UNIVERSITY OF TORONTO
Faculty of Arts and Science

APRIL-MAY 2011 EXAMINATIONS
APM351 Y1Y Differential Equations in Mathematical Physics

Examiner: Professor Almut Burchard

Time: 3 hours. No calculators or other aids allowed.
Please try **all six** *problems; total 100 points.*

1. [15pts] (a) Solve the initial-value problem

$$2u_x + 3xu_y = 0\,, \quad u(0,y) = g(y)\,.$$

(b) Sketch a few **characteristics**. Does your solution exist on the entire plane $\mathbb{R}^2$? Is it unique? Why?

(c) Suppose we try to solve the same PDE with initial values $u(x,0) = g(x)$, what goes wrong?

2. [15pts] (a) Define what it means for a problem to be **well-posed**.

(b) Let f be a smooth function of two variables. Is the PDE $u_{xx} - u_{yy} = f(x,y)$ well-posed on the unit square $0 < x < 1, 0 < y < 1$ with Dirichlet boundary conditions

$$u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0\ ?$$

If no, how should the boundary conditions be modified to obtain a well-posed problem?

(c) Same questions for the PDE $u_{xx} + u_{yy} = f(x,y)$;

(d) same questions for $u_{xy} = f(x,y)$;

(e) same questions for $u_y - u_{xx} = f(x,y)$.

Briefly explain your reasoning in each case. If you recognize the equation, please give its name.

3. [15pts] Let f be a smooth 2π-periodic function with $\int_0^{2\pi} f(x)\,dx = 0$.

(a) Use the Fourier series representation to show that $||f||_{L^2} \le ||f'||_{L^2}$.
(This is called the **Poincaré inequality**.)

(b) Use Schwarz' inequality to show that there exists a constant C such that

$$\sup_{x\in[0,2\pi]} |f(x)| \le C||f'||_{L^2}\,.$$

4. [15pts] (a) Write down a solution of the initial-value problem for the **heat equation**

$$u_t = u_{xx}\,, \quad u(x,0) = \phi(x)\,,$$

where ϕ is a nonnegative smooth function with compact support, not identically zero.

(b) Argue that $0 \le u(x,t) \le \sup \phi$ for all x, t, and that $\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} \phi(x)\,dx$.

(c) Does the heat equation have finite speed of propagation? Please justify your answer.

5. [20pts] Consider the eigenvalue problem for the Neumann Laplacian on the two-dimensional unit disk,

$$\begin{cases} -\Delta u = \lambda u\,, & |x| < 1\,, \\ \frac{\partial u}{\partial n} = o\,, & |x| = 1\,. \end{cases}$$

(a) Use **Separation of Variables** to split the problem into two eigenvalue problems.

(b) Solve the angular problem.

(c) Consider, as a special case, solutions of the form $u(r)$ (i.e., purely radial solutions where the angular part is constant). Express these solutions in terms of the zeroth order Bessel function $J_0(r)$. Please explain your reasoning!

6. [20pts] (a) Define the delta **distribution** δ on $\mathbb{R}^n$, and briefly explain its meaning.

(b) Let $D \subset \mathbb{R}^2$ be a smooth bounded domain, and let $G(x,y;\xi,\eta)$ be its Green's function. Explain the meaning of the equation

$$\Delta G(x,y;\xi,\eta) = \delta(x-y, \xi-\eta)\,.$$

How does this related to the defining properties of the Green's function?

(c) Construct the Green's function for the Laplacian on the positive quadrant

$$\{(x,y) \in \mathbb{R}^2 \ \mid x, y > 0\}\,.$$

Please justify why your construction yields the required properties!

Useful formulas.

- The fundamental solution of **Laplace's equation** on $\mathbb{R}^2$ is $G_0(x,y) = -\frac{1}{4\pi}\log(x^2+y^2)$.
- The fundamental solution of the **diffusion equation** $u_t = \Delta u$ in $\mathbb{R}^n$ is $\Phi(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$.
- The **Bessel functions** $J_n(r)$ are the unique bounded solutions of Bessel's equation

$$J'' + \frac{1}{r}J' + \left(1 - \frac{n^2}{r^2}\right) J = 0$$

for $n = 1, 2, \ldots$. Each J_n is a smooth function that changes sign at an infinite sequence of zeroes $z_{n,1}, z_{n,2}, \cdots \to \infty$, separated by an infinite sequence of local extrema $p_{n,1}, p_{n,2}, \ldots$.