma}$  instead of  $\sigma$ , and setting  $w = \sigma_\epsilon(x)$  and  $z = \sigma_\epsilon(y)$ , it follows that

$$C_1 \|z - w\| \leq \|\tilde{\sigma}_t(z) - \tilde{\sigma}_t(w)\| \leq C_2 \|z - w\|.$$

Therefore,  $\tilde{\sigma}_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is locally Lipschitz. ■

**Proposition 3.3.11.** *With the same set-up as proposition 3.3.10, the map  $\dot{\tilde{\sigma}}_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is locally Hölder-1/2 continuous.*

*Proof.* Suppose that  $x_n \rightarrow x_0$ . Then, Taylor expanding  $\tilde{\sigma}$  in time yields

$$\tilde{\sigma}_s(x_n) - \tilde{\sigma}_s(x_0) = \tilde{\sigma}_t(x_n) - \tilde{\sigma}_t(x_0) + (\dot{\tilde{\sigma}}_t(x_n) - \dot{\tilde{\sigma}}_t(x_0))(t - s) + O((s - t)^2).$$

By the triangle inequality, it follows that

$$\|\tilde{\sigma}_s(x_n) - \tilde{\sigma}_s(x_0)\| \geq \|\dot{\tilde{\sigma}}_t(x_n) - \dot{\tilde{\sigma}}_t(x_0)\| (s - t) - \|\tilde{\sigma}_t(x_n) - \tilde{\sigma}_t(x_0)\| - K(s - t)^2.$$

By the previous proposition (3.3.10), both  $\|\tilde{\sigma}_s(x_n) - \tilde{\sigma}_s(x_0)\|$  and  $\|\tilde{\sigma}_t(x_n) - \tilde{\sigma}_t(x_0)\|$  are bounded

by a scalar multiple of  $\|x_n - x_0\|$ . Hence, there is a constant  $M$  such that

$$M \|x_n - x_0\| \geq \|\dot{\tilde{\sigma}}_t(x_n) - \dot{\tilde{\sigma}}_t(x_0)\| (s - t) - K(s - t)^2. \quad (3.9)$$

Set

$$s - t = \sqrt{\frac{2M + 1}{K}} \|x_n - x_0\|.$$

Substituting this value of  $s - t$  into equation (3.9) and rearranging yields

$$\|\dot{\tilde{\sigma}}_t(x_n) - \dot{\tilde{\sigma}}_t(x_0)\| \leq C \|x_n - x_0\|$$

for a positive constant  $C$ . Therefore,  $\dot{\tilde{\sigma}}_t$  is locally Lipschitz. Since this argument was independent of  $t$ , this is true for every  $t \in [\epsilon, 1 - \epsilon]$ . ■

**Proposition 3.3.12.** *With the same set-up as proposition 3.3.10, the map  $\dot{\tilde{\sigma}}_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is locally Lipschitz for all  $t \in [\epsilon, 1 - \epsilon]$ .*

*Proof.* Fix  $t \in [\epsilon, 1 - \epsilon]$ . Let  $y = \sigma_t(x)$  and let  $N \ni y$  be an open set around  $y$  where all trajectories passing through  $N$  are bounded, as in lemma 3.3.9. Let  $\Omega = \tilde{\sigma}_t^{-1}(N)$  and let  $K \subset \Omega$  be a compact set containing an open set containing  $\tilde{\sigma}_t^{-1}(y)$ . Let  $(x_n)_{n=1}^\infty$  be a sequence in  $K$ . For notational convenience, let

$$y_n = \tilde{\sigma}_t(x_n) \quad \text{and} \quad v_n = \dot{\tilde{\sigma}}_t(x_n).$$

The result will follow upon demonstrating that

$$\sup_{n,m} \frac{\|v_n - v_m\|}{\|x_n - x_m\|} \leq C$$

for some constant  $C$ .

The proof proceeds by contradiction. Suppose that this supremum is infinite. By the previous proposition,  $\dot{\tilde{\sigma}}_t$  is continuous, so the sequence  $(v_n)$  is bounded on compact sets. Hence, the supremum can only diverge along a subsequence  $(x_n)$  where  $\|x_m - x_n\| \rightarrow 0$  as  $m, n \rightarrow \infty$ . After passing to a convergent subsequence, suppose that  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ .

Define  $(q_n, p_n) : [\epsilon, 1 - \epsilon] \rightarrow T^*\mathbb{R}^d$  by taking the global Legendre transform of the sequence  $(y_n, v_n)$ :

$$(q_n, p_n)(t) = (y_n, \nabla_v L(y_n, v_n)).$$

Since the global Legendre transform is a  $C^1$ -diffeomorphism, this sequence is also convergent. (As well, since  $y_n = \tilde{\sigma}_t(x_n)$  and  $v_n = \dot{\tilde{\sigma}}_t(x_n)$ , both may be viewed, for fixed  $n$ , as functions from  $[\epsilon, 1 - \epsilon]$  to  $\mathbb{R}^d$ .)

Let  $\Psi_{t,s} : T^*\mathbb{R}^d \rightarrow T^*\mathbb{R}^d$  denote the Hamiltonian flow from time  $t$  to time  $s$  (see definition 2.4.4). Let  $\pi_q$  denote projection onto position and  $\pi_p$  denote projection onto momentum. The map  $\Psi_{t,s}$  is of class  $C^1$ , so can be Taylor expanded around  $(q_0, p_0)$ :

$$\begin{aligned} \Psi_{t,s}(q_n, p_n) &= \Psi_{t,s}(q_0, p_0) + \begin{pmatrix} \pi_q D_q \Psi_{s,t} & \pi_q D_p \Psi_{s,t} \\ \pi_p D_q \Psi_{s,t} & \pi_p D_p \Psi_{s,t} \end{pmatrix} \begin{pmatrix} q_n - q_0 \\ p_n - p_0 \end{pmatrix} \\ &\quad + o(\|q_n - q_0, p_n - p_0\|). \end{aligned} \quad (3.10)$$

Here, the derivative of the Hamiltonian flow is evaluated at  $(q_0, p_0)$ . Rearranging equation (3.10) and considering only the position component yields

$$q_n(s) - q_0(s) = \begin{pmatrix} \pi_q D_q \Psi_{s,t} & \pi_q D_p \Psi_{s,t} \end{pmatrix} \begin{pmatrix} q_n(t) - q_0(t) \\ p_n(t) - p_0(t) \end{pmatrix} + o(\|q_n - q_0, p_n - p_0\|). \quad (3.11)$$

Recall that  $q_n(s) = \tilde{\sigma}_s(x_n)$  and  $q_0(s) = \tilde{\sigma}_s(x_0)$ . From applying Mather's lemma, it follows that, for some constants  $C_1$  and  $C_2$ ,

$$\begin{aligned} C_1 \|x_n - x_0\| + o(\|q_n - q_0, p_n - p_0\|) \\ \geq \|\pi_q D_q \Psi_{s,t} \cdot (q_n(t) - q_0(t)) + \pi_q D_p \Psi_{s,t} \cdot (p_n(t) - p_0(t))\| \end{aligned} \quad (3.12)$$

$$\geq C_2 \|x_n - x_0\| - o(\|q_n - q_0, p_n - p_0\|). \quad (3.13)$$

Since  $\pi_q D_q \Psi_{s,t}$  is bounded (as  $\Psi_{s,t}$  is of class  $C_1$ ) and  $q_i(t) = \tilde{\sigma}_t(x_i)$  is locally Lipschitz, it follows that, with different constants than above,

$$\begin{aligned} C_1 \|x_n - x_0\| + o(\|q_n - q_0, p_n - p_0\|) &\geq \|\pi_q D_p \Psi_{s,t} \cdot (p_n(t) - p_0(t))\| \\ &\geq C_2 \|x_n - x_0\| - o(\|q_n - q_0, p_n - p_0\|). \end{aligned}$$

If  $\pi_q D_p \Psi_{s,t}$  is positive definite, then this will imply that  $p_i(t) = \nabla_v L(\tilde{\sigma}_t(x_i), \dot{\tilde{\sigma}}_t(x_i))$  is locally Lipschitz with respect to  $x$ , and since the Legendre transform is a  $C^1$ -diffeomorphism, it will follow that  $\dot{\tilde{\sigma}}_t(x)$  is locally Lipschitz with respect to  $x$ , which will complete the proof.

Finally, from proposition 2.4.10,  $\pi_q D_p \Psi_{s,t}(p, q) = (t - s)\nabla_{pp}H + o(t - s)$ . The matrix  $\nabla_{pp}H(\Psi_{s,t}(p, q))$  is positive definite and bounded on each compact set. Since  $(p, q)$  is contained in a compact set (which depends on the set  $K$  from the beginning of the proof), so, the remainder term in the derivative is of the form  $C(t - s)^2$ . Therefore, for  $t - s$  sufficiently small,  $\pi_q D_p \Psi_{s,t}(p, q)$  is positive definite, which concludes the proof. ■

### 3.4 The Eulerian Optimal Transport Problem

In [4], Benamou and Brenier reformulate the classical quadratic cost optimal transport problem in an Eulerian framework, motivated by fluid mechanics. The equivalent formulation in the



setting of costs induced by Tonelli Lagrangians is the following:

**Definition 3.4.1.** Let  $\rho_0$  and  $\rho_1$  be probability densities on  $\mathbb{R}^d$ . The **Eulerian optimal transport problem from  $\rho_0$  to  $\rho_1$  for the cost  $c$  induced by the Tonelli Lagrangian  $L$**  is the minimization problem

$$\min_{\rho, V} \left\{ \int_0^1 \int_{\mathbb{R}^d} L(x, V(t, x)) \rho(t, x) dx dt : \rho|_{t=0} = \rho_0, \rho|_{t=1} = \rho_1, \text{ and } \frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho V) = 0 \right\},$$

where  $[0, 1] \ni t \mapsto \rho_t \in \mathcal{P}^{ac}(\mathbb{R}^d)$  and  $V$  is a time-varying vector field with the property that  $V$  is locally Lipschitz on  $(0, 1) \times \mathbb{R}^d$  and locally bounded on  $[0, 1] \times \mathbb{R}^d$ .

The local Lipschitz condition is imposed so that only vector fields which define flows are considered. A theory of flows of non-smooth vector fields has been developed (see [1] and [16] for example), following from the work of DiPerna-Lions [18], but will not be considered here.

In the above definition, the differential equation  $\partial_t \rho + \nabla_x \cdot (\rho V) = 0$  is understood in the sense of distributions. That is, a pair  $(\rho, V)$  satisfies  $\partial_t \rho + \nabla_x \cdot (\rho V) = 0$  if for all  $\varphi \in C_c^\infty([0, 1] \times \mathbb{R}^d)$ ,

$$0 = \int_{\mathbb{R}^d} \int_0^1 \varphi [\partial_t \rho + \nabla_x \cdot (\rho V)] dt dx \quad (3.14)$$

$$\begin{aligned} &:= \int_{\mathbb{R}^d} \left( \varphi(t, x) \rho(t, x) \Big|_{t=0}^{t=1} - \int_0^1 \rho(t, x) \partial_t \varphi(t, x) dt \right) dx \\ &\quad - \int_{\mathbb{R}^d} \int_0^1 \nabla_x \varphi(t, x) \cdot \rho(t, x) V(t, x) dt dx. \end{aligned} \quad (3.15)$$

**Definition 3.4.2.** The differential equation  $\partial_t \rho + \nabla_x \cdot (\rho V) = 0$  is called the **continuity equation**.

**Proposition 3.4.3.** Let  $V : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a locally bounded locally Lipschitz time-varying vector field in the sense of definition 3.4.1. Let  $\sigma : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the flow of  $V$ , defined by

$$\dot{\sigma}(t, x) = V(t, \sigma(t, x)),$$

where  $\sigma_0 = id$ .

Let  $\rho_0$  be a probability density on  $\mathbb{R}^d$  and let  $\rho_t = (\sigma_t)_\# \rho_0$ . Then, the pair  $(\rho_t, V)$  solves the continuity equation in the sense of distributions.

*Proof.* By definition, for every measurable set  $A$ ,

$$(\sigma_t)_\# \rho_0(A) := \rho_0(\sigma_t^{-1}(A)).$$

It follows from the definition of the pushforward that for  $f$  locally integrable,

$$\int_{\mathbb{R}^d} f(t, x) \rho_t(x) dx = \int_{\mathbb{R}^d} f(t, x) (\sigma_t)_\# \rho_0(x) dx = \int_{\mathbb{R}^d} f(t, \sigma(t, x)) \rho_0(x) dx. \quad (3.16)$$

Let  $\varphi \in C_c^\infty([0, 1] \times \mathbb{R}^d)$ . Then, by equation (3.14),

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_0^1 \varphi [\partial_t \rho + \nabla_x \cdot (\rho V)] dt dx \\ & := \int_{\mathbb{R}^d} \left( \varphi(t, x) \rho(t, x) \Big|_{t=0}^{t=1} - \int_0^1 \rho(t, x) \partial_t \varphi(t, x) dt \right) dx \\ & \quad - \int_{\mathbb{R}^d} \int_0^1 \nabla_x \varphi(t, x) \cdot \rho(t, x) V(t, x) dt dx \end{aligned} \quad (3.17)$$

$$\begin{aligned} & = \int_{\mathbb{R}^d} \varphi(1, x) \rho_1(x) - \varphi(0, x) \rho_0(x) dx \\ & \quad - \int_0^1 \int_{\mathbb{R}^d} [\partial_t \varphi(t, x) + \nabla_x \varphi(t, x) \cdot V(t, x)] \rho(t, x) dx dt. \end{aligned} \quad (3.18)$$

Using the identity from eq. (3.16), the double integral in eq. (3.18) simplifies to

$$\int_0^1 \int_{\mathbb{R}^d} [\partial_t \varphi(t, x) + \nabla_x \varphi(t, x) \cdot V(t, x)] \rho(t, x) dx dt \quad (3.19)$$

$$= \int_0^1 \int_{\mathbb{R}^d} [\partial_t \varphi(t, \sigma(t, x)) + \nabla_x \varphi(t, \sigma(t, x)) \cdot V(t, \sigma(t, x))] \rho_0(x) dx dt \quad (3.20)$$

$$= \int_0^1 \int_{\mathbb{R}^d} \underbrace{[\partial_t \varphi(t, \sigma(t, x)) + \nabla_x \varphi(t, \sigma(t, x)) \cdot \dot{\sigma}(t, x)]}_{= \frac{d}{dt} \varphi(t, \sigma(t, x))} \rho_0(x) dx dt \quad (3.21)$$

Applying the fundamental theorem of calculus to equation (3.21) yields

$$\int_0^1 \int_{\mathbb{R}^d} \frac{d}{dt} \varphi(t, \sigma(t, x)) \rho_0(x) dx dt \quad (3.22)$$

$$= \int_{\mathbb{R}^d} \varphi(t, \sigma(t, x)) \Big|_{t=0}^{t=1} \rho_0(x) dx \quad (3.23)$$

$$= \int_{\mathbb{R}^d} \varphi(1, \sigma(1, x)) \rho_0(x) - \varphi(0, \sigma(0, x)) \rho_0(x) dx \quad (3.24)$$

$$= \int_{\mathbb{R}^d} \varphi(1, x) \rho_1(x) - \varphi(0, x) \rho_0(x) dx. \quad (3.25)$$

The identity (3.16) is applied again to transform (3.24) to (3.25).

Substituting (3.25) back into equation (3.18) yields

$$\begin{aligned} \int_{\mathbb{R}^d} \int_0^1 \varphi [\partial_t \rho + \nabla_x \cdot (\rho V)] dt dx & = \int_{\mathbb{R}^d} \varphi(1, x) \rho_1(x) - \varphi(0, x) \rho_0(x) dx \\ & \quad - \int_{\mathbb{R}^d} \varphi(1, x) \rho_1(x) - \varphi(0, x) \rho_0(x) dx \\ & = 0, \end{aligned}$$

which proves the proposition.  $\blacksquare$

With the regularity results for optimal trajectories in the Lagrangian optimal transport problem in sections 3.3.2, 3.3.1, and 2.2.1, there is a distinguished candidate minimizer for the Eulerian optimal transport problem.

**Proposition 3.4.4.** *Let  $\rho_0$  and  $\rho_1$  be probability densities on  $\mathbb{R}^d$ . Let  $L$  be a Tonelli Lagrangian which is strongly convex in velocity and let  $c$  be the induced cost function. Let  $\sigma : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the family of optimal trajectories solving the Lagrangian optimal transport problem from  $\rho_0$  to  $\rho_1$ . Then, there is a vector field  $V : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  which satisfies*

$$\dot{\sigma}(t, x) = V(t, \sigma(t, x)) \text{ for all } t, x,$$

and  $V$  is locally Lipschitz on  $(0, 1) \times \mathbb{R}^d$  and locally bounded on  $[0, 1] \times \mathbb{R}^d$ .

*Proof.* The vector field  $V$  is constructed directly by  $\dot{\sigma}(t, x) = V(t, \sigma(t, x))$  and then extended to all of  $\mathbb{R}^d$ .

Such a vector field is well defined: from proposition 3.3.7, the trajectories  $\sigma_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$  are injective for all  $t \in [0, 1)$ . Therefore,  $V(t, \cdot)$  is well-defined on  $\{\sigma_t(x) : x \in \mathbb{R}^d\}$ .

At  $t = 0$ ,  $\sigma_t = id$ , so  $\dot{\sigma}_0(x) = V(0, x)$  and from proposition 3.3.7 and 3.3.4, the initial velocities of optimal trajectories are locally bounded. Solving the optimal transport problem backwards, from  $\rho_1$  to  $\rho_0$  yields the same result for  $t = 1$ .

Let  $t \in (0, 1)$ . Let  $0 < \epsilon < t$ . Using the notation from 3.3.10, denote  $z = \sigma_\epsilon(x)$ . Then,

$$\begin{cases} \sigma(t, x) = \tilde{\sigma}(t, z) \\ \dot{\sigma}(t, x) = \dot{\tilde{\sigma}}(t, z) \end{cases}$$

and

$$V(t, \tilde{\sigma}(t, z)) = \dot{\tilde{\sigma}}(t, z).$$

From proposition 3.3.12, the map  $\dot{\tilde{\sigma}}(t, \cdot)$  is locally Lipschitz, and in the proof of proposition 3.3.10, the map  $\tilde{\sigma}(t, \cdot)$  was seen to be bi-Lipschitz. Together, this implies that  $V$  is Lipschitz for  $t \in (0, 1)$ .

This vector field can be extended globally to be locally Lipschitz, which completes the proof. ■

### 3.5 Five Equivalent Formulations of the Kantorovich Cost

Let  $L$  be a Tonelli Lagrangian which is strongly convex in velocity. Let  $c$  be the induced cost function.

The goal of this section is to prove the equivalence of the different forms of the optimal transport problem and understand the relationships between the minimizers in each formulation of the problem.

**Theorem 3.5.1.** *Let  $L$  be a Tonelli Lagrangian which is strongly convex in velocity. Let  $c$  be the induced cost function. Let  $\rho_0$  and  $\rho_1$  be probability densities on  $\mathbb{R}^d$ . Then, the Kantorovich cost*

$$W_c(\rho_0, \rho_1) := \inf_{\gamma} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y) d\gamma(x, y) : \gamma \in \Gamma(\rho_0, \rho_1) \right\} \quad (3.26)$$

satisfies

$$W_c(\rho_0, \rho_1) = \sup_{u, v} \left\{ \int_{\mathbb{R}^d} v(y) \rho_1(y) dy - \int_{\mathbb{R}^d} u(x) \rho_0(x) dx : v(y) - u(x) \leq c(x, y) \right. \\ \left. \text{for } \mu\text{-a.e. } x, \nu\text{-a.e. } y \right\} \quad (3.27)$$

$$= \inf_T \left\{ \int_{\mathbb{R}^d} c(x, T(x)) \rho_0(x) dx : T_{\#} \rho_0 = \rho_1 \right\} \quad (3.28)$$

$$= \inf_{\phi} \left\{ \int_0^1 \int_{\mathbb{R}^d} L(\phi(t, x), \dot{\phi}(t, x)) \rho_0(x) dx dt : \phi(1, \cdot)_{\#} \rho_0 = \rho_1 \right\} \quad (3.29)$$

$$= \inf_{\rho, V} \left\{ \int_0^1 \int_{\mathbb{R}^d} L(x, V(t, x)) \rho(t, x) dx dt : \rho|_{t=0} = \rho_0, \rho|_{t=1} = \rho_1, \right. \\ \left. \frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho V) = 0, V \text{ loc. Lipschitz for } t \in (0, 1), \right. \\ \left. V \text{ loc. bounded for } t \in [0, 1] \right\} \quad (3.30)$$

*Proof.* If  $W_c(\rho_0, \rho_1)$  is infinite, there is nothing to show, so suppose that  $W_c(\rho_0, \rho_1) < +\infty$ .

**(3.26)=(3.27):** From section 4.2, Kantorovich duality holds (so lines (3.26) and (3.27) are equal) whenever the cost function is lower semi-continuous (see 3.2.16). From section 2.2.2, cost function induced by Tonelli Lagrangians are continuous (see 2.2.19). Hence, (3.26) and (3.27) are equal.

**(3.26)=(3.28):** From section 4.1, if the hypothesis of theorem 3.2.13 are satisfied, then an optimal transport map from  $\rho_0$  to  $\rho_1$  exists and the Monge and Kantorovich costs coincide, so (3.26) and (3.28) are equal. Cost functions induced by Tonelli Lagrangians are continuous and bounded below. Since  $\rho_0$  and  $\rho_1$  are probability densities, they are absolutely continuous with respect to Lebesgue measure by assumption. The Kantorovich cost is assumed finite. Thus, all but two of the hypotheses of 3.2.13 are satisfied.

The remaining hypotheses of 3.2.13 are that the family of maps  $x \mapsto c(x, y) = c_y(x)$  is locally semi-concave in  $x$  locally uniformly in  $y$  and the cost  $c$  satisfies the twist condition. By proposition 3.2.11, costs induced from Tonelli Lagrangians satisfy these properties.

Hence, the Monge and Kantorovich costs from  $\rho_0$  to  $\rho_1$  for the cost  $c$  are equal.

**(3.28)=(3.29):** Suppose that  $T$  is a Borel measurable map pushing forward  $\rho_0$  to  $\rho_1$ . For each  $x$ , let  $\phi(\cdot, x)$  be the action minimizing curve from  $x$  to  $T(x)$ . Then,  $\phi(1, x) = T(x)$ , so

$\phi(1, \cdot)$  pushes forward  $\rho_0$  to  $\rho_1$ . Further,

$$c(x, T(x)) = \int_0^1 L(\phi(t, x), \dot{\phi}(t, x)) dt,$$

from which it follows that

$$\begin{aligned} \int_{\mathbb{R}^d} c(x, T(x)) \rho_0(x) dx &= \int_{\mathbb{R}^d} \int_0^1 L(\phi(t, x), \dot{\phi}(t, x)) dt \rho_0(x) dx \\ &= \int_{\mathbb{R}^d} \int_0^1 L(\phi(t, x), \dot{\phi}(t, x)) \rho_0(x) dt dx. \end{aligned}$$

Let  $(T_n)_{n=1}^\infty$  be a minimizing sequence for (3.29). Let  $\phi_n(\cdot, x)$  be the correspond minimizing curve from  $x$  to  $T_n(x)$ . Then,

$$\begin{aligned} &\inf_{\phi(1, \cdot) \# \rho_0 = \rho_1} \left\{ \int_0^1 \int_{\mathbb{R}^d} L(\phi(t, x), \dot{\phi}(t, x)) \rho_0(x) dx dt \right\} \\ &\leq \lim_{n \rightarrow \infty} \int_0^1 \int_{\mathbb{R}^d} L(\phi_n(t, x), \dot{\phi}_n(t, x)) \rho_0(x) dx dt \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} c(x, T_n(x)) \rho_0(x) dx \\ &= \inf_{T \# \rho_0 = \rho_1} \left\{ \int_{\mathbb{R}^d} c(x, T(x)) \rho_0(x) dx \right\}. \end{aligned}$$

Similarly, if  $\phi : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  satisfies  $\phi(1, \cdot)$  pushes forward  $\rho_0$  to  $\rho_1$ , then the map  $T_\phi$  given by  $T_\phi(x) = \phi(1, x)$  pushes forward  $\rho_0$  to  $\rho_1$ , and an analogous argument as above shows that

$$\inf_{T \# \rho_0 = \rho_1} \left\{ \int_{\mathbb{R}^d} c(x, T(x)) \rho_0(x) dx \right\} \leq \inf_{\phi(1, \cdot) \# \rho_0 = \rho_1} \left\{ \int_0^1 \int_{\mathbb{R}^d} L(\phi(t, x), \dot{\phi}(t, x)) \rho_0(x) dx dt \right\}.$$

Hence,

$$\inf_{T \# \rho_0 = \rho_1} \left\{ \int_{\mathbb{R}^d} c(x, T(x)) \rho_0(x) dx \right\} = \inf_{\phi(1, \cdot) \# \rho_0 = \rho_1} \left\{ \int_0^1 \int_{\mathbb{R}^d} L(\phi(t, x), \dot{\phi}(t, x)) \rho_0(x) dx dt \right\}.$$

**(3.29)=(3.30):** Let  $(\rho, V)$  be an admissible pair in the Eulerian optimal transport problem. Let  $\phi : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the flow of  $V$ . Then,  $\phi$  is in the admissible class for the Lagrangian optimal transport problem, and using equation (3.16),

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}^d} L(x, V(t, x)) \rho(t, x) dx dt \\ &= \int_0^1 \int_{\mathbb{R}^d} L(\phi(t, x), V(t, \phi(t, x))) \rho_0(x) dx dt \\ &= \int_0^1 \int_{\mathbb{R}^d} L(\phi(t, x), V(t, \phi(t, x))) \rho_0(x), \end{aligned}$$

so every admissible pair in the Eulerian optimal transport problem yields an admissible family of trajectories in the Lagrangian problem, and both have equal cost. Hence, the Lagrangian cost is no greater than the Eulerian cost. That is, (3.29)  $\leq$  (3.30).

Let  $\sigma : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the family of optimal trajectories for the Lagrangian optimal transport problem, given in proposition 3.3.7. From proposition 3.4.4, this family of trajectories is the flow of an admissible vector field  $V$  in the Eulerian optimal transport problem, together with  $\rho_t := (\sigma_t)_\# \rho_0$ , which are of equal cost in the optimal transport problem. Therefore, (3.30)  $\leq$  (3.29), which concludes the proof. ■

The following theorem describes the relationships between the optimizers in the various optimal transport problems.

**Theorem 3.5.2.** *Let  $\rho_0$  and  $\rho_1$  be probability densities on  $\mathbb{R}^d$ . Let  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  be a Tonelli Lagrangian which is strongly convex in velocity with corresponding Hamiltonian  $H : T^*\mathbb{R}^d \rightarrow \mathbb{R}$ . Let  $c$  be the induced cost function. Let  $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the Monge map for the Monge problem from  $\rho_0$  to  $\rho_1$  for the cost  $c$ . Let  $(u, v)$  be the pair of Kantorovich potentials for the Kantorovich dual problem. Let  $\sigma$  be the family of optimal trajectories for the Lagrangian problem. Let  $(\rho, V)$  be the optimal pair for the Eulerian problem. Then*

(a) *The optimal trajectories satisfy  $\sigma(0, x) = x$ ,  $\sigma(1, x) = T(x)$ , and  $\sigma$  solves*

$$\begin{cases} \dot{x} = v \\ \frac{d}{dt} \nabla_v L(x, v) = \nabla_x L(x, v) \\ (x_0, v_0) = (x_0, \nabla_p H(x_0, -\partial u(x_0))). \end{cases}$$

(b) *The interpolant probability densities satisfy  $\rho_t = (\sigma_t)_\# \rho_0$*

(c) *The vector field satisfies  $\dot{\sigma}(t, x) = V(t, \sigma(t, x))$ , with  $\dot{\sigma}(t, x) = \nabla_p H(x, -\partial u_t(x))$  where  $(u_t, v_t)$  are the Kantorovich potentials for the time  $1 - t$  optimal transport problem from  $\rho_t$  to  $\rho_1$ .*

## Chapter 4

# An Eulerian Calculus for Higher Order Functionals

In 2000, Benamou and Brenier introduced an Eulerian formulation of the optimal transport problem ([4]). With this formulation of the problem, it was realized that Wasserstein space has a structure in many ways analogous to a Riemannian manifold. At the same time, Otto introduced a Riemannian calculus that allowed for computations of derivatives and Hessians in Wasserstein space ([40] [41]). This calculus has come to be called the *Otto calculus*; it has proven to be a useful tool for understanding the geometry of Wasserstein space. In [41], using this calculus, Villani and Otto predicted that in the optimal transportation problem on Riemannian manifolds with the cost given by squared Riemannian distance, there would be a non-negative Ricci curvature condition associated with the displacement convexity of the entropy functional. This Ricci curvature bound was independently discovered and proven, at essentially the same time, by Cordero-Erausquin, McCann, and Schmuckenschläger via distinct methods in [15].

In *Optimal Transport: Old and New*, Villani poses two problems regarding the Otto calculus. He first asks whether it is possible to extend the Otto Calculus to the setting of the optimal transportation problem with costs induced by Tonelli Lagrangians, rather than only the classical cost of distance squared. This problem is resolved; when working in a purely Eulerian framework, rather than in a Riemannian setting, this problem becomes straightforward. Paul Lee has worked in a similar direction in [30].

Villani then asks whether it is possible to extend the Otto calculus to allow formal computations of displacement Hessians of functionals involving derivatives of densities of probability measures. This problem is more subtle; its resolution is the main contribution of this chapter.

In particular, the displacement Hessian of a functional involving any number of derivatives of a probability measure on an  $n$ -dimensional manifold will be computed. In one dimension, this will yield a new class of displacement convex functionals heretofore unknown. This class will include, as a special case, the functionals considered by Carrillo and Slepčev [11], which

thus far were the only known examples of displacement convex functionals involving derivatives of densities.

Some results from this chapter appear in [44], which has been submitted for publication at the time of writing.

In order that this chapter be readable independent of the rest of this thesis, several definitions and basic facts will be restated.

## 4.1 Setup

For this chapter, in order to avoid subtle issues of regularity, all objects will be assumed smooth in both time and position. At the conclusion of the chapter there will be a brief discussion of some of these regularity issues.

Let  $L$  be a Tonelli Lagrangian and let  $\rho_0, \rho_1 \in \mathcal{P}^{ac}(\mathbb{R}^d)$  or in  $\mathcal{P}^{ac}(\mathbb{T}^d)$  where  $\mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d$  be smooth probability densities; with the assumption of smoothness, probability measures and their densities will be treated interchangeably. For the rest of this section, definitions will be stated for  $\mathbb{R}^d$ , but apply equally well on  $\mathbb{T}^d$ . The 1-torus will be denoted by  $S^1$ . A smooth map

$$[0, 1] \ni t \mapsto \rho_t \in \mathcal{P}^{ac}(\mathbb{R}^d)$$

is a **smooth curve** in  $\mathcal{P}^{ac}(\mathbb{R}^d)$ . A curve in  $\mathcal{P}^{ac}(\mathbb{R}^d)$  will be denoted  $(\rho_t)$ , while the probability measure at time  $t$  will be denoted  $\rho_t$ . When there is no ambiguity, the subscripts will be omitted.

The **Eulerian formulation of the optimal transportation problem from  $\rho_0$  to  $\rho_1$  with cost induced by  $L$**  is the infimal problem

$$\inf_{\rho, V} \left\{ \int_{\mathbb{R}^d} \int_0^1 L(x, V(t, x)) \rho(t, x) dt dx \mid \rho|_{t=0} = \rho_0, \rho|_{t=1} = \rho_1, \dot{\rho} + \nabla \cdot (\rho V) = 0 \right\},$$

where  $(\rho_t)$  is a smooth curve in  $\mathcal{P}^{ac}(\mathbb{R}^d)$ , with  $\rho(t, x) = \rho_t(x)$ , and  $V$  is a time-varying vector field. The equation  $\dot{\rho} + \nabla \cdot (\rho V) = 0$  is called the **continuity equation**. The derivative with respect to time is denoted by  $\dot{X} = \frac{\partial}{\partial t} X$ , while  $\nabla$  and  $\nabla \cdot$  refer to the spatial gradient and spatial divergence, respectively. Let  $\sigma$  denote the characteristic curves of  $V$ . That is,

$$\dot{\sigma}(t, x) = V(t, \sigma(t, x)).$$

If  $(\rho, V)$  is an optimal pair in the Eulerian optimal transport problem, then the pair  $(\rho, V)$  also solves the following transport equations:

$$\dot{\rho} = -\nabla \cdot (\rho V) \quad \dot{V} = -V'V + W$$



where  $V'$  refers to the Jacobian matrix of spatial derivatives of  $V$  and

$$W(t, x) = \ddot{\sigma}(t, z) \Big|_{z=\sigma_t^{-1}(x)} = \frac{d^2}{dt^2} \sigma(t, z) \Big|_{z=\sigma_t^{-1}(x)}$$

More generally, these transport equations are satisfied by all pairs of smooth densities and time-varying vector fields which solve the continuity equation. The term  $W$  corresponds to the acceleration of the characteristics of  $V$  and will often be assumed to be 0. For instance, in the classical optimal transportation problem with quadratic cost on  $\mathbb{R}^d$ , it is identically zero.

If  $(\rho, V)$  is a smooth optimizer in the Eulerian optimal transport problem, then  $(\rho_t)$  is called a **smooth displacement interpolant** or **smooth geodesic in Wasserstein space**.

Let  $\mathcal{F} : [0, 1] \rightarrow \mathbb{R}$  be given by

$$\mathcal{F}(t) = \int F((D^\alpha \rho_t(x))_{|\alpha| \leq n}) dx, \quad (4.1)$$

where  $n$  is an integer,  $\alpha$  is a multi-index of length  $d$ ,  $N$  is the number of multi-indices of length  $d$  and order at most  $n$ , and  $F : \mathbb{R}^N \rightarrow \mathbb{R}$  is a smooth function.

In the smooth setting, the second derivative of  $\mathcal{F}$  is given by

$$\frac{d^2}{dt^2} \mathcal{F}(t) = \int_{\mathbb{R}^d} \frac{d^2}{dt^2} F((D^\alpha \rho_t(x))_{|\alpha| \leq n}) dx, \quad (4.2)$$

**Definition 4.1.1.** A functional  $\mathcal{F} : [0, 1] \rightarrow \mathbb{R}$  is **displacement convex** if it is convex along every displacement interpolant  $(\rho)$ .

The **displacement Hessian of  $\mathcal{F}$**  is the second derivative of  $\mathcal{F}$  computed along an arbitrary displacement interpolant.

There are three key identities needed to compute displacement Hessians (two of them are the transport equations from above).

$$\text{Key identities: } \begin{cases} \dot{\rho} = -\nabla \cdot (\rho V) & (4.3) \\ \dot{V} = -V'V + W & (4.4) \\ \ddot{\rho} = \nabla \cdot ((\nabla \cdot (\rho V)) V + \rho V'V - \rho W). & (4.5) \end{cases}$$

## 4.2 One derivative and one dimension

Let  $(\rho)$  be a displacement interpolant in  $\mathcal{P}^{ac}(\mathbb{R})$  or  $\mathcal{P}^{ac}(S^1)$ . Let  $\mathcal{F} : [0, 1] \rightarrow \mathbb{R}$  be the functional given by

$$\mathcal{F}(t) = \int_{\mathbb{R}} F(\rho, \rho') dx,$$

where  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a smooth function. Denote by  $F_0$  the partial derivative of  $F$  with respect to the first argument and by  $F_1$  the partial derivative with respect to the second argument.

The computation of the displacement Hessian for functionals of the above form, involving only one derivative of the probability density, which itself has a one dimensional domain, is simpler than the general computation and yields interesting examples. It will be computed first.

Let  $X$  be a time-varying vector field on  $\mathbb{R}$ . Let  $\Psi_0(\rho, X) = (\rho X)' - \rho' X$  and  $\Psi_1(\rho, X) = (\rho X)'' - \rho'' X$ . That is,  $\Psi_i(\rho, X)$  the remainder of  $(\rho X)^{(i+1)}$  after subtracting off all terms where  $X$  appears undifferentiated.

**Theorem 4.2.1.** *Let  $\mathcal{F} : [0, 1] \rightarrow \mathbb{R}$  be as above. The displacement Hessian of  $\mathcal{F}$  is*

$$\begin{aligned} \mathcal{F}''(t) = & \int F_{00} (\Psi_0(\rho, V))^2 + 2F_{01} (\Psi_0(\rho, V)) (\Psi_1(\rho, V)) + F_{11} (\Psi_1(\rho, V))^2 \\ & + F_1 \left( 2\rho' (V')^2 + 4\rho V'' V' \right) + F(W') \\ & - F_0 (\Psi_0(\rho, W)) - F_1 (\Psi_1(\rho, W)) dx. \end{aligned} \quad (4.6)$$

*Proof.* As all functions are assumed smooth,  $\mathcal{F}$  is differentiated twice:

$$\mathcal{F}''(t) = \int F_{00} (\dot{\rho})^2 + 2F_{01} (\dot{\rho}) (\dot{\rho}') + F_{11} (\dot{\rho}')^2 \quad (4.7)$$

$$+ F_0 (\ddot{\rho}) + F_1 (\ddot{\rho}') dx \quad (4.8)$$

Substituting the first key identity (4.3) into line (4.7) of the displacement Hessian yields

$$F_{00} (\dot{\rho})^2 + 2F_{01} (\dot{\rho}) (\dot{\rho}') + F_{11} (\dot{\rho}')^2 \quad (4.9)$$

$$= F_{00} ((\rho V)')^2 + 2F_{01} ((\rho V)') ((\rho V)'') + F_{11} ((\rho V)'')^2 \quad (4.10)$$

$$= F_{00} (\Psi_0 + \rho' V)^2 + 2F_{01} (\Psi_0 + \rho' V) (\Psi_1 + \rho'' V) + F_{11} (\Psi_1 + \rho'' V)^2 \quad (4.11)$$

$$\begin{aligned} & = F_{00} (\Psi_0)^2 + 2F_{01} (\Psi_0) (\Psi_1) + F_{11} (\Psi_1)^2 \\ & + \{F_{00}(\rho') + F_{01}(\rho'')\} (V(\rho' V + 2\Psi_0)) + \{F_{01}(\rho') + F_{11}(\rho'')\} (V(\rho'' V + 2\Psi_1)). \end{aligned} \quad (4.12)$$

Note that the terms in curly brackets are equal to, respectively,  $F'_0$  and  $F'_1$ . Substituting this back into the displacement Hessian and integrating by parts then yields

$$\begin{aligned} \mathcal{F}''(t) = & \int F_{00} (\Psi_0)^2 + 2F_{01} (\Psi_0) (\Psi_1) + F_{11} (\Psi_1)^2 \\ & + F_0 \left( \ddot{\rho} - (V(\rho' V + 2\Psi_0))' \right) + F_1 \left( \ddot{\rho}' - (V(\rho'' V + 2\Psi_1))' \right) dx. \end{aligned} \quad (4.13)$$

Substituting the third key identity (4.5) into the coefficient on  $F_0$  (and remembering that

$\Psi_0(\rho, V) = \rho V'$  results in

$$\ddot{\rho} - (V(\rho'V + 2\Psi_0))' = ((\rho V)'V + \rho V'V - \rho W)' - (\rho'V^2 + 2\rho V'V)' \quad (4.14)$$

$$= -(\rho W)' \quad (4.15)$$

$$= -(\rho'W + \Psi_0(\rho, W)). \quad (4.16)$$

Similarly, the coefficient on  $F_1$  simplifies to

$$\begin{aligned} (\ddot{\rho}' - (V(\rho''V + 2\Psi_1))') &= ((\rho V)'V + \rho V'V - \rho W)'' \\ &\quad - (\rho''V^2 + 2V(\rho V'' + 2\rho'V'))' \end{aligned} \quad (4.17)$$

$$= (2\rho(V')^2)' - (\rho W)'' \quad (4.18)$$

$$= (2\rho(V')^2)' - (\rho''W + \Psi_1(\rho, W)). \quad (4.19)$$

Substituting both (4.16) and (4.19) into (4.13) results in the following expression for the displacement Hessian

$$\begin{aligned} \mathcal{F}''(t) &= \int F_{00}(\Psi_0(\rho, V))^2 + 2F_{01}(\Psi_0(\rho, V))(\Psi_1(\rho, V)) + F_{11}(\Psi_1(\rho, V))^2 \\ &\quad + F_1(2\rho(V')^2) - F_0(\Psi_0(\rho, w)) - F_1(\Psi_1(\rho, w)) \\ &\quad - \underbrace{(F_0(\rho') + F_1(\rho''))}_{=(F)'} W dx. \end{aligned} \quad (4.20)$$

Integrating by parts completes the computation.  $\blacksquare$

In [11], Carrillo and Slepčev prove that the functional

$$E(\rho) = \int \left( \frac{d}{dx} \rho^\beta(t, x) \right)^2 dx$$

is displacement convex on  $S^1 \cong \mathbb{R}/\mathbb{Z}$  with cost given by Euclidean distance squared whenever  $\beta \in [-\frac{3}{2}, -1]$ .

**Example 4.2.1** (Carrillo-Slepčev functionals). Let  $F(x, y) = \beta^2 x^{2\beta-2} y^2$ . Then,

$$E(\rho) = \int F(\rho, \rho') dx.$$

Let  $\theta = 2\beta - 2$ . The derivatives of  $F$  are

$$\begin{cases} F_0 = \beta^2 \theta x^{\theta-1} y^2 \\ F_{00} = \beta^2 \theta (\theta - 1) x^{\theta-2} y^2 \\ F_{01} = 2\beta^2 \theta x^{\theta-1} y \\ F_1 = 2\beta x^\theta y \\ F_{11} = 2x^\theta. \end{cases}$$

Note that in the classical optimal transportation problem with Euclidean distance squared, the acceleration term  $W$  is identically 0. Then, using the formula from theorem 4.2.1 (eq. (4.6)), the displacement Hessian of  $E$  is

$$\begin{aligned} \frac{d^2}{dt^2} E(\rho) = \int \beta^2 \left\{ \theta(\theta - 1) \rho^{\theta-2} (\rho')^2 (\rho V')^2 + 2(2\theta \rho^{\theta-1} \rho') (\rho V') (\rho V'' + 2\rho' V') \right. \\ \left. + 2\rho^\theta (\rho V'' + 2\rho' V')^2 + 2\rho^\theta \rho' (2\rho' (V')^2 + 4\rho V'' V') \right\} dx. \end{aligned} \quad (4.21)$$

This is, in fact, a quadratic form in  $(\rho' V')$  and  $(\rho V'')$ . After collecting like terms, it can be rewritten as

$$\frac{d^2}{dt^2} E(\rho) = \int \beta^2 \rho^\theta \left\{ (\theta + 3)(\theta + 4) (\rho' V')^2 + 4(\theta + 4) (\rho' V') (\rho V'') + 2(\rho V'')^2 \right\} dx \quad (4.22)$$

$$= \int \beta^2 \rho^\theta \begin{pmatrix} \rho' V' \\ \rho V'' \end{pmatrix}^T \begin{pmatrix} (\theta + 3)(\theta + 4) & 2(\theta + 4) \\ 2(\theta + 4) & 2 \end{pmatrix} \begin{pmatrix} \rho' V' \\ \rho V'' \end{pmatrix} dx. \quad (4.23)$$

It is straightforward to check that

$$\begin{pmatrix} (\theta + 3)(\theta + 4) & 2(\theta + 4) \\ 2(\theta + 4) & 2 \end{pmatrix}$$

is a positive definite matrix whenever  $\theta \in (-5, -4)$  and is positive semi-definite for  $\theta = -5, -4$ . Hence, the functional  $E$  is displacement convex whenever  $\theta \in [-5, -4] \iff \beta \in [-3/2, -1]$ .  $\blacklozenge$

Following this example, it is straightforward to construct a new class of displacement convex functionals involving derivatives of densities. A simple convexity lemma will be needed.

**Lemma 4.2.2.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be non-negative and of class  $C^2$ . Let  $\gamma \in \mathbb{R} \setminus \{0\}$ . Then,*

$$f''(x)f(x) + (\gamma - 1)(f'(x))^2 \geq 0$$

*if  $g(x) = \frac{1}{\gamma} f^\gamma(x)$  is convex.*

*Proof.* The second derivative of  $g$  is

$$\begin{aligned} g''(x) &= \frac{1}{\gamma} (\gamma f^{\gamma-1}(x) f'(x))' \\ &= (\gamma - 1) f^{\gamma-2}(x) (f'(x))^2 + f^{\gamma-1}(x) f''(x) \\ &= f^{\gamma-2} (f''(x) f(x) + (\gamma - 1) (f'(x))^2). \end{aligned}$$

Since  $f$  is non-negative, the sign of  $g$  is the same as that of  $f''(x)f(x) + (\gamma - 1)(f'(x))^2$ . ■

A new class of displacement convex functionals is now presented.

**Theorem 4.2.3.** *Let  $\alpha > 1$ . Let  $H : \mathbb{R} \rightarrow \mathbb{R}$ . Suppose that  $H$  is of class  $C^2$ . If  $H(x) \geq 0$  for all  $x \geq 0$  and*

$$H^{-\frac{1}{\alpha-1}}$$

*is concave on  $[0, \infty)$ , then the functional*

$$\mathcal{F}(t) = \int H(\rho) \left( \frac{|\rho'|}{\rho^2} \right)^\alpha dx \quad (4.24)$$

*is displacement convex for the optimal transport problem on  $S^1$  with cost given by Euclidean distance squared.*

Note that, for instance, every functional of the form

$$\mathcal{F}(t) = \int \left( \frac{d}{dx} G(\rho) \right)^2 dx$$

can be written in the form of eq. (4.24) with  $\alpha = 2$ . The functionals considered by Carrillo and Slepčev in [11] are of this form, with  $G(\rho) = \rho^\beta$  and  $\beta \in [-3/2, -1]$ . A corollary of the main theorem extends Carrillo-Slepčev's result.

**Corollary 4.2.4** (Negative power-laws). *Let  $\alpha > 1$ . The functional*

$$\mathcal{E}(t) = \int \left| \frac{d}{dx} (\rho^{-\beta}(t, x)) \right|^\alpha dx$$

*is displacement convex on  $S^1$  if  $\beta \in [1, 2 - \frac{1}{\alpha}]$ .*

Taking limits of  $\mathcal{E}(t)$  as  $\alpha \rightarrow 1$  and  $\alpha \rightarrow \infty$  yields two additional results.

**Corollary 4.2.5** (Total variation norm). *The functional*

$$\mathcal{G}(t) = \|\rho^{-1}\|_{\text{Total Variation}} = \int \left| \frac{d}{dx} (\rho^{-1}(t, x)) \right| dx$$

*is displacement convex on  $S^1$ .*

Closely related functionals appear in [10, eq. 1.1].

**Corollary 4.2.6** (Lipschitz norm). *The functional*

$$\mathcal{H}(t) = \text{Lip} \left( \left| \rho^{-\beta}(t, x) \right| \right) = \left\| \frac{d}{dx} \left( \rho^{-\beta} \right) \right\|_{L^\infty}$$

is displacement quasiconvex on  $S^1$  for  $\beta \in [1, 2]$ . That is,  $\mathcal{H}(t) \leq \max\{\mathcal{H}(0), \mathcal{H}(1)\}$  for  $t \in [0, 1]$ .

*Proof of theorem 4.2.3.* Let  $\mathcal{F} = \int H(\rho) |\rho^{-2}\rho'|^\alpha dx$ . Write  $F(x, y) = x^{-2\alpha}H(x)|y|^\alpha$ . The derivatives of  $F$  are

$$\begin{cases} F_0 = [-2\alpha x^{-2\alpha-1}H(x) + x^{-2\alpha}H'(x)] |y|^\alpha \\ F_{00} = [2\alpha(2\alpha+1)x^{-2\alpha-2}H(x) - 4\alpha x^{-2\alpha-1}H'(x) + x^{-2\alpha}H''(x)] |y|^\alpha \\ F_{01} = \alpha [-2\alpha x^{-2\alpha-1}H(x) + x^{-2\alpha}H'(x)] y^{\alpha-1} \text{sign}(y) \\ F_1 = \alpha x^{-2\alpha}H(x) y^{\alpha-1} \text{sign}(y) \\ F_{11} = \alpha(\alpha-1)x^{-2\alpha}H(x)|y|^{\alpha-2}. \end{cases}$$

Then, using theorem 4.2.1, the displacement Hessian of  $\mathcal{F}$  is

$$\begin{aligned} \mathcal{F}''(t) &= \int |\rho'|^{\alpha-2} \rho^{-2\alpha} \left\{ H''(\rho)\rho^2 (\rho'V')^2 + (2\alpha H'(\rho)\rho) (\rho'V')(\rho V'') \right. \\ &\quad \left. + (\alpha(\alpha-1)H(\rho)) (\rho V'')^2 \right\} dx \end{aligned} \quad (4.25)$$

$$= \int |\rho'|^{\alpha-2} \begin{pmatrix} \rho'V' \\ \rho V'' \end{pmatrix}^T A \begin{pmatrix} \rho'V' \\ \rho V'' \end{pmatrix} dx. \quad (4.26)$$

$A$  is the  $2 \times 2$  matrix

$$A = \rho^{-2\alpha} \begin{pmatrix} H''(\rho)\rho^2 & \alpha H'(\rho)\rho \\ \alpha H'(\rho)\rho & \alpha(\alpha-1)H(\rho) \end{pmatrix}. \quad (4.27)$$

The functional  $\mathcal{F}$  is displacement convex whenever  $H$  is chosen such that for all  $x > 0$ , the following two conditions hold

$$\begin{cases} x^{-2\alpha}H(x) > 0 & \iff H(x) > 0 \\ \frac{\alpha(\alpha-1)H''(x)H(x) - \alpha^2(H'(x))^2}{x^{2\alpha-2}} > 0 & \iff H''(x)H(x) - \frac{\alpha}{\alpha-1}(H'(x))^2 > 0. \end{cases}$$

Applying lemma 4.2.2 to the second expression completes the proof.  $\blacksquare$

*Proof of corollary 4.2.4.* Apply theorem 4.2.3 with  $H(x) = x^{\alpha(1-\beta)}$ .  $\blacksquare$

*Proof of corollary 4.2.5.* From corollary 4.2.4, the functional  $\mathcal{E}(t)$  is displacement convex. Take the limit as  $\alpha \rightarrow 1$ .  $\blacksquare$

*Proof of corollary 4.2.6.* From corollary 4.2.4, the functional  $\mathcal{E}(t)$  is displacement convex. Taking the limit as  $\alpha \rightarrow \infty$  yields

$$\lim_{\alpha \rightarrow \infty} |\mathcal{E}(t)|^{1/\alpha} \leq \lim_{\alpha \rightarrow \infty} |(1-t)\mathcal{E}(0) + t\mathcal{E}(1)|^{1/\alpha} \leq \lim_{\alpha \rightarrow \infty} |\max\{\mathcal{E}(0), \mathcal{E}(1)\}|^{1/\alpha}. \quad \blacksquare$$

In the proof of theorem 4.2.3, the positive definiteness of the matrix  $A$ , from equation (4.27), was found to be a sufficient condition for displacement convexity. It is an open problem to find necessary conditions for displacement convexity.

There is, however, a partial converse to theorem 4.2.3: if the concavity of  $H^{-1/(\alpha-1)}$  is replaced with the hypothesis that  $H''(c) < 0$  for some  $c > 0$ , keeping all other hypotheses fixed, then the functional  $\int H(\rho) |\rho^{-2}\rho'|^\alpha dx$  will not be displacement convex. (This is a strictly stronger assumption than  $H^{-1/(\alpha-1)}$  failing to be convex.) This partial converse is sketched in the following example.

**Example 4.2.2.** Let  $\mathcal{F}(t) = \int H(\rho) |\rho^{-2}\rho'|^\alpha dx$ . Suppose the hypotheses of theorem 4.2.3 hold except for the concavity of  $H^{-1/(\alpha-1)}$ . Suppose, instead, that  $H''(c) < 0$  for some  $c > 0$ . Then,  $\mathcal{F}$  is not displacement convex.

The key idea of the counterexample is to construct a displacement interpolant  $(\rho, V)$  with the following property: on an interval  $I$ ,

$$\begin{cases} \rho|_I(x) = c + \delta \sin\left(\frac{x}{\epsilon}\right) \\ V''|_I \equiv 0. \end{cases} \quad (4.28)$$

The numbers  $\delta$  and  $\epsilon$  will be specified later.

Let  $\xi(x) = x^{2-2\alpha}H''(x)$ , the (1,1) entry in the matrix  $A$  given in equation (4.27). Suppose that  $(\rho, V)$  has the desired properties from (4.28). Using the formula from equation (4.26) for the displacement Hessian of  $\mathcal{F}$  and restricting attention to the interval  $I$  (and abusing notation),

$$\mathcal{F}''(t)|_I = \int_I |\rho'|^{\alpha-2} \begin{pmatrix} \rho'V' \\ \rho V'' \end{pmatrix}^T A \begin{pmatrix} \rho'V' \\ \rho V'' \end{pmatrix} dx = \int_I |\rho'|^{\alpha-2} (\rho')^2 \xi(\rho) dx. \quad (4.29)$$

By taking  $\delta$  sufficiently small,  $\rho_0(x)$  can be made arbitrarily close to  $c$  and by taking  $\epsilon$  sufficiently small,  $|\rho'_0|$  can be made as large as desired. Then, at time  $t = 0$ ,

$$\mathcal{F}''(0)|_I = \int_I |\rho'_0|^{\alpha-2} (\rho'_0)^2 \xi(\rho_0) dx \approx \int_I |\rho'_0|^\alpha \xi(c) dx \quad (4.30)$$

which is negative and, by the choice of  $\epsilon$ , can be chosen to have arbitrarily large magnitude; the ‘‘approximately equals’’ can be made precise as the limit of the integral as  $\delta, \epsilon \rightarrow 0$ . In particular, equation (4.30) can be made sufficiently large and negative to ensure that  $\mathcal{F}''(0)$ , which also depends on  $\rho_0$  and  $V_0$  on  $S^1 \setminus I$ , is negative.

A displacement interpolant  $(\rho, V)$  with the desired properties from (4.28) can indeed be

found: a displacement interpolant  $(\rho, V)$  is fully characterized by an initial density  $\rho_0$  and an admissible Kantorovich potential  $\phi$  which defines the vector field  $V$ ; this potential function can be chosen independently of the initial density ([36], [46], theorem 3.5.2). On  $S^1 \cong \mathbb{R}/\mathbb{Z}$ , admissible potentials define vector fields via the equation  $V = \phi' - id$  ([36]). Because there is no restriction on the initial measure, all that remains is to show that for an interval  $I$  on  $S^1$  there is admissible potential function  $\phi$  such that its vector field  $V$  is linear on  $I$ .

The desired optimal potential will be constructed as follows. Let  $\eta_0$  be identically  $c$  on an interval  $I$  and smoothly extend  $\eta_0$  on  $S^1$  to ensure that it is a probability density (and if desired, bounded away from 0). Let  $\eta_1$  be uniform measure on  $S^1$ . Then, the potential  $\phi$  for the optimal transport problem from  $\eta_0$  to  $\eta_1$  satisfies the Monge-Ampere equation on  $S^1$ :

$$\eta_1(\phi(x))\phi'(x) = \eta_0(x).$$

Since  $\eta_0$  is constant on  $I$  and  $\eta_1$  is constant on all of  $S^1$ , it follows that  $\phi'(x)$  must be constant on  $I$ . Hence, the corresponding vector field  $V = \phi' - id$  is linear on  $I$ .  $\blacklozenge$

### 4.3 A General Formula for the Displacement Hessian

As in the beginning of the chapter, let  $\mathcal{F} : [0, 1] \rightarrow \mathbb{R}$  be given by

$$\mathcal{F}(t) = \int_{\mathbb{R}^d} F((D^\alpha \rho_t(x))_{|\alpha| \leq n}) dx,$$

where  $n$  is an integer,  $\alpha$  is a multi-index of length  $d$ ,  $N$  is the number of multi-indices of length  $d$  and order at most  $n$ , and  $F : \mathbb{R}^N \rightarrow \mathbb{R}$  is a smooth function.

Let  $X : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . Let

$$\Psi_\alpha(\rho, X) = D^\alpha \nabla \cdot (\rho X) - (\nabla D^\alpha \rho) X. \quad (4.31)$$

That is,  $\Psi_\alpha(\rho, X)$  is the remainder of  $D^\alpha \nabla \cdot (\rho X)$  after subtracting off all terms where  $X$  appears undifferentiated. Similarly, define

$$\begin{aligned} \Phi_\alpha(\rho, X) &= D^\alpha \nabla \cdot ((\nabla \cdot (\rho X) X) + \rho X' X) \\ &\quad - 2X \cdot \nabla D^\alpha (\nabla \cdot (\rho X)) + X \cdot (D^\alpha \rho'') X. \end{aligned} \quad (4.32)$$

Less obviously,  $\Phi_\alpha(\rho, X)$  is the remainder of  $D^\alpha \nabla \cdot ((\nabla \cdot (\rho X) X) + \rho X' X)$  after subtracting off all terms where  $X$  appears undifferentiated.

**Proposition 4.3.1.** *Let  $X : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be smooth. Then,*

$$\Phi_\alpha(\rho, X) = D^\alpha \nabla \cdot ((\nabla \cdot (\rho X) X) + \rho X' X) - 2X \cdot \nabla D^\alpha (\nabla \cdot (\rho X)) + X \cdot (D^\alpha \rho'') X$$



and

$$\Psi_\alpha(\rho, X) = D^\alpha \nabla \cdot (\rho X) - (\nabla D^\alpha \rho) X.$$

contains no terms where  $X$  is undifferentiated.

*Proof.* Let  $[a, b] = ab - ba$  denote the commutator of two linear operators  $a$  and  $b$ . Writing out  $\Psi$  and  $\Phi$  in coordinates reveals them to be sums of commutators:

$$\begin{aligned} \Psi_\alpha(\rho, V) &= \sum_{i=1}^d \{D^\alpha D_i V_i - V_i D^\alpha D_i\}(\rho) \\ &= \sum_{i=1}^d [D^\alpha D_i, V_i](\rho), \end{aligned}$$

and

$$\begin{aligned} \Phi_\alpha(\rho, V) &= \sum_{i,j=1}^d \{D^\alpha D_i D_j V_i V_j - 2V_i D^\alpha D_i D_j V_j + V_i V_j D^\alpha D_i D_j\}(\rho) \\ &= \sum_{i,j=1}^d [[D^\alpha D_i D_j, V_i], V_j](\rho). \end{aligned}$$

Given any differential operator  $L$  of order  $k$  and any smooth function  $f$ , the commutator  $[L, f]$  is a differential operator of order  $k - 1$ . (This can be seen, for instance, by induction on the degree of multi-indices, with the base case  $[\frac{\partial}{\partial x_i}, f](g) = g \frac{\partial f}{\partial x_i}$ .)

Each summand  $[D^\alpha D_i, V_i](\rho)$  of  $\Psi$  therefore contains no terms where  $V_i$  appears undifferentiated, which can also be seen by simply expanding out the commutator.

Since  $D^\alpha D_i$  is a differential operator of order at least 2,  $[D^\alpha D_i D_j, V_i]$  is a differential operator of order at least 1, and therefore  $[[D^\alpha D_i D_j, V_i], V_j]$  contains no terms where  $V_i$  or  $V_j$  appear undifferentiated. ■

**Theorem 4.3.2** (The Hessian of  $\mathcal{F}$ ). *Let  $(\rho_t)$  be a smooth curve in  $\mathcal{P}^{ac}(\mathbb{R}^d)$ , let  $V_t$  be a time-dependent smooth vector field on  $\mathbb{R}^d$ , and let  $\mathcal{F}$  be a functional of the form as described above. If  $(\rho_t, V_t)$  solve the continuity equation, eq. (4.3), then the displacement Hessian of  $\mathcal{F}$  is*

$$\begin{aligned} \frac{d^2}{dt^2} \mathcal{F} &= \int_{\mathbb{R}^d} \sum_{|\alpha|, |\beta| \leq n} F_{\alpha, \beta} \Psi_\alpha(\rho, V) \Psi_\beta(\rho, V) dx \\ &\quad + \int_{\mathbb{R}^d} \sum_{|\alpha| \leq n} F_\alpha \{ \Phi_\alpha(\rho, V) - 2(\nabla \cdot V) \Psi_\alpha(\rho, V) - \Psi_\alpha(\rho, W) \} dx \\ &\quad + \int_{\mathbb{R}^d} F \{ (\nabla \cdot V)^2 - \text{tr}((V')^2) + \nabla \cdot W \} dx. \end{aligned} \tag{4.33}$$

*Remark.* In the Riemannian setting, with the transport cost given by squared Riemannian

distance, the term  $-\text{tr}((V')^2) + \nabla \cdot W$  corresponds to the Bakry-Emery tensor (see appendix A). This is where the non-negative Ricci curvature condition associated to the displacement convexity of the entropy functional can be seen.

Hence, in the Riemannian setting, the displacement Hessian of  $\mathcal{F}$  is a quadratic form in the derivatives of  $V$  of order  $0, \dots, n+1$  whose coefficients depend on  $(D^\alpha \rho)_{|\alpha| \leq n}$ , the function  $F$ , and its first and second partial derivatives. The 0<sup>th</sup> order terms of  $V$  only appear in  $\nabla \cdot W$ .

*Proof of theorem 4.3.2.* The integral formula for the displacement Hessian of  $\mathcal{F}$  is given in eq. (4.2).

By the chain rule, the integrand is can be rewritten as

$$\frac{d^2}{dt^2} \mathcal{F}((D^\alpha \rho)_{|\alpha| \leq n}) = \sum_{|\alpha|, |\beta| \leq n}^n F_{\alpha, \beta} D^\alpha(\dot{\rho}) D^\beta(\dot{\rho}) + \sum_{|\alpha| \leq n} F_\alpha D^\alpha(\ddot{\rho}). \quad (4.34)$$

From the key identities eq. (4.3) and (4.5), the time derivatives  $\dot{\rho}$  and  $\ddot{\rho}$  can be rewritten as expressions involving spatial gradients and Hessians of  $\rho$ . Upon substituting these identities into eq. (4.34), the double sum becomes an expression involving spatial derivatives of  $\rho$  up to order  $n+1$ , and up to order  $n+2$  in the other sum.

The displacement Hessian will be rewritten to eliminate all derivatives of  $\rho$  of order greater than  $n$ .

By equations (4.3), (4.4), and (4.31), the factors in the double sum are given by

$$\begin{aligned} D^\alpha \dot{\rho} &= -D^\alpha \nabla \cdot (\rho V) \\ &= -\Psi_\alpha(\rho, V) - V \cdot \nabla D^\alpha \rho. \end{aligned}$$

By the equality of mixed partials,  $F_{\alpha, \beta} = F_{\beta, \alpha}$ ,

$$\begin{aligned} \sum_{|\alpha|, |\beta| \leq n}^n F_{\alpha, \beta} D^\alpha(\dot{\rho}) D^\beta(\dot{\rho}) &= \sum_{|\alpha|, |\beta| \leq n}^n F_{\alpha, \beta} \Psi_\alpha(\rho, V) \Psi_\beta(\rho, V) \\ &+ \sum_{|\alpha| \leq n} \left( \sum_{|\beta| \leq n} F_{\alpha, \beta} V \cdot \nabla D^\beta \rho \right) (2D^\alpha \nabla \cdot (\rho V) - V \cdot \nabla D^\alpha \rho). \end{aligned} \quad (4.35)$$

The sum in the large parentheses can be simplified with the help of the chain rule,

$$\sum_{|\beta| \leq n} F_{\alpha, \beta} (V \cdot \nabla D^\beta \rho) = \nabla F_\alpha \cdot V. \quad (4.36)$$

where  $\nabla F$  denotes the spatial gradient of the composition  $F((D^\alpha \rho)_{|\alpha| \leq n})$ . Integrating equa-

tion (4.35) yields, with an integration by parts on the last term,

$$\begin{aligned}
& \int_{\mathbb{R}^d} \sum_{|\alpha|, |\beta| \leq n} F_{\alpha, \beta} D^\alpha(\dot{\rho}) D^\beta(\dot{\rho}) dx \\
&= \int_{\mathbb{R}^d} \sum_{|\alpha|, |\beta| \leq n} F_{\alpha, \beta} \Psi_\alpha(\rho, V) \Psi_\beta(\rho, V) dx \\
&\quad - \int_{\mathbb{R}^d} \sum_{|\alpha| \leq n} F_\alpha \nabla \cdot \{ (2D^\alpha \nabla \cdot (\rho V) - V \cdot \nabla D^\alpha \rho) V \} dx. \tag{4.37}
\end{aligned}$$

The contribution of  $\ddot{\rho}$  to the integrand in equation (4.34) is given by equation (4.5):

$$D^\alpha \ddot{\rho} = D^\alpha \nabla \cdot (\nabla \cdot (\rho V) V + \rho V' V - \rho W).$$

This term is added to the summands in last integral of Eq. (4.37). Expanding in components yields

$$\begin{aligned}
& D^\alpha \ddot{\rho} - 2\nabla \cdot ((D^\alpha \nabla \cdot (\rho V) V) + \nabla \cdot (V \cdot \nabla D^\alpha \rho) V) \\
&= \sum_{i, j=1}^d D_i \{ D^\alpha D_j V_i V_j - 2V_i D^\alpha D_j V_j + V_i V_j D^\alpha D_j \}(\rho) - \sum_{i=1}^d D^\alpha D_i W_i(\rho) \tag{4.38}
\end{aligned}$$

$$\begin{aligned}
&= \Phi_\alpha(\rho, V) - \Psi_\alpha(\rho, W) - W \cdot \nabla D^\alpha \rho \\
&\quad + \sum_{i, j=1}^d \left\{ -2[D_i, V_i] D^\alpha D_j V_j + [D_i, V_i] V_j D^\alpha D_j + V_i [D_i, V_j] D^\alpha D_j \right\}(\rho). \tag{4.39}
\end{aligned}$$

Singling out the terms where  $V$  appear without derivatives,

$$\sum_{i, j=1}^d \left\{ -[D_i, V_i] V_j D^\alpha D_j + V_i [D_i, V_j] D^\alpha D_j \right\}(\rho) = ((\nabla \cdot V) V - V' V) \cdot \nabla D^\alpha \rho,$$

equation (4.38) becomes

$$\begin{aligned}
& D^\alpha \ddot{\rho} - 2\nabla \cdot ((D^\alpha \nabla \cdot (\rho V) V) + \nabla \cdot (V \cdot \nabla D^\alpha \rho) V) \\
&= \Phi_\alpha(\rho, V) - 2(\nabla \cdot V) \Psi_\alpha(\rho, V) - \Psi_\alpha(\rho, W) - ((\nabla \cdot V) V - V' V + W) \cdot \nabla D^\alpha \rho. \tag{4.40}
\end{aligned}$$

Inserting equations (4.37) and (4.40) into equation (4.34) yields

$$\frac{d^2}{dt^2} \mathcal{F} = \int_{\mathbb{R}^d} \sum_{|\alpha|, |\beta| \leq n} F_{\alpha, \beta} D^\alpha(\dot{\rho}) D^\beta(\dot{\rho}) + \sum_{|\alpha| \leq n} F_\alpha D^\alpha(\ddot{\rho}) dx \quad (4.41)$$

$$\begin{aligned} &= \int_{\mathbb{R}^d} \sum_{|\alpha|, |\beta| \leq n} F_{\alpha, \beta} \Psi_\alpha(\rho, V) \Psi_\beta(\rho, V) dx \\ &\quad + \int_{\mathbb{R}^d} \sum_{|\alpha| \leq n} F_\alpha \{ \Phi_\alpha(\rho, V) - 2(\nabla \cdot V) \Psi_\alpha(\rho, V) - \Psi_\alpha(\rho, W) \} dx \\ &\quad - \int_{\mathbb{R}^d} \sum_{|\alpha| \leq n} F_\alpha ((\nabla \cdot V)V - V'V + W) \cdot \nabla D^\alpha \rho dx. \end{aligned} \quad (4.42)$$

Since

$$\sum_{|\alpha| \leq n} F_\alpha \nabla D^\alpha \rho = \nabla F,$$

by the chain rule, the final integral in equation (4.42) equals

$$- \int_{\mathbb{R}^d} \nabla F \cdot ((\nabla \cdot V)V - V'V + W) dx = \int_{\mathbb{R}^d} F \nabla \cdot ((\nabla \cdot V)V - V'V + W) dx.$$

The identity  $\nabla \cdot ((\nabla \cdot V)V - V'V) = (\nabla \cdot V)^2 - \text{tr}((V')^2)$  completes the proof of Eq. (4.33).  $\blacksquare$

**Example 4.3.1** (Carrillo-Slepčev functionals on  $\mathbb{R}^d$ ). Let  $(\rho)$  be a geodesic in  $\mathcal{P}^{ac}(\mathbb{R}^d)$ . Let  $\mathcal{F}$  be given by

$$\mathcal{F}(t) = \int \rho^\beta \|\nabla \rho\|^{2\alpha} dx. \quad (4.43)$$

Using the displacement Hessian formula (4.33), it can be (very tediously) checked that the

displacement Hessian of  $\mathcal{F}$  is, assuming that the  $W$  term is identically zero, given by

$$\begin{aligned}
\mathcal{F}''(t) = & \int \rho^\beta \|\nabla \rho\|^{2\alpha} (1 - 2\alpha - \beta) ((\nabla \cdot V)^2 - \text{tr}((V')^2)) \\
& + \rho^\beta \|\nabla \rho\|^{2\alpha} \left\{ (\beta(\beta - 1) + 2\alpha\beta + 2\alpha + 8\alpha(\alpha - 1)) \|\nabla \rho\|^2 (\nabla \cdot V)^2 \right. \\
& \quad + 2\alpha \nabla \rho^T ((V'^T)^2) \nabla \rho + (2\alpha(1 + \beta) + 16) (\nabla \cdot V) \nabla \rho^T V'^T \nabla \rho \\
& \quad + 2\alpha \|V'^T \nabla \rho\|^2 + 2\alpha \rho \nabla \rho \cdot \nabla \text{tr}((V')^2) + 4\alpha \rho \nabla \rho^T V'^T \nabla (\nabla \cdot V) \\
& \quad + 4\alpha \rho \nabla (\nabla \cdot V)^T V'^T \nabla \rho + [2\alpha(2 + \beta) + 8\alpha(\alpha + 1)] \rho (\nabla \cdot V) \nabla \rho^T \nabla (\nabla \cdot V) \\
& \quad \left. + 2\alpha \rho^2 \|\nabla (\nabla \cdot V)\|^2 \right\} \\
& + \rho^\beta \|\nabla \rho\|^{2\alpha-4} 8\alpha(\alpha - 1) \left\{ (\nabla \rho^T V'^T \nabla \rho) (\nabla \rho^T V'^T \nabla \rho) \right. \\
& \quad \left. + (\rho \nabla \rho^T \nabla (\nabla \cdot V)) (\rho \nabla \rho^T \nabla (\nabla \cdot V)) + 2\rho \nabla \rho^T V' \nabla \rho \nabla \rho^T \nabla (\nabla \cdot V) \right\} dx.
\end{aligned}$$

Experiments with Mathematica suggest that there are no choices of  $\alpha$  and  $\beta$  for which the integrand is a positive definite quadratic form. In particular, the term  $\nabla \text{tr}((V')^2)$  has terms of the form  $\partial_i V \partial_i \partial_j V$ , which occur linearly in the integrand, so it is likely  $V$  may be chosen to make the integrand negative. Certainly, this ensures that the integrand is not a positive definite quadratic form. However, the additional structure of the optimal transport problem may place constraints on admissible vector fields  $V$ .

Perhaps in the classical case on  $\mathbb{R}^d$ , where  $V = T - I = \nabla \phi$  for some function  $\phi$ , there is a simplification, as  $V'$  is a symmetric matrix, but such a simplification is not apparent to me.

◆

From working through small examples, it appears that, in general, displacement convexity fails to hold in higher dimensions for functionals involving derivatives of densities. Of course, working through small examples does not constitute a proof. In addition, even if the *integrand* fails to be positive, it is possible that cancellations occur so the *integral* may be positive. Regardless, I am sufficiently convinced that there are no displacement convex functionals involving derivatives in higher dimensions that I will record it as a conjecture.

**Conjecture 4.3.1.** There are no displacement convex functionals of the form given in equation (4.1) involving derivatives of densities in dimension  $d \geq 2$ .

## 4.4 Epilogue: Issues of regularity

This chapter constructs an Eulerian extension of the Otto calculus which allows for the computation of displacement Hessians of functionals depending on derivatives of densities, and shows how such a concept is fruitful, by constructing a new class of displacement convex functionals. But, the question remains as to whether this is merely a formal exercise.

For the functionals found in theorem 4.2.3, the answer is, thankfully, no: these functionals really are displacement convex. An analogous approximation scheme as that used in [11] carries through for these functionals. If  $(\rho)$  is a geodesic on  $\mathcal{P}^{ac}(S^1)$  for which the functional  $\int G(\rho)|\rho'|^\alpha dx$  is finite, then  $\rho$  can be checked to be bounded away from 0 and  $\infty$ , and can be approximated by densities in  $H^1$ , which can then be approximated by smooth densities.

With some qualifications, the general Hessian formula from theorem 4.3.2 also holds in a rigorous sense. In the “Ma-Trudinger Wang” setting, displacement interpolants are sufficiently smooth and the computation in the proof of theorem 4.3.2 carries through unchanged. In general, however, displacement interpolants lack sufficient smoothness for the computation in the proof of theorem 4.3.2 to carry through. Nevertheless, the result holds in the sense of distributions; the proof in this case relies on the DiPerna-Lions theory of renormalized solutions ([18]).

The computation in the proof of theorem 4.3.2 is valid when displacement interpolants are sufficiently smooth, which occurs in the “Ma-Trudinger-Wang” setting: smooth displacement interpolants are ensured under appropriate hypotheses on the cost function, underlying geometry, and initial and final endpoint densities. The first step in this direction was due to Caffarelli in [9]: with the cost given by squared Euclidean distance on  $\mathbb{R}^d$ , if both the initial and final densities are smooth, supported on convex sets, and are bounded above and below by strictly positive constants on their support, then the displacement interpolant between them will be smooth as well. This work was then extended by Ma, Trudinger, and Wang, in [32], to more general cost functions satisfying particular fourth derivative conditions, with similar convexity and positivity hypotheses on the endpoint densities as in the Caffarelli regularity setting. An overview of the Ma-Trudinger-Wang theory can be found in the survey paper [20] and in [46, Chapter 12]. Lee and McCann, in [31], verify that several cost functions induced by Lagrangians satisfy the Ma-Trudinger-Wang conditions.

In the setting of  $\mathbb{R}^d$  with the transport cost induced by a Tonelli Lagrangian, the optimal vector field  $V$  corresponding to a geodesic  $(\rho)$  is only locally Lipschitz in position. Given densities  $\rho_0$  and  $\rho_1$ , the corresponding optimal vector field  $V$ , flowing  $\rho_0$  to  $\rho_1$ , preserves absolute continuity, so intermediate measures  $\rho_t$  do not have singularities. In general, though, no additional regularity properties held by endpoint densities are preserved at intermediate times. Even if  $\rho_0$  and  $\rho_1$  are both smooth, an intermediate density  $\rho_t$  along a geodesic will not have any regularity beyond remaining absolutely continuous, both because general Tonelli Lagrangian induced costs may fail to satisfy the Ma-Trudinger-Wang conditions and because most displacement interpolants do not have endpoints satisfying the necessary convexity and positivity hypotheses.

There are obstructions to standard techniques for smoothly approximating displacement interpolants, carrying out the computation in the proof of theorem 4.3.2 in the smooth setting, and arguing that the result holds in general. For example, there are at least two obstructions to approximating a displacement interpolant by smoothing the Kantorovich potential correspond-

ing to the optimal vector field. First, for some cost functions, smooth functions fail to be dense in the space of Kantorovich potentials [46, Theorem 12.20]. Second, even when they are dense (for instance, in the classical case on  $\mathbb{R}^d$ ), smooth approximations of displacement interpolants only converge to the true displacement interpolants weakly as measures ([2]).

A more direct approach is to approximate a displacement interpolant  $(\rho, V)$  by the pair

$$\rho_\epsilon = \rho * \eta_\epsilon \quad \text{and} \quad V_\epsilon = \frac{(\rho V) * \eta_\epsilon}{\rho_\epsilon},$$

where  $\eta$  is a standard mollifier. Then,  $(\rho_\epsilon, V_\epsilon)$  solves the continuity equation on the support of  $\rho_\epsilon$ . In trying to understand the convergence of a functional  $\int F(\rho_\epsilon)$  as  $\epsilon \rightarrow 0$ , one is led, by considering the difference between  $\nabla \cdot V_\epsilon$  and  $\nabla \cdot V$ , to the term

$$(\nabla \rho \cdot V) * \eta_\epsilon \rho_\epsilon - (\rho V) * \eta_\epsilon \nabla \rho_\epsilon.$$

Happily, a “commutator estimate” for closely related differences is understood. It is a useful technical lemma in [18, Lemma 2.1]; in this paper, DiPerna and Lions develop (among other things) a theory of *renormalized solutions*. A renormalized solution of a differential equation is a weak solution which also satisfies the chain rule; DiPerna and Lions specify minimal hypotheses on a differential equation to ensure that a weak solution will also be a renormalized solution.

In particular, displacement interpolants with endpoint densities which are compactly supported and bounded in  $L^\infty$  satisfy the hypotheses to be renormalized solutions to the continuity equation for intermediate times  $t \in (0, 1)$ . Applying the DiPerna-Lions theory then proves theorem 4.3.2 for zeroth order functionals, understanding equation (4.33) to hold in the sense of distributions on  $(0, 1)$ .

The DiPerna-Lions theory also makes rigorous theorem 4.3.2 for higher order functionals with an additional hypothesis. With the hypothesis that the endpoint densities of a displacement interpolant are compactly supported and bounded in  $L^\infty$ , equation (4.33) holds in the sense of distribution on  $(0, 1)$  with the extra hypothesis that the right hand side of the equation is well defined. This extra hypothesis is necessary, because, regardless of the degree of regularity of the endpoint densities of a displacement interpolant, intermediate densities along a displacement interpolant need not have any derivatives.

Finally, although not an issue of regularity, there is one question I would like to highlight.

**Question 4.4.1.** Let  $(\rho, V)$  be a smooth minimizer in the Eulerian optimal transport problem. Do the terms  $\Phi_\alpha(\rho, V)$ ,  $\Psi_\alpha(\rho, V)$ , and

$$\nabla \cdot ((\nabla \cdot V)V - V'V) = (\nabla \cdot V)^2 - \text{tr}((V')^2)$$

have a geometric interpretation?

# Appendix A

## Eulerian and Otto Calculus Computations - Riemannian Setting

### A.1 Preliminaries

Let  $g$  be a Riemannian metric on  $\mathbb{R}^d$ . That is,  $g$  is a smooth map from  $\mathbb{R}^d$  to the space of positive definite  $d \times d$  real matrices. The  $(i, j)$ -entry of  $g(x)$  will be denoted  $g_{ij}(x)$ . When there is no ambiguity, the argument  $x$  will be omitted. In this appendix, Einstein summation notation will be used. The  $(i, j)$ -entry of the inverse matrix of  $g$  will be denoted  $g^{ij}$ .

Let  $L : T\mathbb{R}^d \rightarrow \mathbb{R}$  be given by

$$L(x, v) = \frac{1}{2}v^T g(x)v = \frac{1}{2}v^i v^j g_{ij}. \quad (\text{A.1})$$

Let  $(\rho, V)$  be a geodesic in Wasserstein space. Let  $\varphi$  be the corresponding Kantorovich potential, defined via

$$-D\varphi_t(x) = D_v L(x, V_t(x)). \quad (\text{A.2})$$

Consider the entropy functional

$$\mathcal{F}(t) = \int \rho \log(\rho) dx. \quad (\text{A.3})$$

If  $F(x) = x \log(x)$ , then, for  $x \geq 0$ ,

$$\begin{cases} p(x) = xF'(x) - F(x) = x \\ p_2(x) = x^2 F''(x) - xF'(x) + F(x) = 0. \end{cases} \quad (\text{A.4})$$



My formal displacement Hessian of  $\mathcal{F}$  is

$$\begin{aligned} \mathcal{F}''(t) &= \int \rho^2 F''(\rho) (\nabla \cdot V)^2 \\ &\quad + [F(\rho) - \rho F'(\rho)] [(\nabla \cdot V)^2 - \text{tr}(DV^2) + \nabla \cdot W] dx \end{aligned} \quad (\text{A.5})$$

$$= \int p_2(\rho) (\nabla \cdot V)^2 + p(\rho) [\text{tr}(DV^2) - \nabla \cdot W] dx \quad (\text{A.6})$$

$$= \int \rho [\text{tr}(DV^2) - \nabla \cdot W] dx. \quad (\text{A.7})$$

where various derivatives are the standard derivatives are  $\mathbb{R}^d$ . The function  $W$  is defined as follows:

Let  $\sigma$  be a solution to the Euler-Lagrange equation

$$\begin{cases} \frac{d}{dt} \nabla_v L(\sigma, \dot{\sigma}) = \nabla_x L(\sigma, \dot{\sigma}) \\ (\sigma, \dot{\sigma})(x, 0) = (x, V_0(x)) \end{cases}$$

Expanding and rearranging the Euler-Lagrange equation yields,

$$\begin{aligned} \ddot{\sigma}(x, t) &= (D_v \nabla_v L(\sigma(x, t), \dot{\sigma}(x, t)))^{-1} \\ &\quad \times [\nabla_x L(\sigma(x, t), \dot{\sigma}(x, t)) - D_x \nabla_v L(\sigma(x, t), \dot{\sigma}(x, t)) \dot{\sigma}(x, t)]. \end{aligned} \quad (\text{A.8})$$

Finally, noting that

$$\begin{cases} \sigma(\sigma_t^{-1}(x), t) = x \\ \dot{\sigma}(x, t) = V(\sigma(x, t), t) \iff \dot{\sigma}(\sigma_t^{-1}(x), t) = V(x, t), \end{cases}$$

the map  $W$  is defined

$$W(x, t) = \ddot{\sigma}(z, t) \Big|_{z=\sigma_t^{-1}(x)} \quad (\text{A.9})$$

$$= (D_v \nabla_v L(x, V_t(x)))^{-1} [\nabla_x L(x, V_t(x)) - D_x \nabla_v L(x, V_t(x)) V_t(x)]. \quad (\text{A.10})$$

When there is no ambiguity, the time subscripts will be omitted.

**Villani's formal displacement Hessian** of  $\mathcal{F}$  is

$$\begin{aligned} \mathcal{F}''(t) &= \int (-\Delta \varphi + \nabla A \cdot \nabla \varphi)^2 p_2(\rho) dx \\ &\quad + \int [\|\nabla^2 \varphi\|_{HS}^2 + (\text{Ric} + \nabla^2 A) (\nabla \varphi)] p(\rho) dx \end{aligned} \quad (\text{A.11})$$

$$= \int \rho [\|\nabla^2 \varphi\|_{HS}^2 + (\text{Ric} + \nabla^2 A) (\nabla \varphi)] dx \quad (\text{A.12})$$

where  $\|\cdot\|_{HS}$  denotes the Hilbert-Schmidt norm of a matrix and the function  $A$  is defined via

$$dx = e^{-A} dvol. \quad (\text{A.13})$$

Since  $dvol = \sqrt{\det g} dx$ , it follows that  $A = \frac{1}{2} \log \det g$ .

The derivatives in Villani's formal displacement Hessian are derivatives with respect to the Riemannian metric.

## A.2 Coordinate computations

### A.2.1 My displacement Hessian

In this subsection, the derivatives are standard derivatives on  $\mathbb{R}^d$ .

#### Derivatives of $L$ in coordinates

Various derivatives of  $L$  are computed in coordinates. The gradient with respect to velocity is

$$\nabla_v L(x, v) = \nabla_v \left( \frac{1}{2} v^i v^j g_{ij} \right) = \begin{pmatrix} v^j g_{1j} \\ \vdots \\ v^j g_{dj} \end{pmatrix}. \quad (\text{A.14})$$

Then,

$$D_v \nabla_v L(x, v) = g. \quad (\text{A.15})$$

The gradient in position is

$$\nabla_x L(x, v) = \nabla_x \left( \frac{1}{2} v^i v^j g_{ij} \right) = \frac{1}{2} \begin{pmatrix} v^i v^j \partial_1 g_{ij} \\ \vdots \\ v^i v^j \partial_d g_{ij} \end{pmatrix}. \quad (\text{A.16})$$

and

$$D_x \nabla_v L(x, v) = D_x \frac{1}{2} \begin{pmatrix} v^j g_{1j} \\ \vdots \\ v^j g_{dj} \end{pmatrix} \quad (\text{A.17})$$

$$= \begin{pmatrix} v^j \partial_1 g_{1j} & \cdots & v^j \partial_d g_{1j} \\ \vdots & & \vdots \\ v^j \partial_1 g_{dj} & \cdots & v^j \partial_d g_{dj} \end{pmatrix}. \quad (\text{A.18})$$

Finally,

$$D_x \nabla_v L(x, v)v = \begin{pmatrix} v^j \partial_1 g_{1j} & \cdots & v^j \partial_d g_{1j} \\ \vdots & & \vdots \\ v^j \partial_1 g_{dj} & \cdots & v^j \partial_d g_{dj} \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^d \end{pmatrix} \quad (\text{A.19})$$

$$= \begin{pmatrix} v^i v^j \partial_i g_{1j} \\ \vdots \\ v^i v^j \partial_i g_{dj} \end{pmatrix}. \quad (\text{A.20})$$

### **$W$ in coordinates**

In coordinates, the map  $W$  is given by

$$W(x, t) = (D_v \nabla_v L(x, V(x)))^{-1} [\nabla_x L(x, V(x)) - D_v \nabla_x L(x, V(x))V(x)] \quad (\text{A.21})$$

$$= \begin{pmatrix} g^{11} & \cdots & g^{1d} \\ \vdots & & \vdots \\ g^{d1} & \cdots & g^{dd} \end{pmatrix} \left[ \frac{1}{2} \begin{pmatrix} V^i V^j \partial_1 g_{ij} \\ \vdots \\ V^i V^j \partial_d g_{ij} \end{pmatrix} - \begin{pmatrix} V^i V^j \partial_i g_{1j} \\ \vdots \\ V^i V^j \partial_i g_{dj} \end{pmatrix} \right] \quad (\text{A.22})$$

$$= \frac{1}{2} \begin{pmatrix} V^i V^j g^{1k} \partial_k g_{ij} \\ \vdots \\ V^i V^j g^{dk} \partial_k g_{ij} \end{pmatrix} - \begin{pmatrix} V^i V^j g^{1k} \partial_i g_{kj} \\ \vdots \\ V^i V^j g^{dk} \partial_i g_{kj} \end{pmatrix}. \quad (\text{A.23})$$

Noting that  $2v^i v^j \partial_i g_{kj} = 2v^i v^j \partial_j g_{ki} = v^i v^j (\partial_i g_{kj} + \partial_j g_{ki})$ , equation (A.23) can be rewritten as

$$W(x, t) = \frac{1}{2} V^i V^j \begin{pmatrix} g^{1k} (\partial_k g_{ij} - \partial_i g_{kj} - \partial_j g_{ki}) \\ \vdots \\ g^{dk} (\partial_k g_{ij} - \partial_i g_{kj} - \partial_j g_{ki}) \end{pmatrix} \quad (\text{A.24})$$

$$= V^i V^j \begin{pmatrix} -\Gamma_{ij}^1 \\ \vdots \\ -\Gamma_{ij}^d \end{pmatrix}, \quad (\text{A.25})$$

where  $\Gamma_{ij}^l$  is the Christoffel symbol

$$\Gamma_{ij}^l = \frac{1}{2} g^{lk} (\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}). \quad (\text{A.26})$$

### **My Displacement Hessian in coordinates**

From equation (A.7), for  $\mathcal{F} = \int \rho \log \rho dx$ , my displacement Hessian is

$$\mathcal{F}'' = \int \rho [\text{tr}(DV^2) - \nabla \cdot W] dx. \quad (\text{A.27})$$

In coordinates,  $\text{tr}(DV^2) = (\partial_i V^j)(\partial_j V^i)$ . From equation (A.25), the formula for  $W$  is

$$W(x, t) = V^i V^j \begin{pmatrix} -\Gamma_{ij}^1 \\ \vdots \\ -\Gamma_{ij}^d \end{pmatrix}. \quad (\text{A.28})$$

Therefore,  $\nabla \cdot W$  is

$$\nabla \cdot W = -\partial_k (V^i V^j \Gamma_{ij}^k) \quad (\text{A.29})$$

$$= -2(\partial_k V^i) V^j \Gamma_{ij}^k - V^i V^j \partial_k \Gamma_{ij}^k. \quad (\text{A.30})$$

Substituting equation (A.30) into (A.7) yields

$$[\text{tr}(DV^2) - \nabla \cdot W] = (\partial_i V^j)(\partial_j V^i) + 2(\partial_k V^i) V^j \Gamma_{ij}^k + V^i V^j \partial_k \Gamma_{ij}^k \quad (\text{A.31})$$

## A.2.2 Villani's displacement Hessian

In this section, the derivatives are with respect to the Riemannian metric. Several formulas are recalled. The **Christoffel symbol of the second kind** is

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}) \quad (\text{A.32})$$

and, when  $j = k$ , this simplifies to

$$\Gamma_{ik}^k = \frac{1}{2} g^{jk} \partial_i g_{kj}. \quad (\text{A.33})$$

The **Christoffel symbol of the first kind** is

$$\Gamma_{lij} = g_{lk} \Gamma_{ij}^k. \quad (\text{A.34})$$

Christoffel symbols satisfy the symmetry relations

$$\Gamma_{ij}^k = \Gamma_{ji}^k \quad \text{and} \quad \Gamma_{lij} = \Gamma_{lji}. \quad (\text{A.35})$$

The vanishing of the covariant derivative of the Riemannian metric yields the identity

$$0 = \partial_l g_{ik} - g_{km} \Gamma_{il}^m - g_{im} \Gamma_{kl}^m \iff \partial_l g_{ik} = g_{km} \Gamma_{il}^m + g_{im} \Gamma_{kl}^m = \Gamma_{kil} + \Gamma_{ikl}. \quad (\text{A.36})$$

The **Ricci tensor** is

$$R_{ji} = \partial_k \Gamma_{ji}^k - \partial_i \Gamma_{jk}^k + \Gamma_{kl}^k \Gamma_{ji}^l - \Gamma_{il}^k \Gamma_{jk}^l. \quad (\text{A.37})$$

### Derivatives of $A$ in coordinates

The volume distortion map  $A$  is given by  $A = \frac{1}{2} \log \det g = \log \sqrt{\det g}$ , so its  $i^{\text{th}}$  partial derivative is

$$\partial_i A = \frac{1}{\sqrt{\det g}} \frac{1}{2\sqrt{\det g}} \partial_i \det g. \quad (\text{A.38})$$

By Jacobi's formula,

$$\partial_i (\det g) = \det g (\text{tr}(g^{-1} \partial_i g)), \quad (\text{A.39})$$

where  $\partial_i g_{kl}$  is the  $(k, l)$ -entry of  $\partial_i g$ . Substituting (A.39) into (A.38) yields

$$\partial_i A = \frac{1}{2} \text{tr}(g^{-1} \partial_i g) \quad (\text{A.40})$$

$$= \frac{1}{2} g^{jk} \partial_i g_{kj} \quad (\text{A.41})$$

Equation (A.41) follows from equation (A.40) because

$$\text{tr}(g^{-1} \partial_i g) = g^{jk} \partial_i g_{kj}. \quad (\text{A.42})$$

Equation (A.41) can be rewritten as

$$\partial_i A = \Gamma_{ik}^k. \quad (\text{A.43})$$

Then, using the formula for the Hessian of a scalar function with respect to a Riemannian metric, the Hessian of  $A$  is

$$\nabla^2 A = \partial_i \partial_j A - \Gamma_{ij}^k \partial_k A = \partial_i \Gamma_{jl}^l - \Gamma_{ij}^k \Gamma_{kl}^l \quad (\text{A.44})$$

The term  $\text{Ric} + \nabla^2 A$  in (A.12) is called the **Bakry-Emery tensor**. Using equation (A.44) to write  $\nabla^2 A$  in coordinates, the Bakry-Emery tensor can be written in the form

$$\text{Ric} + \nabla^2 A = R_{ij} + \left( \partial_i \Gamma_{jl}^l - \Gamma_{ij}^k \Gamma_{kl}^l \right) \quad (\text{A.45})$$

$$= \left( \partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{kl}^k \Gamma_{ij}^l - \Gamma_{jl}^k \Gamma_{ik}^l \right) + \left( \partial_i \Gamma_{jl}^l - \Gamma_{ij}^k \Gamma_{kl}^l \right) \quad (\text{A.46})$$

$$= \partial_k \Gamma_{ij}^k - \Gamma_{jl}^k \Gamma_{ik}^l \quad (\text{A.47})$$

### Derivatives of $\varphi$ in coordinates

The potential  $\varphi$  is defined via

$$V_t(x) = \nabla_p H(x, -D\varphi_t(x)) \iff -D\varphi = D_v L(x, V(x)). \quad (\text{A.48})$$

Then, the  $i^{\text{th}}$  coordinate of  $\nabla\varphi$  is given by raising indices:

$$(\nabla\varphi)^i = g^{ij}\partial_j\varphi. \quad (\text{A.49})$$

Since  $L(x, v) = \frac{1}{2}v^i v^j g_{ij}(x)$ , the  $j^{\text{th}}$  component of  $D\varphi$  is

$$-\partial_j\varphi = (-D\varphi)_j = (D_v L(x, V(x)))_j = g_{ji}V^i. \quad (\text{A.50})$$

Therefore,

$$(\nabla\varphi)^i = -V^i \quad (\text{A.51})$$

Using equation (A.50), the Hessian of  $\varphi$  is

$$\nabla^2\varphi = \partial_i\partial_j\varphi - \Gamma_{ij}^k\partial_k\varphi = -\partial_i(g_{jl}V^l) + \Gamma_{ij}^k g_{kl}V^l \quad (\text{A.52})$$

**Lemma A.2.1.** *Let  $g$  and  $V$  be as above. Then,*

$$(\partial_i g_{jl})V^l - (\partial_j g_{il})V^l = g_{il}(\partial_j V^l) - g_{jl}(\partial_i V^l) \quad (\text{A.53})$$

*Proof.* By the symmetry of mixed partial derivatives,

$$\partial_i\partial_j\varphi = \partial_j\partial_i\varphi. \quad (\text{A.54})$$

Substituting equation (A.50) into equation (A.54) yields

$$\partial_i(g_{jl}V^l) = \partial_j(g_{il}V^l).$$

Applying the product rule and rearranging completes the proof.  $\blacksquare$

The adjoint of  $\nabla^2\varphi$ , corresponding to equation (A.52), is seen to be

$$\nabla^2\varphi^* = g^{is} \left[ -\partial_s(g_{rl}V^l) + \Gamma_{sr}^k g_{kl}V^l \right] g^{rj} \quad (\text{A.55})$$

The Hilbert-Schmidt norm of  $\nabla^2\varphi$  is given by

$$\|\nabla^2\varphi\|_{HS}^2 := \text{tr}(\nabla^2\varphi^*\nabla^2\varphi) \quad (\text{A.56})$$

$$= (\nabla^2\varphi^*)^{ij} (\nabla^2\varphi)_{ji} \quad (\text{A.57})$$

Substituting equations (A.52) and (A.55) into (A.57) yields

$$\|\nabla^2 \varphi\|_{HS}^2 = g^{is} \left[ -\partial_s (g_{rl} V^l) + \Gamma_{sr}^k g_{kl} V^l \right] g^{rj} \left[ -\partial_j (g_{in} V^n) + \Gamma_{ji}^k g_{kn} V^n \right] \quad (\text{A.58})$$

$$\begin{aligned} &= g^{is} \left[ -(\partial_s g_{rl}) V^l - g_{rl} (\partial_s V^l) + g_{kl} \Gamma_{sr}^k V^l \right] g^{rj} \\ &\quad \times \left[ -(\partial_j g_{in}) V^n - g_{in} (\partial_j V^n) + g_{kn} \Gamma_{ji}^k V^n \right] \end{aligned} \quad (\text{A.59})$$

The summand in (A.59) consisting of only derivatives of  $V$  is

$$g^{is} \left( -g_{rl} (\partial_s V^l) \right) g^{rj} \left( -g_{in} (\partial_j V^n) \right) = g^{is} g_{il} (\partial_j V^l) g^{rj} g_{rn} (\partial_s V^n) \quad (\text{A.60})$$

$$= (\partial_j V^s) (\partial_s V^j). \quad (\text{A.61})$$

Recalling the definition of the Christoffel symbol of the first kind from equation (A.34), the summand in (A.59) containing only undifferentiated  $V$  terms is

$$g^{is} \left[ -(\partial_s g_{rl}) V^l + g_{kl} \Gamma_{sr}^k V^l \right] g^{rj} \left[ -(\partial_j g_{in}) V^n + g_{kn} \Gamma_{ji}^k V^n \right] \quad (\text{A.62})$$

$$= g^{is} \left[ -(\partial_s g_{rl}) V^l + \Gamma_{lsr} V^l \right] g^{rj} \left[ -(\partial_j g_{in}) V^n + \Gamma_{nji} V^n \right] \quad (\text{A.63})$$

From the identity given in (A.36),

$$\begin{cases} -(\partial_s g_{rl}) = -\Gamma_{lrs} - \Gamma_{rls} \\ -(\partial_j g_{in}) = -\Gamma_{nij} - \Gamma_{inj}. \end{cases} \quad (\text{A.64})$$

Substituting (A.64) into equation (A.63) yields

$$g^{is} \left[ -(\partial_s g_{rl}) V^l + \Gamma_{lsr} V^l \right] g^{rj} \left[ -(\partial_j g_{in}) V^n + \Gamma_{nji} V^n \right] \quad (\text{A.65})$$

$$= g^{is} \left[ -\Gamma_{lrs} - \Gamma_{rls} + \Gamma_{lsr} \right] V^l g^{rj} \left[ -\Gamma_{nij} - \Gamma_{inj} + \Gamma_{nji} \right] V^n \quad (\text{A.66})$$

$$= \left[ -g^{rj} \Gamma_{rls} \right] V^l \left[ -g^{is} \Gamma_{inj} \right] V^n \quad (\text{A.67})$$

$$= \Gamma_{ls}^j \Gamma_{nj}^s V^l V^n. \quad (\text{A.68})$$

The summand in equation (A.59) containing cross terms (products of  $V$  with derivatives of  $V$ ) is

$$\begin{aligned} &-g^{is} \left[ -(\partial_s g_{rl}) V^l + g_{kl} \Gamma_{sr}^k V^l \right] g^{rj} g_{in} (\partial_j V^n) \\ &\quad + g^{is} \left[ -g_{rl} (\partial_s V^l) \right] g^{rj} \left[ -(\partial_j g_{in}) V^n + g_{kn} \Gamma_{ji}^k V^n \right]. \end{aligned} \quad (\text{A.69})$$

In the first summand, there is a product of the form  $g^{is} g_{in}$  and in the second summand, there is a product of the form  $g^{rj} g_{rl}$ . Contracting indices and substituting in Christoffel symbols of

the first kind allows (A.69) to be rewritten

$$\begin{aligned} & -g^{is} \left[ -(\partial_s g_{rl}) V^l + g_{kl} \Gamma_{sr}^k V^l \right] g^{rj} g_{in} (\partial_j V^n) \\ & + g^{is} \left[ -g_{rl} (\partial_s V^l) \right] g^{rj} \left[ -(\partial_j g_{in}) V^n + g_{kn} \Gamma_{ji}^k V^n \right] \end{aligned} \quad (\text{A.70})$$

$$\begin{aligned} & = - \left[ -(\partial_n g_{rl}) V^l + g_{kl} \Gamma_{nr}^k V^l \right] g^{rj} (\partial_j V^n) \\ & + g^{is} \left[ -(\partial_s V^j) \right] \left[ -(\partial_j g_{in}) V^n + g_{kn} \Gamma_{ji}^k V^n \right] \end{aligned} \quad (\text{A.71})$$

$$= -g^{rj} (\partial_j V^n) \left[ -(\partial_n g_{rl}) + \Gamma_{lnr} \right] V^l - g^{is} \left[ (\partial_s V^j) \right] \left[ -(\partial_j g_{in}) + \Gamma_{nji} \right] V^n. \quad (\text{A.72})$$

Using the identities in (A.64) and the symmetry relation (A.35), it follows that

$$\begin{cases} -(\partial_n g_{rl}) + \Gamma_{lnr} = -\Gamma_{lrn} - \Gamma_{rln} + \Gamma_{lnr} = -\Gamma_{rln} \\ -(\partial_j g_{in}) + \Gamma_{nji} = -\Gamma_{nij} - \Gamma_{inj} + \Gamma_{nji} = -\Gamma_{inj}. \end{cases} \quad (\text{A.73})$$

Substituting the identities in (A.73) into equation (A.72) yields

$$-g^{rj} (\partial_j V^n) \left[ -(\partial_n g_{rl}) + \Gamma_{lnr} \right] V^l - g^{is} \left[ (\partial_s V^j) \right] \left[ -(\partial_j g_{in}) + \Gamma_{nji} \right] V^n \quad (\text{A.74})$$

$$= g^{rj} \Gamma_{rln} (\partial_j V^n) V^l + g^{is} \Gamma_{inj} (\partial_s V^j) V^n \quad (\text{A.75})$$

$$= \Gamma_{ln}^j (\partial_j V^n) V^l + \Gamma_{nj}^s (\partial_s V^j) V^n. \quad (\text{A.76})$$

Relabelling indices and using the symmetry of the Christoffel symbol in equation (A.76) allows for a simplification:

$$\Gamma_{ln}^j (\partial_j V^n) V^l + \Gamma_{nj}^s (\partial_s V^j) V^n = 2\Gamma_{ij}^k (\partial_k V^j) V^i. \quad (\text{A.77})$$

The Hilbert Schmidt norm of  $\nabla^2 \varphi$  is the sum of the terms (A.68), (A.68), and (A.77). That is,

$$\|\nabla^2 \varphi\|_{HS}^2 = \underbrace{(\partial_j V^s) (\partial_s V^j)}_{\text{derivative terms}} + \underbrace{\Gamma_{jl}^k \Gamma_{ik}^l V^j V^i}_{\text{no derivative terms}} + \underbrace{2\Gamma_{ij}^k (\partial_k V^j) V^i}_{\text{cross terms}}$$

### Villani's displacement Hessian in coordinates

Recall from (A.12) that, for  $\mathcal{F} = \int \rho \log \rho dx$ , Villani's displacement Hessian is

$$\mathcal{F}''(t) = \int \rho \left[ \|\nabla^2 \varphi\|_{HS}^2 + (\text{Ric} + \nabla^2 A) (\nabla \varphi) \right] dx. \quad (\text{A.78})$$

From equation (A.47), in coordinates,

$$\text{Ric} + \nabla^2 A = \partial_k \Gamma_{ij}^k - \Gamma_{jl}^k \Gamma_{ik}^l.$$



Since  $(\text{Ric} + \nabla^2 A)(\nabla\varphi) := (\text{Ric} + \nabla^2 A)_{ij}(\nabla\varphi)^i(\nabla\varphi)^j$  and  $(\nabla\varphi)^i = -V^i$ ,

$$(\text{Ric} + \nabla^2 A)(\nabla\varphi) = \left( \partial_k \Gamma_{ij}^k - \Gamma_{jl}^k \Gamma_{ik}^l \right) V^i V^j.$$

From equation (A.78),

$$\|\nabla^2\varphi\|_{HS}^2 = (\partial_j V^s)(\partial_s V^j) + \Gamma_{jl}^k \Gamma_{ik}^l V^j V^i + 2\Gamma_{ij}^k (\partial_k V^j) V^i.$$

Therefore, the term  $\|\nabla^2\varphi\|_{HS}^2 + (\text{Ric} + \nabla^2 A)(\nabla\varphi)$  in the integrand in Villani's displacement Hessian (A.12) is

$$\begin{aligned} \|\nabla^2\varphi\|_{HS}^2 + (\text{Ric} + \nabla^2 A)(\nabla\varphi) &= \left[ (\partial_j V^s)(\partial_s V^j) + \Gamma_{jl}^k \Gamma_{ik}^l V^j V^i + 2\Gamma_{ij}^k (\partial_k V^j) V^i \right] \\ &\quad + \left( \partial_k \Gamma_{ij}^k - \Gamma_{jl}^k \Gamma_{ik}^l \right) V^i V^j \end{aligned} \quad (\text{A.79})$$

$$= (\partial_j V^s)(\partial_s V^j) + 2\Gamma_{ij}^k (\partial_k V^j) V^i + \partial_k \Gamma_{ij}^k V^i V^j \quad (\text{A.80})$$

### A.3 Comparison of Displacement Hessians

From equations (A.12) and (A.80), Villani's displacement Hessian takes the form

$$\mathcal{F}''(t) = \int \rho \left[ \|\nabla^2\varphi\|_{HS}^2 + (\text{Ric} + \nabla^2 A)(\nabla\varphi) \right] dx \quad (\text{A.81})$$

$$= \int \rho \left[ (\partial_j V^s)(\partial_s V^j) + 2\Gamma_{ij}^k (\partial_k V^j) V^i + \partial_k \Gamma_{ij}^k V^i V^j \right] dx \quad (\text{A.82})$$

And from equations (A.7) and (A.31), my displacement Hessian takes the form

$$\mathcal{F}'' = \int \rho \left[ \text{tr}(DV^2) - \nabla \cdot W \right] dx \quad (\text{A.83})$$

$$= \int \rho \left[ (\partial_i V^j)(\partial_j V^i) + 2\Gamma_{ij}^k (\partial_k V^i) V^j + \left( \partial_k \Gamma_{ij}^k \right) V^i V^j \right] dx. \quad (\text{A.84})$$

This result is recorded in a theorem.

**Theorem A.3.1.** *In the Riemannian setting, my displacement Hessian for the entropy functional agrees with Villani's formula for the displacement Hessian in Formula 15.7 in [46].*

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