UNIVERSITY OF TORONTO
Faculty of Arts and Science

APRIL 2017 EXAMINATIONS
MAT351Y1Y Partial Differential Equations

Examiner: Professor Almut Burchard

Time: 3 hours. No calculators or other aids allowed. Six problems, 20 points each.

1. For each of the following terms, give a definition and an example where it applies.

   (a) *well-posed* problem;

   (b) *Neumann* boundary condition;

   (c) *finite speed of propagation*;

   (d) *orthonormal basis* (of a Hilbert space);

   (e) *spherical harmonic*.

2. For **Burger's** equation $u_t + uu_x = 0$ on the real line ($x \in \mathbb{R}, t > 0$):

   (a) If u is a smooth solution, show that $v(x,t) = u(-x,-t)$ is also a solution.

   (b) Please write down the characteristic equations.

   (c) State the Rankine-Hugoniot condition and Lax' entropy condition.
   How do these conditions behave under the time reversal in part (a)?

   (d) Given initial conditions

   $$\phi(x) = \begin{cases} 0\,, & x < -1\,, \\ \frac{1}{2}\,, & -1 < x < 0\,, \\ -1\,, & 0 < x < 1\,, \\ 0\,, & x > 1\,, \end{cases}$$

   how many shocks develop? Determine their location, height, and speed up to time $t = 1$.

   (e) Sketch the characteristics in the x-t-plane up to time $t \approx 1.5$. Also sketch $u(x,t)$ at time $t = 0$, $t = 1$. What will happen as $t \to \infty$?

3. (a) Let $D \subset \mathbb{R}^d$ be a bounded smooth domain, let f be a given function on D, and g a function on the bounday of D. Assuming that **Poisson problem**

   $$\Delta u = f \quad \text{on } D\,, \qquad u\big|_{\partial D} = g$$

   has a solution, prove that it is unique. Give two different arguments,
   i. using the maximum principle; ii. using Dirichlet's principle.

   (b) Construct a radial solution (depending only on $|x|$) of the Poisson problem

   $$-\Delta u = \begin{cases} 1\,, & |x| < 1\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

   on $\mathbb{R}^3$. (*Hint:* Solve separately inside and outside the ball, then match the values of u and its first derivative on the unit sphere.)

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4. For the the **wave** equation in three space dimensions

$$u_{tt} = c^2 \Delta u\,, \quad x \in \mathbb{R}^3, t > 0$$

with initial values $u(x,0) = \phi(x)$ and $u_t(x, 0 = \psi(x)$:

   (a) Define the terms *light cone*, *causality principle*, and *conservation of energy*.

   (b) Suppose ϕ, ψ are harmonic. Prove directly from Kirchhoff's formula that

   $$u(x,t) = \phi(x) + t\psi(x)\,.$$

   (*Hint:* Use the Mean Value Property).

5. (a) Let T be a distribution on $\mathbb{R}^d$, and g a smooth function. Define their product, gT, and argue that it is a distribution.

   (b) Define the distributional derivative $\partial_{x_i} T$. Prove the product rule

   $$\partial_{x_i}(gT) = (\partial_{x_i} g)\, T + g\, (\partial_{x_i} T)\,.$$

   (c) Do distributional partial derivatives commute, that is,

   $$\partial_{x_i}\partial_{x_j} T = \partial_{x_j}\partial_{x_i} T \qquad \text{for all } i \neq j\ ?$$

   Please prove your claim!

6. Consider the **Fourier transform** on the real line.

   (a) On a recent assignment, you showed that the **Hermite polynomials** $(H_n)_{n\geq 0}$ satisfy

   $$\psi_n(x) := H_n(x) e^{-x^2} = (-1)^n \frac{d^n}{dx^n} e^{-x^2}\,.$$

   Use this to find the Fourier transform of ψ_n.

   (b) Prove **Heisenberg's uncertainty relation**

   $$||xf||_{L^2}\, ||k\hat{f}||_2 \geq \frac{1}{4\pi} ||f||_2^2\,.$$

   Hint: Apply Schwarz' inequality to $\int x f f'\, dx$. Please justify your computation !
   (Do not worry about differentiability and integrability issues.)

Useful formulas

- The **Laplacian** on $\mathbb{R}^3$ in spherical coordinates: $\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2 \sin^2\theta}\big((\sin\theta\partial_\theta)^2 + \partial_\phi^2\big)$.
- The fundamental solution of **Laplace's equation** on $\mathbb{R}^3$ is $G_0(x,y) = -\frac{1}{4\pi|y-x|}$.
- **Kirchhoff's formula** for waves in three space dimensions:

$$u(x,t) = \frac{\partial}{\partial t}\left\{\frac{1}{4\pi c^2 t}\int_{|y-x|=ct} \phi(y)\, dS(y)\right\} + \frac{1}{4\pi c^2 t}\int_{|y-x|=ct} \psi(y)\, dS(y)\,.$$

- The **Fourier transform** on $\mathbb{R}^d$ is defined by $\hat{f}(k) = \int_{\mathbb{R}^d} e^{-2\pi i k\cdot x} f(x)\, dx$.
  With this convention, the Gaussian $f(x) = e^{-\pi|x|^2}$ satisfies $\hat{f} = f$.

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