MAT 1001 / 458 : Real Analysis II
Final Exam, April 13, 2020

(Instructor: Burchard. Six problems; 20 points each. 3 hours, no aids allowed)

Please be brief but justify your answers, citing relevant theorems.

1. State ...

   (a) ...the Riesz-Thorin interpolation theorem (for $1 < p, q < \infty$);

   (b) ...the closed graph theorem;

   (c) ...the uniform boundedness principle;

   (d) ...the Banach-Alaoglu theorem (say how it applies to L^p).

   Remember to give the assumptions!

2. *(Dual characterization of the norm.)*
   Let $L : L^p \to L^q$ be a bounded linear transformation. Justify the formula

$$||L|| = \sup_{||f||_p = ||g||_{q'} = 1} \int (Lf) g \, dy$$

   for $1 \le p, q < \infty$. Here, q' is the Hölder dual exponent to q.

3. *(An application of the Hahn-Banach theorem.)*
   Let X be a normed vector space, and let $S = \{x \in X : \ ||x|| = 1\}$ be its closed unit sphere.

   (a) If Y is a proper closed subspace of X, then for every $\varepsilon > 0$ there is $x \in S$ such that $\mathrm{dist}(x, Y) \ge 1 - \epsilon$.

   (b) Deduce that X is finite-dimensional if and only if the unit ball of X is compact.

4. *(Convolution operators on S^1.)*
   Let K be a 2π-periodic function that is integrable over $(-\pi, \pi)$. If f is 2π-periodic and square integrable, define

$$(Lf)(x) := \int_{-\pi}^{\pi} K(x - y) f(y) \, dy \, ,$$

   that is, $Lf = K * f$.

(a) For each integer k, show that $e_k(x) = e^{ikx}$ is an eigenfunction L, and determine the corresponding eigenvalue λ_k.

(b) Show that L is a bounded linear operator on $L^2(-\pi, \pi)$, with norm $||L|| \leq \sup_k |\lambda_k| \leq ||K||_{L^1}$.

(c) Prove that L is compact (using, for example, the Riemann-Lebesgue lemma.)

5. *(Algebra property of Sobolev spaces.)*
For $s \in \mathbb{R}$, we have defined (in Folland Section 9.3) the spaces

$$H_s := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; (1 + |\xi|^2)^{\frac{s}{2}} \hat{f} \in L^2(\mathbb{R}^n) \right\} ,$$

with the norm given by $||u||_{H_s} := \left(\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi \right)^{\frac{1}{2}}$. Here, $\mathcal{S}$ is the Schwarz space, and $\mathcal{S}'$ is its dual. As discussed in class, for integer values of s this space agrees with $W^{s,2}(\mathbb{R}^n)$ (you are not asked to prove this.)

Assume $s > \frac{n}{2}$**.** You will show that the product of two functions in H_s is again in H_s.

(a) Prove that there exists a constant $C_1 > 0$ (depending on s and n) such that

$$||\hat{u}||_{L^1} \leq C_1 ||u||_{H_s} \qquad \text{for all } u \in \mathcal{S} .$$

(b) Let $u, v \in \mathcal{S}$. Write down the Fourier transform of their product, $\widehat{uv}$.

(c) Deduce that

$$||uv||_s^2 \leq 2C_2 C_1^2 \, ||u||_{H_s}^2 \, ||v||_{H_s}^2 \qquad \text{for all } u, v \in \mathcal{S} .$$

You may use without proof that there is a constant $C_2 > 0$ (depending on s) such that

$$(1 + |\xi|^2)^s \leq C_2 \big((1 + |\xi - \eta|^2)^s + (1 + |\eta|^2)^s \big) \qquad \text{for all } \xi, \eta \in \mathbb{R}^n .$$

(d) Conclude that $u, v \in H_s \Rightarrow (uv) \in H_s$. Please explain and justify your conclusion!

6. *(The direct method.)*
Consider the problem of minimizing

$$\mathcal{E}(u) := \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 + f(x) u(x) \, dx$$

over $u \in W^{1,2}(\mathbb{R}^3)$. Here, $f \in L^{\frac{6}{5}}$ is a given function, and $W^{1,2}$ is the Sobolev space of L^2-functions whose distributional gradient also lies in L^2.

(a) Apply a Sobolev inequality to the gradient term to obtain a lower bound on $\mathcal{E}$.

(b) Let $(u_k)_{k \geq 1}$ be a minimizing sequence, i.e., $\mathcal{E}(u_k) \to \inf \mathcal{E}$ as $k \to \infty$.
Prove that this sequence is bounded in $W^{1,2}$.

(c) After passing to a subsequence, there exist L^2 functions $u^*, v_1^*, \ldots, v_n^*$ such that

$$u_k \rightharpoonup u^* , \qquad \partial_{x_i} u_k \rightharpoonup v_i^* \quad (i = 1, \ldots, n)$$

weakly in L^2. (Why?) Prove that $v^* = \nabla u^*$.

(d) Conclude that $\mathcal{E}$ attains its minimum at u^*.