MAT 351: Partial Differential Equations
Assignment 5, due Sept. 24, 2016

Summary

We turn to inhomogeneous equations and the role of boundary conditions for the heat and wave equation. We will formally write these equations as

$$u_t = Lu + f(\cdot, t)\,, \quad u(\cdot, 0) = \phi\,, \tag{1}$$

where L is a linear differential operator that involves only the x variable, f is the inhomogeneity or **source term**, and ϕ is the initial condition.

- To ground ourselves, let us first consider a linear ODE

  $$\frac{dy}{dt} = Ay + f(t) \quad y(0) = y_0\,, \tag{2}$$

  where A is a $n \times n$ matrix f is a given function, $y_0 \in \mathbb{R}^n$ a given vector, and the unknown function $y(t)$ takes values in $\mathbb{R}^n$. **Duhamel's principle** says that the solution is given by

  $$y(t) = e^{At} y_0 + \int_0^t e^{A(t-s)} f(s)\, ds\,.$$

  Here, e^{At} denotes the solution operator for the homogeneous equation $\frac{d}{dt}y = Ay$. By definition, $y(t) = e^{At}y_0$ solves the initial-value problem

  $$\frac{dy}{dt} = Ay \quad y(0) = y_0\,.$$

  There are many equivalent ways to define and compute the matrix-valued function e^{At}, by its power series, by diagonalizing A, or by special techniques such as contour integrals. Duhamel's formula is proved by the method of **variation of constants**.

- For the linear transport equation (Problem 2 on Assignment 1),

  $$u_t + bu_x = f(x, t)\,, \quad u(x, 0) = \phi(x)\,, \tag{3}$$

  we write $L = -b\partial_x$. The solution of the homogeneous problem is $e^{tL}\phi(x) = \phi(x - bt)$. By Duhamel's principle,

  $$\begin{aligned} u(x,t) &= \left(e^{tL}\phi\right)(x) + \int_0^t e^{(t-s)L} f(\cdot, s)\, ds \\ &= \phi(x - bt) + \int_0^t f(x - b(t-s), s)\, ds\,. \end{aligned}$$

- Let us now try to solve the **heat equation with sources**

$$u_t = k u_{xx}\,, \quad u(x,0) = \phi(x)\,. \tag{4}$$

Here, $L = k\partial_x^2$. Denote by e^{tL} the solution operator for the heat equation, given by

$$\left(e^{tL}\phi\right)(x) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy\,,$$

where $S(x,t) = (4\pi kt)^{-1/2}e^{-x^2/(4t)}$ is the fundamental solution. By Duhamel's principle, the solution of the inhomogeneous equation is given by

$$u(x,t) = \left(e^{tL}\phi\right)(x) + \int_0^t \left(e^{(t-s)L} f(\cdot,s)\right)(x)\,ds\,,$$

that is,

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\,f(y,s)\,dyds\,,$$

- Consider finally the initial-value problem for the wave equation with sources,

$$u_{tt} = c^2 u_{xx} + f(x,t)\,, \quad u(x,u) = \phi(x), u_t(x,0) = \psi(x)\,. \tag{5}$$

We rewrite this equation as a system of first-order PDE for u and $v = u_t$. Then $L = (\partial_y, c^2\partial_x^2)$. We then use D'Alembert's formula to write the solution operator $e^{tL}(\phi,\psi)$ for the homogeneous equation. Plugging this into Duhamel's formula gives

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct) + \phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dyds\,.$$

We've found a recipe how to construct solutions of inhomogeneous evolution equations. Our heuristic derivation does not constitute a proof. But once we guess the correct formulas, we can check by direct computation that they indeed satisfies Eqs. (3), (4), and (5).

Assignments:

Read Chapter 3 of Strauss.

Hand-in (due Monday, October 24):

25. (a) Use the method of reflections to write the solution of the boundary-value problem

$$\begin{array}{ll} u_t = u_{xx}\,, & 0 < x < 1, t > 0\,, \\ u_x(0,t) = u_x(1,t) = 0\,, & t > 0\,, u(x,0) = \phi(x) \end{array}$$

as a series. Here, ϕ is a continuously differentiable function with $\phi'(0) = \phi'(1) = 0$.
(b) Does the series converge? In what sense?

26. If $u(x,t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$, prove the identity

$$u(x+h,t+k) + u(x-h,t-k) = u(x+k,t+h) + u(x-k,t-h)$$

for all x, t, h, and k. Sketch the quadrilateral Q in the x-t-plane whose vertices appear in the identity.