MAT 1600 : Probability I
Assignment 2, due September 28, 2016

7. *The section property for Borel sets*
   Let A be a Borel set in $\mathbb{R}^2$. Prove that, for every $y \in \mathbb{R}$, the cross section
   $$A(y) = \{x \in \mathbb{R} : (x, y) \in A\}$$
   is a Borel set. (*Hint:* Consider the collection of sets whose cross sections are Borel.)

8. *(Durrett 1.3.8)* Let X, Y be random variables, and let $\sigma(X)$ be the σ-field generated by X. Show that Y is measurable with respect to $\sigma(X)$, if and only if $Y = f(X)$ for some measurable function $f : \mathbb{R} \to \mathbb{R}$. (*Hint:* First consider the case where Y is a simple function.)

9. (a) *Inclusion-exclusion (Durrett 1.6.9)*
   For any collection of events $A_1, \dots, A_n$, show that
   $$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\emptyset \neq F \subset \{1,\dots,n\}} (-1)^{|F|+1} P\left(\bigcap_{i\in F} A_i\right) .$$
   Here, $|F|$ denotes the number of elements of the set F.

   Hint: Expand the indicator function
   $$1_{\bigcup A_i} = 1 - \prod (1 - 1_{A_i})$$
   and take expectations. (Induction over n is a hassle.)

   (b) *The forgetful secretary*
   Consider the uniform probability measure on S_n, the set of permutations of $\{1, \dots, n\}$. Find the probability that a random permutation has no fixed point,
   $$p_n := P\left(\pi(i) \neq i \text{ for all } i = 1, \dots, n\right) ,$$
   and compute $p = \lim p_n$. (*Hint:* Take $A_i = \{\pi \in S_n : \pi(i) = i\}$.)

10. *Chebyshev's inequality is and is not sharp (Durrett 1.6.2)*

    (a) For fixed $a > 0$, find a random variable X such that Eq. (1.6.1) holds with equality,
    $$P(|X| \geq a) = \frac{E(X^2)}{a^2} .$$

    (b) On the other hand, if X is a random variable with $0 < E(X^2) < \infty$, then
    $$\lim_{a\to\infty} a^2 P(|X| \geq a) = 0 .$$

11. Use Jensen's inequality to give an alternative prove of Hölder's inequality: If $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ are Hölder dual exponents, then

$$E|XY| \leq ||X||_p \, ||Y||_q \, ,$$

in particular, the left hand side is finite whenever the right hand side is.

Hint: In the special case $||Y||_q = 1$, use Y to define a probability measure; then rescale to get the general statement.

12. *The Vitali-Hahn-Saks theorem*
Let $(P_n)_{n\geq 1}$ be a sequence of probability measures on $(\Omega, \mathcal{F})$. Suppose that

$$m(A) := \lim_{n\to\infty} P_n(A)$$

exists for each $A \in \mathcal{F}$. Prove that m is a measure.