MAT 351: Partial Differential Equations
Test 1, November 24 2017

(Four problems; 20 points each.)

1. Use the method of characteristics to solve the equation $xu_x - yu_y = 2u$, with initial values $u(x, 1) = f(x)$.

   In which region of the plane is the solution uniquely determined by the initial conditions? Outside that region, what fails — existence or uniqueness? Please support your answer with a sketch of the characteristics.

2. Consider the wave equation on the positive real half-line, with Dirichlet boundary conditions $u(0, t) = 0$ for all t.

   (a) Please formulate the initial-value problem.

   (b) Use the method of reflections and D'Alembert's formula to express its solution in terms of the initial values.

   (c) Suppose you know that the initial amplitude (ϕ) and the inital velocity (ψ) both vanish outside the interval $[1, 2]$. Sketch the region in the x-t-plane where u must vanish. Briefly explain, using the term *'domain of influence'*.

3. Consider Laplace's equation $u_{xx} + u_{yy} = 0$ on the strip $(0, \pi) \times \mathbb{R}$, with Dirichlet boundary conditions $u(0, y) = u(\pi, y) = 0$.

   (a) Use Separation of Variables to construct solutions of the form $u(x, y) = X(x)Y(y)$.

   (b) Assuming that u is a smooth solution of the equation (not necessarily of product form), compute
   $$\frac{d^2}{dy^2} \int_0^\pi u^2(x, y)\, dx\,.$$
   Conclude that every smooth solution, except for $u \equiv 0$, is unbounded on the strip.

4. Let H be a Hilbert space.

   (a) State *Bessel's inequality* and *Parseval's identity*. Don't forget to give the assumptions!

   (b) Let $(w_n)_{n\geq 1}$ be an orthonormal basis for H. We have shown in class that every $u \in H$ can be represented as a series $u = \sum_{n\geq 1} a_n w_n$ for a suitable sequence of coefficients $(a_n)_{n\geq 1}$ in $\mathbb{C}$. Prove that the coefficients are uniquely determined by u, i.e.,
   $$\sum_{n=1}^{\infty} a_n w_n = \sum_{n=1}^{\infty} b_n w_n \qquad \Longrightarrow \qquad a_n = b_n \text{ for all } n \geq 1\,.$$

   (c) Conversely, given a sequence of complex numbers $(c_n)_{n\geq 1}$. Prove that
   $$\sum_{n=1}^{\infty} |c_n|^2 < \infty \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} c_n w_n \text{ converges in } H\,,$$
   i.e., every square summable sequence can be realized as the coefficients of an element $u \in H$ with respect to the orthonormal basis.