MAT 1001 / 458 : Real Analysis II
Final Exam, April 8, 2015

(Instructor: Burchard. Six problems; 20 points each. 3 hours, no aids allowed)

Please be brief but justify your answers, citing relevant theorems.

1. Please state ...

   (a) ... Bessel's inequality;

   (b) ... the open mapping theorem;

   (c) ... the Krein-Milman theorem;

   (d) ... the isoperimetric inequality.

2. Let L be a compact linear operator on an infinite-dimensional Banach space X.
   Prove, from the definitions:

   (a) L is bounded;

   (b) L cannot have a bounded inverse.

3. Let $L^2 = L^2(-\pi, \pi)$ be the space of 2π-periodic square integrable functions on the real line. Given $f \in L^2$ and $0 \leq r \leq 1$, define a function $P_r f \in L^2$ by

$$P_r f(x) = \sum_{k \geq 0} \hat{f}(k) r^k e^{ikx} .$$

   (a) Show that the series converges absolutely for $r < 1$.

   (b) For $r < 1$, find a 2π-periodic function H_r such that

$$P_r f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_r(x-y) f(y)\, dy .$$

   (c) Prove that

$$\lim_{r \to 1, r<1} ||P_r f - P_1 f||_2 = 0 \quad \text{for all } f \in L^2 .$$

   (d) However, P_r does not converge to P_1 in the operator norm.

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4. True or False? Please provide reasons and missing assumptions (as needed).

(a) If $F \in \mathcal{D}'$ is a distribution on $\mathbb{R}^d$, then

$$\partial_{x_i}\partial_{x_j}F = \partial_{x_j}\partial_{x_i}F\,.$$

(b) If (f_n) is a sequence in $L^2(-\pi,\pi)$, and $(u_k)_{k\geq 1}$ is an orthonormal basis, then

$$f_n \rightharpoonup f \text{ weakly in } L^2 \quad \Longleftrightarrow \quad \lim_{n\to\infty}\langle f_n, u_k\rangle = \langle f, u_k\rangle \quad \text{for all } k \geq 1\,.$$

5. (a) Show that the rationals, $\mathbb{Q}$, cannot be written as a countable intersection of open sets in $\mathbb{R}$.

(b) Doesn't this contradict the outer regularity of Lebesgue measure? Please explain!

6. (a) Define the Sobolev space $W^{1,p}(\mathbb{R}^d)$.

(b) *Morrey's inequality* states that, for suitable values of p, d, and α, there exists a constant $C = C(p,d,\alpha)$ such that

$$\sup_{x\neq y}\frac{f(x)-f(y)}{|x-y|^\alpha} \leq C||\nabla f||_p$$

for every smooth function f with compact support in $\mathbb{R}^d$. Derive a necessary condition on the relation between d, p, and α. (*Hint:* Scaling)

It turns out that Morrey's inequality indeed holds, so long as $0 < \alpha \leq 1$. Moreover

$$\sup_{x\in\mathbb{R}^d}|f(x)| \leq C||f||_{W^{1,p}}$$

for every smooth function f with compact support. (You may use this without proof.)

(c) Show that the identity map on C_c^∞ extends to a bounded linear transformation from $W^{1,p}$ to $C(\mathbb{R}^d)$, the space of bounded continuous functions on $\mathbb{R}^d$ endowed with the sup norm.

— *end* —