MAT 1600 : Probability I
Assignment 6, due October 28, 2020

1. *(Panchenko 2.2.12)* Suppose that $(X_n)_{n\geq 1}$ are i.i.d. such that $\mathbb{E}|X_1| < \infty$ and $\mathbb{E}X_1 = 0$. If (c_n) is a bounded sequence of real numbers, prove that
$$\frac{1}{n}\sum_{i=1}^{n} c_i X_i \to 0 \quad \text{almost surely.}$$
Hint: Either group the coefficients c_i according to their values, or examine the proof of the strong law of large numbers.

2. For $0 < p < 1$, let $\mathbb{P}_p$ be the probability measure on $\Omega = \{0,1\}^{\mathbb{N}}$ (the space of binary sequences) defined by repeatedly tossing a coin with bias p.
   (a) Prove that $\mathbb{P}_p$ and $\mathbb{P}_q$ are mutually singular when $p \neq q$.
   (Use the Strong Law of Large Numbers.)
   (b) Conclude that the measures μ_p on $I := (0,1)$, obtained by pushing $\mathbb{P}_p$ forward with $\omega \mapsto X(\omega) := \sum_{i\geq 1} \omega_i 2^{-i}$ are mutually singular ($\mu_p \perp \mu_q$) when $p \neq q$).
   (c) Prove that for each p, the distribution function
$$F_p(x) := \mathbb{P}_p(X \leq x) = \mu_p((0,x])$$
   is continuous in x, that is, X is continuously distributed with respect to $\mathbb{P}_p$.

   Remark. We have seen (Panchenko p.14) that $\mu_{\frac{1}{2}}$ agrees with Lebesgue measure on I, in other words, $F_{\frac{1}{2}} = x$.

3. *(Panchenko 2.2.9)* If $(X_n)_{n\geq 1}$ are i.i.d. random variables that are not a.s. constant, then $\mathbb{P}(X_n \text{ converges}) = 0$.

4. *(Panchenko 2.3.3)*
Given any sequence of random variables $(Y_i)_{i\geq 1}$, let A be the event that $\lim_{i\to\infty} Y_i$ exists. Show that for any $\varepsilon > 0$ there exist integers $k, n, m \geq 1$ such that
$$A_{k,n,m} = \bigcap_{n\leq i<j\leq m} \left\{|Y_j - Y_i| \leq \tfrac{1}{k}\right\}$$
satisfies $\mathbb{P}(A \Delta A_{k,n,m}) \leq \varepsilon$. (Use the continuity of measure and the Cauchy criterion.)

5. *Simple random walk (Panchenko 2.3.1)*
   For $n \geq 1$, let $S_n = \sum_{i=1}^{n} X_i$, where $\{X_i\}_{i \geq 1}$ are i.i.d. random variables taking values ± 1, with $\mathbb{P}(X_i = 1) = p$.

   (a) Show that the event $A_0 := \{S_n = 0 \text{ infinitely often}\}$ has probability 0 or 1.

   (b) Panchenko suggests to use the Hewitt-Savage 0-1 law.
   Is A_0 a tail event? symmetric? Why/why not?

6. *Simple random walk, cont'd (Panchenko 2.3.2)*
   In the setting of the previous problem, show that ...

   (a) ... if $p \neq \frac{1}{2}$ then $\mathbb{P}(A_0) = 0$;

   (b) ... if $p = \frac{1}{2}$ then $\mathbb{P}(A_0) = 1$.

   Hint: Consider the events $A_- := \{\liminf S_n \leq -\frac{1}{2}\}$ and $A_+ := \{\limsup S_n \geq \frac{1}{2}\}$.