MAT 351: Partial Differential Equations
January 19, 2018

Summary

We have defined the Green's function of a (smooth, bounded, connected) domain $D \subset \mathbb{R}^n$ by the properties that for every fixed $a \in D$,

$$G(x, a) = \Phi(x - a) + H(x)\,,$$

where Φ is the fundamental solution of Laplace's equation, given by

$$\Phi(x) = \begin{cases} \frac{1}{2\pi} \log |x|\,, & \text{dimension } n = 2\,, \\ -\frac{1}{4\pi|x|}\,, & n = 3\,, \end{cases}$$

and H is a harmonic function chosen such that $G(x, a) = 0$ whenever x lies in the boundary ∂D. (Of course, H depends on a as well). You proved that the Green's function is uniquely determined by these properties, and that G is symmetric ($G(x, y) = G(y, x)$.

The Green's function is used to solve the **Dirichlet problem**

$$\Delta u = f \text{ on } D\,, \qquad u|_{\partial D} = 0\,.$$

The fomula for the solution is

$$u(a) = \int_D G(x, a) f(x)\, dx\,.$$

In accordance with the maximum principle, the Green's function is negative for all $x, y \in D$ with $x \neq y$.

The Green's function is also used to solve the **Poisson problem** for harmonic functions

$$\Delta v = 0 \text{ on } D\,, \qquad v|_{\partial D} = g\,.$$

The formula for the solution is

$$v(a) = \int_{\partial D} g(x) \nabla_x G(x, a) \cdot N(x)\, dS(x)\,.$$

The function $K_a(x) = \nabla_x G(x, a) \cdot N(x)$ is called the **Poisson kernel**. It is defined for $a \in D, x \in \partial D$. In accordance wirth the maximum principle, the Poisson kernel K_a is strictly positive. In accordance with the mean value property, it defines a probability density on the boundary ∂D.

There are only few domains where the Green's function can be computed explicitly. The two most important ones are the upper half-space and the unit ball in $\mathbb{R}^n$. For these, we can use a **reflection principle** to find the harmonic function H.

- **Upper half-space:** Let $D = \{x \in \mathbb{R}^n \mid x_n > 0\}$. For $x_n > 0$, we define its reflection at the boundary $\{x_n = 0\}$ by $\bar{x} = (x_1, x_2, \ldots, -x_n)$, and set

$$H(x) = -\Phi(\bar{a} - x)\,.$$

Clearly, H is harmonic on the entire positive half-space (since $\bar{a}$ lies in the negative half-space). If $x_n = 0$, then $H(x) = -\Phi(a - x)$, because in that case $|\bar{a} - x| = |a - x|$. So for $n > 2$, the Green's function is given by

$$G(x, a) = \frac{1}{n(n-2)\omega_n}\left(\frac{1}{|\bar{a} - x|^{n-2}} - \frac{1}{|a - x|^{n-2}}\right), \quad x, a \in D\,.$$

The Poisson kernel for the upper half space is given by $K_a(x) = \frac{1}{n\omega_n}\frac{2a_n}{|a-x|^n}$.

- **Unit ball:** Let $D = \{x \in \mathbb{R}^n \mid |x| < 1\}$. For $x \in D$, we define its reflection at the unit sphere by $\bar{x} = \frac{x}{|x|^2}$. A quick computation shows that

$$|\bar{a} - \bar{x}|^2 = \frac{|x - a|^2}{|x|^2|a|^2}\,,$$

in particular, if $x \in \partial D$, then $|\bar{a} - x| = \frac{|a-x|}{|a|}$. For $a \in D$, the function

$$H(y) = -\Phi(|a| \cdot |\bar{a} - x|)$$

is clearly harmonic in x on D (since $\bar{a}$ lies outside D), and its boundary values agree with those of $\Phi(x - y)$. So the Green's function is given by

$$G(x, a) = \frac{1}{n(n-2)\omega_n}\left(\frac{1}{(|a| \cdot |\bar{a} - x|)^{n-2}} - \frac{1}{|a - x|^{n-2}}\right).$$

For the Poisson kernel, we obtain $K_a(x) = \frac{1}{n\omega_n}\frac{1-|a|^2}{|a-x|^n}$, where $|a| < 1$ and $|x| = 1$.

Read: Chapter 7.

Please remember: Our second midterm test takes place Friday January 26, in tutorial/class. The test covers lectures and tutorials up to January 19, Assignments 1-11, and Chapters 1-7 of Strauss (with an emphasis on Chapters 5-7).

For discussion and practice:

1. Find the Green's function for the unit disk in the plane.
2. Compute the Poisson kernel for the unit disk from the Green's function, and verify that it agrees with our previous formula.
3. Find the Green's function for the the positive half-disk.