APM 351: Differential Equations in Mathematical Physics
Test 2, March 16, 2011

(Five problems; 20 points each. Time: 2 hours.)

1. (a) Assume that f is a smooth function that minimizes the Rayleigh quotient

$$\frac{\int_{-1}^{1}(1+x^2)\,|f'(x)|^2\,dx}{\int_{-1}^{1}|f(x)|^2\,dx}$$

among *all* smooth functions on $[-1,1]$. Write down the **Sturm-Liouville** eigenvalue problem that f solves.

(b) Let λ, μ be two different eigenvalues of this Sturm-Liouville problem, and let f, g be the corresponding eigenfunctions. Prove that f and g are *orthogonal*.

(c) It is known that the eigenfunction corresponding to the lowest eigenvalue of a Sturm-Liouville problem is always strictly positive. Prove that the lowest eigenvalue is *simple*, i.e., the corresponding eigenspace is one-dimensional.

(d) Conclude that the eigenfunction corresponding to the lowest eigenvalue is *even*.

2. (a) What are the defining properties of a **Green's function** for a domain $U \subset \mathbb{R}^2$?

(b) Let $G(x,y)$ be the Green's function for U. How can G be used to solve Laplace's equation

$$\begin{cases} -\Delta u = 0\,, & \text{in } U\,, \\ u = g\,, & \text{on } \partial U \ ? \end{cases}$$

(c) Construct the Green's function for the Laplacian on the complement of the *unit disc*

$$\{(x_1,x_2)\in\mathbb{R}^2 \ \mid x_1^2+x_2^2>1\}\,.$$

Please justify why your construction yields the required properties!

3. (a) State the strong maximum principle for the **heat equation** $u_t = \Delta u$ on $\mathbb{R}^d$.

(b) Use the maximum principle to show that if the initial-value problem

$$\begin{aligned} u_t &= \Delta u, && (x\in U) \\ u(x,t) &= 0, && (x \in \partial U) \\ u(x,0) &= g(x) \end{aligned}$$

has a solution, then it must be *unique*.

(c) Let f be a bounded function on $\mathbb{R}$ with compact support. Verify that

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy$$

solves the heat equation for $t>0$ and $x\in\mathbb{R}$.

(d) If f is continuous, except for a single jump at $x = a$, prove that

$$\lim_{t \to 0} u(a,t) = \frac{1}{2}\{f(a^-) + f(a^+)\}.$$

Here, $f(a^-)$ and $f(a^+)$ are the limits of f as $x \to a$ from the left and right hand side, respectively.

4. Consider a solution u of the **wave equation**

$$\begin{cases} u_{tt} = \Delta u\,, & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = \phi(x), u_t(x,0) = 0\,, & x \in \mathbb{R}^n\,. \end{cases}$$

Assume that ϕ is smooth, bounded, and vanishes for $|x| > 1$.

(a) In dimension $n = 1, 2, 3$, where does u have to vanish? (A sketch would be useful.)

(b) State *Huygens' principle.*

5. [(20pts] (a) Construct a basis for the **spherical harmonics** (harmonic polynomials) of degree four in three variables (x, y, z). Briefly explain your method.

(b) What is the dimension of the space of *all* homogenenous polynomials of degree 4 in x, y, z?

(c) Argue by dimension-counting that every homogeneous polynomial of degree four can be written as

$$P = P_4 + (x^2 + y^2 + z^2)P_2 + (x^2 + y^2 + z^2)^2\,,$$

where P_4 and P_2 are harmonic polynomials.

(d) Use this to solve

$$\begin{aligned} \Delta u &= 0, & (x^2 + y^2 + z^2 < 1) \\ u &= (2x^2 + y^2)(x^2 + y^2 + z^2), & (x^2 + y^2 + z^2 = 1)\,. \end{aligned}$$

Useful formulas.

- The fundamental solution of **Laplace's equation** on $\mathbb{R}^2$ is $G_0(x) = -\frac{1}{2\pi}\log|x|$.
- The fundamental solution of the **heat equation** $u_t = \Delta u$ in $\mathbb{R}^n$ is $\Phi(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$.
- **Kirchhoff's formula:** The solution of the three-dimensional wave equation $u_{tt} = \Delta u$ with initial values (ϕ, ψ) is given by

$$u(x_0, t_0) = \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi t_0}\int_{|x-x_0|=t_0} \phi(x)\, dS(x)\right] + \frac{1}{4\pi t_0}\int_{|x-x_0|=t_0} \psi(x)\, dS(x)\,.$$

- The inversion at the unit sphere in $\mathbb{R}^n$, given by $x \mapsto \bar{x} = \frac{x}{|x|^2}$, satisfies $|\bar{x} - \bar{y}| = \frac{|x-y|}{|x||y|}$.