MAT 351: Partial Differential Equations
February 16, 2018

We are considering eigenvalue problems of the form $-\Delta u + V(x)u = \lambda u$ for $x \in \mathbb{R}^n$. Here, the linear operator $-\Delta + V(x)$ is called a **Schrödinger operator** with **potential** V. In all examples that we consider, V takes its minimum at $x = 0$ and increases radially from there.

- **Harmonic oscillator** $-\Delta u + |x|^2 u = \lambda u$.

  In dimension $n = 1$, the eigenfunctions and eigenvalues are given by

  $$u_k(x) = H_k(x)e^{-\frac{x^2}{2}}\,, \quad \lambda_k = 2k + 1 \qquad (k = 0, 1, \dots)\,,$$

  where H_k is a polynomial of degree k. These are the **Hermite polynomials**. The family $\{u_k\}$ forms an orthogonal basis for $L^2(\mathbb{R})$. Although the Hermite polynomials do not have an explicit formula, they can be computed in many different ways, using recursion relations, Gram-Schmidt orthogonalization, or generating functions.

  The eigenfunctions and eigenvalues of the harmonic oscillator in dimension $n > 1$ are given by

  $$u = \prod_{j=1}^{n} H_{k_j}(x_j)e^{-\frac{|x|^2}{2}}\,, \qquad \lambda = \sum_{j=1}^{n}(2k_j + 1)$$

  (this follows by separation of variables).

- **Hydrogen atom** $-\Delta u - \frac{2}{|x|}u = \lambda u$, where $x \in \mathbb{R}^3$.

  We split the eigenvalue problem into a radial and an angular part, using separation of variables. We will later see that the eigenfunctions of the full problem are given by $u(x) = v(r)Y(\phi, \theta)$, where Y is a spherical harmonic. In the special case where the eigenfunction is radial (i.e., if Y is constant) then we have $-v'' - \frac{2}{r}v' - \frac{2}{r}v = \lambda v$, and obtain for the eigenfunctions and eigenvalues

  $$v_k(r) = w_k(r)e^{-\frac{r}{k}}\,, \quad \lambda_k = -\frac{1}{k^2} \qquad (k = 1, 2, \dots)\,,$$

  where w_k is a polynomial of degree k. The coefficients of these polynomials are determined by a recursion.

  It turns out that these eigenfunctions do not form an orthogonal basis for L^2 — eigenfunctions for distinct eigenvalues are orthogonal, but their span is a subspace that fails to be dense in L^2.

- **Dirichlet eigenvalue problem** $-\Delta u = \lambda u$ on the unit ball $\{|x| < 1\}$, with boundary conditions $u(x) = 0$ for $|x| = 1$. We again separate variables.

  In two dimensions, the angular part of an eigenfunction is $\sin(n\theta)$ or $\cos(n\theta)$ for some integer n, and the radial part satisfies

  $$v'' + \frac{1}{r}v' + \left(\lambda - \frac{n^2}{r^2}\right)v\,,$$

  where γ is an eigenvalue of the angular part. If we rescale the problem so that $\lambda = 1$, this becomes **Bessel's equation** of order n, and its solution is given by the corresponding Bessel function J_n. This is again a special function that does not have an explicit formula. But there are recursion formulas for its Taylor series, and precise asymptotic expansions as $r \to \infty$. The eigenvalue is determined by the requirement that $J_n(\sqrt{\lambda}) = 0$, i.e., λ is the square of a zero of a Bessel function.

In dimension three and above, the angular part of an eigenfunction is a spherical harmonic. The basic strategy is the same but the radial equation becomes (after some change of variables) a Bessel equation of non-integer order. Specifically, in three dimensions, we set $v(r) = r^{-\frac{1}{2}} w(r)$ and obtain

$$w'' + \frac{1}{r} w' + \left(\lambda - \frac{\gamma + \frac{1}{4}}{r^2} \right) w = 0 \, .$$

Read: Sections 9.4, 9.5 and 10.1.

Hand-in (due March 2):

(H1) Starting from the zeroth Hermite polynomial $H_0(x) = 1$, derive the first four Hermite polynomials from the recursion formula for the coefficients.

(H2) (a) Verify that the Hermite polynomials have the orthogonality property

$$\int H_k(x) H_\ell(x) \, e^{-|x|^2} \, dx = 0 \, , \quad k \neq \ell \, .$$

Hint: Start from Hermite's differential equation $v'' + (\lambda - x^2)v = 0$.

(b) Explain how to use the Gram-Schmidt method to determine the Hermite polynomials recursively. (The integrals arising from the orthogonal projections can be computed explicitly, but you're not asked to do that here.)

(H3) Consider the eigenvalue problem $w'' - 2xw' + (\lambda - 1)w = 0$ that determines the Hermite polynomials.

(a) Show that every solution with $\lambda \neq 2k + 1$ is a power series but not a polynomial.

(b) Deduce that for every such solution, $v(x) = w(x)e^{-\frac{x^2}{2}}$ grows rapidly as $|x| \to \infty$.
(*Hint:* Use the recursion relation for the Taylor coefficients a_k of w as $k \to \infty$, and compare with the power series expansion for e^{x^2}.)

(H4) Show that all Hermite polynomials are given by $H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$.

For discussion and practice:

1. Use Kirchhoff's formula to find the solution of the three-dimensional wave equation with initial data $u(x, u) = 0$,$u_t(x, 0) = x_2$.

2. Use the Euler-Poisson-Darboux equation to solve the three-dimensional wave equation with initial data $u(x, 0) = 0$, $u_t(x, u) = |x|^2$.