MAT 1600 : Probability I
Assignment 3, due October 5, 2016

13. *(Durrett 1.6.14)* Let $X \geq 0$ be a nonnegative random variable. Without assuming that $E(1/X) < \infty$, show that

$$\lim_{y\to\infty} yE(1/X; X > y) = 0\,, \qquad \lim_{y\to 0} yE(1/X; X > y) = 0\,.$$

14. *(Durrett 2.1.11)* Find four random variables taking values in $\{-1, 0, 1\}$ so that any three are independent but all four are not. (*Hint:* Consider products of independent random variables.)

15. *Uncorrelated but not independent (Durrett 2.1.7)* Consider $\Omega = (0, 1)$, with the standard Borel σ-algebra and Lebesgue measure. For positive integers $n \neq m$, show that the random variables $X(\omega) = \sin(2\pi n\omega)$ and $Y(\omega) = \sin(2\pi m\omega)$ satisfy $E(XY) = EX = EY = 0$. But X and Y are not independent.

16. A standard deck of cards contains 52 cards. Each card is uniquely characterized by its face value (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King) and its suit (Spades=$\spadesuit$, Hearts=$\heartsuit$, Diamonds=$\diamondsuit$, Clubs=$\clubsuit$). Suppose you are dealt a hand of five cards. Let N be the number of Hearts you are holding.

    (a) Compute $P(N \geq 3)$, the probability that you are holding at least three Hearts.

    (b) Compute the mean and variance of N. (*Hint:* Use indicator functions.)

17. *Sum of independent Poisson variables (Durrett 2.1.14)*
    The **Poisson** distribution with parameter $\lambda > 0$ is given by

$$P(Z = k) = e^{-\lambda}\frac{\lambda^k}{k!} \quad \text{for } k = 0, 1, \dots\,.$$

We say that $X = \text{Poisson}(\lambda)$. Use the convolution formula to show that if $X = \text{Poisson}(\lambda)$ and $Y = \text{Poisson}(\mu)$ are independent, then $X + Y = \text{Poisson}(\lambda + \mu)$.

18. *Probability spaces for the coin toss (see Durrett 2.1.18)*
Let Ω be the unit interval $(0,1)$ equipped with the Borel σ-algebra and Lebesgue measure.

(a) For $\omega \in (0,1)$, define $Y_n(\omega)$ as the n-th digit in the binary representation of ω, that is,

$$Y_n(\omega) = 1_{[2^n\omega] \text{ is even}} ,$$

where $[t]$ denotes the largest integer less or equal to t. (When ω has more than one binary representation, choose the one that terminates in zeroes.)

Show that $Y_1, Y_2, \ldots$ are independent with $P(Y_n = 0) = P(Y_n = 1) = \frac{1}{2}$. Thus, Y_n can be used to represent the outcome of the n-th toss of a fair coin.

(b) On the other hand, Kolmogorov's theorem yields a measure P on $\{0,1\}^{\mathbb{N}}$ (endowed with the product σ-algebra $\mathcal{F}$ indiced by the uniform distribution on $\{0,1\}$) that does much the same job for the coordinate projections $X_1, X_2, \ldots$.

To compare the two constructions, consider the function $G : (0,1) \to \{0,1\}^{\mathbb{N}}$ defined by

$$G(\omega) = (Y_n(\omega))_{n \geq 1} .$$

Show that G is measurable and one-to-one, and that its range has full measure in $\{0,1\}^{\mathbb{N}}$. Thus G pushes the Borel σ-algebra on $(0,1)$ forward to a σ-algebra that contains $\mathcal{F}$. Do the two σ-algebras agree? If yes, why? Otherwise, what is their precise relationship?