MAT 351: Partial Differential Equations
Test 3, March 9 2018

(Five problems; 80 points in total.)

1. (10pts) Construct the *Green's function* for the three-dimensional half-ball

$$B_+ = \left\{ x \in \mathbb{R}^3 \;\middle|\; |x| < 1\,, x_3 > 0 \right\}\,.$$

2. (20pts) Let u be a smooth function on $\mathbb{R}^3$ with $\Delta u \geq 0$.

   (a) Prove that u is *subharmonic*, that is, for all $x \in \mathbb{R}^3$ and all $r > 0$,

$$u(x) \;\leq\; \frac{1}{\frac{4}{3}\pi r^3} \int_{B_r(x)} u(y)\, dy\,.$$

   Hint: Compare with the harmonic function that has the same boundary values on $B_r(x)$.

   (b) Let $D \subset \mathbb{R}^3$ be a bounded domain. What can you say about $\sup_D u$ and $\inf_D u$? Can u assume its maximum in the interior of D? How about its minimum?

   Please justify your answers briefly.

3. (10pts) For the wave equation $u_{tt} = c^2 \Delta u$ in two spatial dimensions ($x \in \mathbb{R}^2$), find all solutions of the form $u(x,t) = e^{-i\omega t} f(|x|)$ that are finite at $x = 0$.

4. (20pts) Let ρ be a positive continuous function on $[0,1]$ with $\int_0^1 \rho(x)\, dx = 1$. Define an inner product on the space of real-valued continuous functions by

$$\langle u, v \rangle := \int_0^1 u(x) v(x)\, \rho(x)\, dx\,.$$

   (a) Starting from the monomials $m_k(x) = x^k$, $k = 0, 1, 2, \ldots$, explain how to construct a sequence of mutually orthogonal polynomials $P_k(x)$, where each P_k is of degree k. (You don't need to use formulas, but please be specific.)

   (b) *(Recursion relation)* Prove that there exist constants a_k, b_k, and c_k, such that

$$P_k(x) = (a_k x + b_k) P_{k-1}(x) + c_k P_{k-2}(x)\,.$$

   Hint: First choose a_k so that $Q = P_k - a_k x P_{k-1}$ has degree $< k$. Then expand Q in terms of the orthogonal polynomials.

5. (20pts) Consider the *Klein-Gordon equation* $u_{tt} - c^2\Delta u + m^2 u = 0$, where $m > 0$. (Note that the standard wave equation corresponds to taking $m = 0$.)

   (a) What is the energy associated with this equation? Show that it is conserved. *Hint:* Start from the energy for the wave equation.

   (b) State the *causality principle* for the wave equation ($m = 0$).

   (c) Use a sketch to argue that the Klein-Gordon equation should obey the same causality principle. (Make the case that the same energy methods apply.)

Some formulas. (Only a few of them will be needed.)

- The Laplacian in **spherical coordinates** has the form

$$\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[\frac{1}{\sin^2\theta}u_{\phi\phi} + \frac{1}{\sin\theta}(\sin\theta\, u_\theta)_\theta\right].$$

  Here, $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$.

- The **fundamental solution** of Laplace's equation on $\mathbb{R}^3$ is $\Phi(x) = -\frac{1}{4\pi}|x|$.

- The **Green's function** for the unit ball in $\mathbb{R}^3$ is given by

$$G(x,y) = \frac{1}{4\pi}\left(\frac{1}{(|y|\,|\bar{y}-x|)} - \frac{1}{|y-x|}\right).$$

- The **inversion** at the unit sphere, given by $x \mapsto \bar{x} = \frac{x}{|x|^2}$, satisfies $|\bar{x}-\bar{y}| = \frac{|x-y|}{|x||y|}$.

- **Kirchhoff's formula** for the solution of the wave equation in $\mathbb{R}^3$ is

$$u(x,t) = \frac{\partial}{\partial t}\left\{\frac{1}{4\pi c^2 t}\int_{|y-x|=ct}\phi(y)\,dS(y)\right\} + \frac{1}{4\pi c^2 t}\int_{|y-x|=ct}\psi(y)\,dS(y).$$

- **Poisson's formula** for the solution of the wave equation in $\mathbb{R}^2$ is

$$u(x,t) = \frac{\partial}{\partial t}\left\{\frac{1}{2\pi c}\int_{|y-x|<ct}\frac{\phi(y)}{(c^2t^2 - |y-x|^2)^{1/2}}\,dy\right\} + \frac{1}{2\pi c}\int_{|y-x|<ct}\frac{\psi(y)}{(c^2t^2 - |y-x|^2)^{1/2}}\,dy.$$

- The **Bessel functions** $J_n(r)$ are the bounded solutions of Bessel's equations

$$J'' + \frac{1}{r}J' + \left(1 - \frac{n^2}{r^2}\right)J = 0$$

  for $n = 0, 1, \ldots$ (after suitable normalization). Each J_n is a smooth function that changes sign at an infinite sequence of zeroes $z_{n,1}, z_{n,2}, \cdots \to \infty$, separated by an infinite sequence of critical points $p_{n,1}, p_{n,2}, \ldots$.