MAT 351: Partial Differential Equations
Assignment 12, due January 23, 2017

Summary

By definition, the Green's function of a domain D can be constructed (for every fixed x) by

$$G(x,y) = \Phi(x-y) - h(y)\,,$$

where Φ is the fundamental solution of Laplace's equation, given by

$$\Phi(x) = \begin{cases} \frac{1}{2\pi}\log|x|\,, & \text{dimension } n=2\,, \\ -\frac{1}{4\pi|x|}\,, & n=3\,, \end{cases}$$

and h is a harmonic function such that $h(y) = \Phi(x-y)$ whenever y lies in the boundary of D. (Of course, h depends on x as well). It turns out that the Green's function is uniquely determined by these properties. Moreover, G is symmetric ($G(x,y) = G(y,x)$. It can be used to solve the Poisson problem $\Delta u = f$ on D with Dirichlet boundary conditions $u|_{\partial D} = g$ by the formula

$$u(x) = \int_D G(x,y) f(y)\, dy + \int_{\partial D} u(y) \nabla_y G(x,y)\, dS(y)\,.$$

The function $K(x,y) = \nabla_y G(x,y)$ that appears in the boundary integral is called the **Poisson kernel**. It is defined for $x]inD, y \in \partial D$.

There are only few domains where the Green's function can be computed explicitly. The two most important ones are the upper half-space and the unit ball in $\mathbb{R}^n$. For these, we can use a **reflection principle** to find the harmonic function h.

- **Upper half-space:** Let $D = \{x \in \mathbb{R}^3 \mid x_3 > 0\}$. For $x \in D$, we define its reflection at the boundary $\{x_3 = 0\}$ by $\bar{x} = (x_1, x_2, -x_3)$, and set

  $$h(y) = \Phi(\bar{x} - y) = \frac{1}{4\pi}\big((x_1-y_1)^2 + (x_2-y_2)^2 + (x_3+y_3)^2\big)^{-\frac{1}{2}}\,.$$

  Clearly, h is harmonic in y on the entire positive half-space (since $\bar{x}$ lies in the negative half-space). If $y_3 = 0$, then $h(x,y) = \Phi(x-y)$, because in that case $|\bar{x} - y| = |x-y|$. So the Green's function is given by

  $$G(x,y) = \frac{1}{4\pi}\left(\frac{1}{|\bar{x}-y|} - \frac{1}{|x-y|}\right)\,, \quad x,y \in D\,,$$

  and the Poisson kernel for D is given by $K(x,y) = \frac{1}{2\pi}\frac{x_3}{|x-y|^3}$.

- **Unit ball:** Let $D = \{x \in \mathbb{R}^3 \mid |x| < 1\}$. For $x \in D$, we define its reflection at the unit sphere by $\bar{x} = \frac{x}{|x|^2}$. A quick computation shows that

  $$|\bar{x} - \bar{y}|^2 = \frac{|x-y|^2}{|x|^2|y|^2}\,,$$

in particular, if $y \in \partial D$, then $|\bar{x} - y| = \frac{|x-y|}{|x|}$. For $x \in D$, the function

$$h(y) = \Phi(|x| \cdot |\bar{x} - y|) = -\frac{1}{4\pi|x| \cdot |\bar{x} - y|}$$

is clearly harmonic in y on D (since $\bar{x}$ lies outside D), and its boundary values agree with those of $\Phi(x - y)$. So the Green's function is given by

$$G(x, y) = \frac{1}{4\pi} \left(\frac{1}{|x| \cdot |\bar{x} - y|} - \frac{1}{|x - y|} \right) .$$

For the Poisson kernel, we obtain $K(x, y) = \frac{1}{4\pi} \frac{1-|x|^2}{|x-y|^2}$, where $x \in \mathbb{R}^3$ and $|y| = 1$.

In both cases, we have found a formula for the solution of Poisson's equation. Note that the Green's function is negative, and the Poisson kernel positive, in agreement with the maximum principle.

Hand-in (due Monday, January 23):

50. Find the Green's function for the Laplacian

    (a) for the positive quadrant in $\mathbb{R}^2$;

    (b) for the upper half of the unit ball in $\mathbb{R}^3$.

    Hint: Recall the fundamental solution of the heat equation, and use reflections.

51. How many linearly independent polynomials of degree k are there in three variables? How many linearly independent *harmonic* polynomials of degree k are there? (*Hint:* Consider the Laplacian as a linear transformation that maps polynomials of degree k to polynomials of degree $k - 2$. You may assume that this map is onto.)

    Remark: The restriction of the harmonic polynomials to the unit sphere are called the **spherical harmonics**. They are widely used to solve rotationally symmetric problems in Theoretical Physics (you may have seen them in connection with angular momentum in Quantum Mechanics).

52. (a) Determine the dimension of the space of homogeneous polynomials of degree ℓ in n variables.
    Hint: Establish a one-to-one correspondence between monomials $x_1^{k_1} \cdots x_n^{k_n}$ and binary strings of length $\ell + n - 1$.

    (b) Compute the dimension of the space of spherical harmonics in n variables.
    Hint: Use the recursion formula for harmonic polynomials.

    (c) Assuming that Δ maps the space of homogeneous polynomials of degree ℓ *onto* the space of homogeneous polynomials of degree $\ell - 2$, check that your formulas from (a) and (b) are consistent with the rank-nullity theorem of linear algebra.