MAT 351: Partial Differential Equations
Test 3, March 15 2017

(Four problems; 20 points each.)

1. For the wave equation $u_t t = c^2 \Delta u$ on $\mathbb{R}^d$:
   (a) Define the terms *finite speed of propagation* and *domain of dependence*.
   (b) State **Huygens' principle** in three dimensions, and justify it in terms of Kirchhoff's formula.
   (c) Explain why Huygens' principle fails in dimension one and two.

2. Let $D \subset \mathbb{R}^d$ a smooth bounded connected domain.
   (a) Write down the **Rayleigh principle** for the lowest eigenvalue λ_1 of the Laplacian with Dirichlet boundary condititions. How does it determine the corresponding eigenfunction, v_1?
   (b) State the min-max principle for the higher eigenvalues λ_n, $n > 1$.
   (c) It is known that $v_1 > 0$ on D. Show that all higher eigenfunctions v_n change sign.
   (d) Define the subdomain $A = \{x \in D \mid v_2(x) > 0\}$. Show that its lowest Dirichlet eigenvalue $\lambda_1(A)$ is given by λ_2, the second-lowest Dirichlet eigenvalue of D.
   (*Hint:* What is the corresponding eigenfunction?)
   (e) If E, F is *any* pair of disjoint subdomains of D with $\lambda_1(E) = \lambda_1(F)$, argue that
   $$\lambda_1(E) \geq \lambda_2 \,.$$
   (*Hint:* Construct a suitable trial function for the variational principle that defines λ_2. Make sure your function is continuous but ignore differentiability issues.)

3. Consider **Legendre's differential equation**
   $$((1 - x^2)u')' + \gamma u = 0 \,, \quad x \in (-1, 1) \,.$$
   (a) Assuming that u is a power series,
   $$u(x) = \sum_{k \geq 1} a_k x^k \,,$$
   find a recursion formula for the coefficients.
   (b) For what values of γ is the solution a polynomial? Of which degree?

(c) If the solution is not a polynomial, show that the power series diverges at $x = \pm 1$.

(d) Let $(\gamma_n)_{n\geq 0}$ be the values of γ you found in Part (b). Show that the corresponding polynomials (u_n) satisfy the orthogonality relation

$$\int_{-1}^{1} u_n(x) u_m(x)\,(1-x^2)\,dx = 0 \qquad n \neq m\,.$$

(e) Conclude $(u_n)_{n\geq 0}$ (suitably normalized) form an orthonormal basis for $L^2(-1,1)$.

4. Consider the Dirichlet eigenvalue problem for the Laplacian on the unit disc

$$-\Delta u = \lambda u \quad \text{for } x^2+y^2<1\,, \qquad u\big|_{x^2+y^2=1} = 0\,.$$

(a) Express the eigenvalues in terms of the **Bessel functions** J_n.

(b) Give the corresponding description for the Neumann problem.

(c) Let $N(\lambda)$ be the number of eigenvalues up to λ, with the respective boundary conditions. Use Parts (a) and (b) to show that

$$N_{Dirichlet}(\lambda) \geq N_{Neumann}(\lambda)$$

for all $\lambda > 0$. Please support your argument with a sketch of the J_n.

Useful formulas