MAT 351: Partial Differential Equations
Assignment 4, due Oct. 17, 2016

The **wave equation**

$$u_{tt} = c^2 \Delta u$$

is the prototype of a **hyperbolic equation**. It is used to describe the propagation of vibrations in an elastic medium such as a string or a membrane, as well as the propagation of electromagnetic waves in vacuum. In many cases, it is an approximation to a nonlinear wave equation that is valid for small amplitudes. The parameter c is called the **wave speed**. The wave equation is invariant under **time reversal**: If $u(x,t)$ solves the wave equation, then so does then $u(x,-t)$.

In one dimension, the general solution of the wave equation has the form $u(x,t) = f(x-ct) + g(x+ct)$. It can be expressed in terms of its initial ampliture $\phi(x) = u(x,0)$ and initial velocity $\psi(x) = u_t(x,0)$ by **d'Alembert's formula**

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct) + \phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy\,.$$

The lines $x - ct = const.$ and $x + ct = const.$ are called the **characteristics** of the equation. The region between the characteristics that emanate from a point (x_0, t_0) with $t < t_0$ is called the **domain of dependence**, and the corresponding region with $t > t_0$ is called the **domain of influence**; together, they form the (solid) **light cone**. D'Alembert's formula implies that waves have **finite speed of propagation**, i.e., no signal can travel faster than at speed c. This is closely related with the idea of causality.

An important feature of the wave equation is that **energy is conserved**:

$$\frac{d}{dt}\int \frac{1}{2}|u_t(x,t)|^2 + \frac{c^2}{2}|u_x|^2\, dx = 0$$

(assuming that the integral is finite). Here, the first term in the integrand represents kinetic energy, and the second term represents the potential energy of the wave. Conservation of energy is useful for proving that the initial-value problem is well-posed in a sutiable space of square integrable functions.

Solutions of the wave equation can be oscillatory (like $u(x,t) = \cos(ct)\cos x$) or traveling waves (given by $u(x,t) = f(x \pm ct)$); in higher dimensions, we will see examples of focusing (wave packets that are initially far apart collide in a small area) and dispersion (a wave packet separates into pieces that run off in different directions).

The **diffusion equation**

$$u_t = k\Delta u$$

is the prototype of a **parabolic** equation. is used to describe the diffusion of a chemical substance by Brownian motion, or the flow of heat in a body. A variant of this equation appears in ther Black-Scholes equation for the price of a stock option. The parameter $k > 0$ is called the **diffusion constant** or the **volatility**. The diffusion equation is not time reversible; we will see that the initial-value problem is well-posed forward in time, but the backwards heat equation is ill-posed in most commonly used function spaces.

The most striking property of the heat equation is the **maximum principle**: If we consider a solution on a region $x \in D$, $t_0 \leq t \leq T$, then its maximal value is assumed either at at the initial time ($t = t_0$), or at the boundary of D. The **strong maximum principle** says that the maximum *cannot* be asssumed at some point (x_1, t_1) with x_1 in the interior of the interval and $t_1 > t_0$ unless u is constant up to time t_1. (We have proved the maximum principle on one dimension but not the strong maximum principle.) One consequence is that the solution of the heat equation with nonnegative data remains nonnegative. In fact, unless the data are zero, the solution will immediately become positive everywhere — the diffusion equation has **infinite speed of propagation** !

Typical solutions of the diffusion equation on the real line spread out and decay over time. An example of this is the function $u(x,t) = (4\pi kt)^{-\frac{1}{2}} e^{-\frac{x^2}{4kt}}$. One manifestation of this is that **energy decreases**:

$$\frac{d}{dt} \int \frac{1}{2} u^2(x,t)\, dx \leq 0$$

(assuming that the integral is finite). This is useful for understanding well-posedness and anayzing the long-time behavior.

Hand-in (due Monday, October 17):

12. Use the graphical method to sketch the solution of **Burger's equation** $u_t + uu_x = 0$ with initial values

$$u(x,0) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

that satisfies both the Rankine-Hugoniot jump condition and the entropy condition. Be sure to indicate the location of the shocks and rarefaction waves.

13. Consider the initial-value problem for the wave equation with initial amplitude $\phi(x) = 0$ and initial velocity $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the profile of the solution $u(x,t)$ as a function of x for $t = j\frac{a}{2c}$, $j = 0, \ldots, 5$.

14. Consider the wave equation $u_{tt} - c^2 u_{xx} = 0$ with initial values $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$. If both ϕ and ψ are odd in x, prove that u is odd in x.

15. Solve $u_{xx} - 3u_{xt} - 4u_{tt}$ with initial values $u(x,0) = x^2$, $u_t(x,0) = e^x$.
Hint: Factor the differential operator into two first order operators, as we did for the wave equation.

16. The PDE for **damped string** is given by $u_{tt} - c^2 u_{xx} + ru_t = 0$, where $r > 0$ is a parameter related to friction. Let $u(x,t)$ be a solution of this equation for $-\infty < x < \infty$ (i.e., for an infinitely long string) and assume that $u(x,t) \to 0$ as $x \to \pm\infty$. What is the energy for this equation? Prove that energy decreases with time.

17. Solve the diffusion equation on the real line with initial condition $\phi(x) = 1$ for $|x| < a$, and $\phi(x) = 0$ for $|x| \geq a$.