MAT 1600 : Probability I
Assignment 1, due September 23, 2020

1. *Monotone limits*
   Let $\mathcal{A}$ be an algebra. Suppose that $\mathcal{A}$ is closed under countable *increasing* unions, i.e., $\bigcup_{j=1}^{\infty} B_j \in \mathcal{A}$ whenever $B_j \in \mathcal{A}$ and $B_j \subset B_{j+1}$ for each $j = 1, 2, \ldots$.

   Prove that $\mathcal{A}$ is a σ-algebra, i.e., $\mathcal{A}$ is in fact closed under general countable unions.

2. *(Durrett 1.1.6)* A set $A \subset \{1, 2, \ldots\}$ is said to have **asymptotic density** θ if

$$\lim_{n\to\infty} \frac{1}{n} |A \cap \{1, 2, \ldots, n\}| = \theta .$$

   Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists.

   (a) Is $\mathcal{A}$ a σ-algebra? an algebra?

   (b) Is θ additive? σ-additive on $\mathcal{A}$?

3. *Section property (Lieb & Loss, p. 8)*

   (a) Let A be a Borel set in $\mathbb{R}^2$. Prove that, for every $y \in \mathbb{R}$, the cross section

$$A(y) = \{x \in \mathbb{R} : (x, y) \in A\}$$

   is a Borel set. (*Hint:* Consider the collection of sets whose cross sections are Borel, and use Dynkin's theorem.)

   (b) The corresponding statement fails for Lebesgue measurable sets. Briefly explain why!

4. *Tail sum formula (b-Panchenko, Problem 1.2.5)*
   For any nonnegative random variable $X \geq 0$, and any $p > 0$, prove that

$$\mathbb{E}(X^n) = \int_0^\infty pt^{p-1}\, \mathbb{P}(X > t)\, dt .$$

5. *Uniformization*

   (a) *(Panchenko 1.2.1)* If a random variable X has continuous c.d.f. $F(t)$, show that the random variable $F(X)$ is uniform on $[0,1]$, i.e., the law of $F(X)$ is Lebesgue measure on $[0,1]$.

   (b) *(Panchenko 1.2.2)* Conclude that $\int F(t)\,dF(t) = \frac{1}{2}$.

6. (a) *Inclusion-exclusion (Durrett 1.6.9)*
   For any collection of events $A_1, \ldots, A_n$, show that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\emptyset \neq F \subset \{1,\ldots,n\}} (-1)^{|F|+1} \mathbb{P}\left(\bigcap_{i\in F} A_i\right) .$$

   Here, $|F|$ denotes the number of elements of the set F.

   Hint: Expand the indicator function

$$I_{\bigcup A_i} = 1 - \prod (1 - I_{A_i})$$

   and take expectations. (Induction over n is a hassle.)

   (b) *The forgetful secretary*
   Consider the uniform probability measure on S_n, the set of permutations of $\{1,\ldots,n\}$. Find the probability that a random permutation has no fixed point,

$$p_n := \mathbb{P}\left(\pi(i) \neq i \text{ for all } i = 1,\ldots,n\right) ,$$

   and compute $p = \lim p_n$. (*Hint:* Take $A_i = \{\pi \in S_n : \pi(i) = i\}$.)