MAT 351: Partial Differential Equations
Test 2, January 26 2018

(Five problems; 80 points in total.)

1. (20pts) Consider the Fourier series

$$\sin\frac{x}{2} = \sum_{k=-\infty}^{\infty} A_k e^{ikx}, \quad (-\pi < x < \pi).$$

   (a) Argue that the Fourier coefficients A_k are purely imaginary, and $A_{-k} = -A_k$.

   (b) Find the coefficients A_k. (Useful fact: $e^{i\pi/2} = i$.)

   (c) Find the value of $\sum_{k=1}^{\infty} |A_k|^2$.

   (d) Briefly describe in which sense the series converges.

2. (20pts) Suppose u solves $\Delta u = 0$ on the unit disc $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, with boundary values $u(\cos\theta, \sin\theta) = (\cos\theta)^4$. Find ...

   (a) ...the maximum and minimum value of u on the unit disc;

   (b) ...the value of $u(0,0)$;

   (c) ...an explicit formula for u (in polar or Cartesian coordinates).

   Please explain your reasoning!

3. (10pts) Consider the inhomogeneous heat equation

$$u_t = u_{xx} + f(x,t), \quad (x \in \mathbb{R}, t > 0)$$

   with initial values $u(x,0) = 0$. Write down a solution of this problem, using Duhamel's formula and the fundamental solution.

4. (10pts) State and prove Dirichlet's principle.

5. (20pts) Let D be a bounded open subset of $\mathbb{R}^3$, with smooth boundary.

   (a) Define the Green's function $G(x,a)$ of D.

   (b) Let u be a harmonic function on D, with boundary values $u|_{\partial D} = g$, where g is a given continuous function. Use the Green's function to give a formula for u.

   (c) In your formula, indicate the Poisson kernel, $K_a(x)$, defined for $a \in D$, $x \in \partial D$. Prove that

$$\int_{\partial D} K_a(x)\, dS(x) = 1$$

   for each $a \in D$. (*Hint:* Consider the case where g is constant.)

   (d) Also prove that $K_a(x) > 0$ for all $x \in \partial D$.