MAT 1600 : Probability I
Assignment 2, due September 30, 2020

1. *(Durrett 2.1.11)* Find four random variables taking values in $\{-1, 0, 1\}$ so that any three are independent but all four are not. (*Hint:* Consider products of independent random variables.)

2. *The Vitali-Hahn-Saks theorem*
   Let $(\mathbb{P}_n)_{n\geq 1}$ be a sequence of probability measures on $(\Omega, \mathcal{A})$. Suppose that
   $$m(A) := \lim_{n\to\infty} \mathbb{P}_n(A)$$
   exists for each $A \in \mathcal{A}$. Prove that m is a measure. *Remark:* In this generality, the theorem is hard (though it has an elementary proof). I will post an outline on Friday.

3. *(Durrett 1.3.8)* Let X, Y be random variables, and let $\sigma(X)$ be the σ-algebragenerated by X. Show that Y is measurable with respect to $\sigma(X)$, if and only if $Y = f(X)$ for some measurable function $f : \mathbb{R} \to \mathbb{R}$. (*Hint:* First consider the case where Y is a simple function.)

4. *Continuous functions (Panchenko 1.3.1)*
   Does the set $C([0, 1])$ belong to the cylindrical σ-algebra $\mathcal{A}$ on $\mathbb{R}^{[0,1]}$?
   (*Panchenko's hint:* An exercise in Section 1 might be helpful.)

5. *Uncorrelated but not independent (Durrett 2.1.7)* Consider $\Omega = (0, 1)$, with the standard Borel σ-algebra and Lebesgue measure. For positive integers $n \neq m$, show that the random variables $X(\omega) = \sin(2\pi n\omega)$ and $Y(\omega) = \sin(2\pi m\omega)$ satisfy $E(XY) = EX = EY = 0$. But X and Y are not independent.

6. Let $X \sim \text{Poiss}(\lambda)$ and $Y \sim \text{Poiss}(\mu)$ be independent random variables with Poisson distribution, as defined in Example 1.1.1.

   (a) Determine the distribution of $Z = X + Y$.

   Conversely, let $N \sim \text{Poiss}(\lambda)$ be a Poisson-distributed random variable. If $N = n$, toss a biased coin n times and let N_1 be the number of Heads you get.
   Then $P(N_1 = k \big| N = n) = \binom{n}{k} p^k (1-p)^{n-k}$, where p is the bias.

   (b) Argue that $N_0 = N - N_1$ and N_1 are independent, and determine their distribution. (Avoid excessive computation.)