MAT 351: Partial Differential Equations
Assignment 10, due December 5, 2016

Summary

We continue our study of harmonic functions in the plane. By **Poisson's formula**, the solution of Laplace's equation on a ball of radius a

$$u_{xx} + u_{yy} = 0\,, \qquad (x^2 + y^2 < a)\,,$$

with Dirichlet boundary condititons $u(a\cos\theta, a\sin\theta) = g(\theta)$ is given by

$$u(r\cos\theta, r\sin\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2}\, g(\phi)\, d\pi\,.$$

(You showed in Problem 24 that the solution is unique. We have derived this formula by separation of variables.)

The properties of harmonic functions can be understood by analogy with holomorphic functions. Holomorphic functions in two real variables are in fact precisely the real parts of holomorphic functions, and their properties follow directly from the corresponding properties of holomorphic functions. In particular, Poisson's formula on the disc follows from Cauchy's integral formula. The analogy persists in higher dimensions, but the properties require different proofs.

Here are a few consequences of Poisson's formula:

1. **Mean value property.** If u is harmonic on a domain $D \subset \mathbb{R}^n$, then its value at a point x equals its average over any ball $B_r(x)$ contained in D; it also agrees with the average over the boundary sphere $\partial B_r(x)$.

2. **Strong maximum principle.** If a harmonic function u on D assumes its maximum or minimum in the interior of D, then u is constant.

   (In two dimensions, this follows from the open mapping principle for holomorphic functions.)

3. **Liouville's theorem.** If a harmonic function on $\mathbb{R}^2$ is bounded, then it is constant.

4. **Smoothness.** Every harmonic function is smooth, in fact, real-analytic on the interior of its domain.

5. **Unique continuation.** If two harmonic functions agree on an open set, they agree on the intersection of their domains.

6. A continuous function on a domain D that has the mean value property is harmonic.

   (In two dimensions, this follows from Morera's theorem.)

7. **Completeness of the standard Fourier basis.** The functions $(2\pi)^{-1/2}(e^{-kx})_{k\in\mathbb{Z}}$ form a complete orthonormal system of $L^2(-\pi, \pi)$.

Assignments:

Read Chapter 6.3 of Strauss.

Hand-in (due Monday, December 5):

41. Suppose that u is a harmonic function in the disk $D = \{r < 2\}$ in two dimensions, and that $u = 2016 + (\sin\theta)^{17}$ for $r = 2$. Without computing the solution, find
(a) the maximum of u on D;
(b) the value of u at the origin;
(c) the integral of u over the disk.

42. Solve $u_{xx} + u_{yy} = 0$ in the disk $(r < a)$ with the boundary condition $u = \sin^3\theta$.

43. Derive Poisson's formula for the exterior of the unit disc $(r > 1)$.

44. Let D be the unit disc in the plane, and denote by D_+ its intersection with the half-space $y > 0$. Let u be a harmonic function D_+ that is continuous on the closure $\overline{D_+}$. Assume that u vanishes on the flat part of the boundary $\{(x, 0) \mid -1 \leq x \leq 1\}$, and extend it to a function $\tilde{u}$ on the whole disc by odd reflection,

$$\tilde{u}(x,y) = \begin{cases} u(x,y)\,, & (x,y) \in \overline{D}\,, y \geq 0 \\ -u(x,y) & (x,y) \in \overline{D}, y \leq 0\,. \end{cases}$$

Prove that $\tilde{u}$ is harmonic on D, in two ways:

(a) Show directly that $\Delta\tilde{u}(x,y) = 0$ when $y = 0$.
Note: You need to assume here that the second derivatives of u are continuous on $\overline{D}_+$.

(b) Identify $\tilde{u}$ as the solution of a suitable boundary-value problem.
Hint: You will need to use uniqueness twice.