MAT 1001 / 458 : Real Analysis II
Final Exam, April 9, 2014

(Instructor: Burchard. Six problems; 20 points each. 3 hours, no aids allowed)

Please be brief but justify your answers, citing relevant theorems.

1. Please state ...

   (a) ... the Open Mapping Theorem;

   (b) ... the Riesz Representation Theorem for L^p (with $1 < p < \infty$);

   (c) ... the Spectral Theorem.

2. Let X be a Banach space.

   (a) Define the *dual space*, X^*, and its norm.

   (b) Define *weak convergence* in X.

   (c) Suppose that X is *reflexive*, i.e., $X^{**} = X$.
   Let $x \in X$ be a point, let $V \subset X$ be a closed subspace, and let

   $$d(x, V) := \inf_{y \in V} ||x - y||$$

   be the distance from x to V. Prove that there exists a point $v \in V$ such that

   $$d(x, V) = ||x - v|| \, .$$

   Hint: Consider a *minimizing sequence* (v_n) in V, i.e., a sequence with

   $$\lim ||x - v_n|| = d(x, V) \, ,$$

   and extract a weakly convergent subsequence.

3. (a) Define the *Fourier transform* $\hat{f}$ of an integrable function f on $\mathbb{R}^d$.

   (b) Give an example of a function on $\mathbb{R}^d$ that lies in L^2 but not in L^1. How do you compute the Fourier transform of such a function? Please justify why your procedure works!

   (c) The Fourier transform $\mathcal{F} : f \mapsto \hat{f}$ defines a linear transformation from L^1 to L^∞.
   Is it continuous? injective? surjective?

— continued on next page —

4. (a) Define the Dirac δ-distribution (the point mass at the origin) on $\mathbb{R}^d$.
Briefly explain why it lies in $\mathcal{S}'$, the space of tempered distributions (the dual of Schwartz space).

(b) Determine its Fourier transform $\hat{\delta}(k)$. Use it to "solve" the equation

$$\Delta\Phi = \delta$$

on $\mathbb{R}^d$ by finding a formula for $\hat{\Phi}(k)$. Argue that $\Phi \in \mathcal{S}'$.
Note: Do not try to find a formula for $\Phi(x)$.

(c) If f is a Schwarz function, prove that the function $u = \Phi * f$ solves Poisson's equation

$$\Delta u = f\ .$$

Be sure to explain what you mean by "solve".

Remark: It turns out that Φ is represented by a function, the *fundamental solution* of Laplace's equation, given by $\Phi(x) = C_x|x|^{2-d}$ in dimension $d > 2$. (You are not asked to prove this.)

5. (a) Let f be a smooth function on $\mathbb{R}^d$ with compact support. Define the translation of f by a vector $v \in \mathbb{R}^d$ as $\tau_v f(x) = f(x - v)$. Prove that, for $p \in [1, \infty)$,

$$||f - \tau_v f||_p \leq ||\nabla f||_p |v|\ .$$

Hint: Use that $f(x) - f(x - v) = \int_0^1 \nabla f(x - tv) \cdot v\, dt$, and integrate over $x \in \mathbb{R}^d$.

(b) Argue that the inequality extends to all $f \in W^{1,p}$.

(c) Setting $p = 1$, conclude that $W^{1,1} \subset L^q$ for all q with $1 \leq q \leq d/(d - 1)$.
Hint: Use the Sobolev inequality.

6. Let $C \subset \mathbb{R}^{d+1}$ be a compact convex set of positive measure. Denote by

$$C(h) = \big\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \,\big|\, (x_1, \ldots, x_d, h) \in C\big\}$$

its *cross section* at height h. Clearly, each cross section is again compact and convex.
Let $\gamma(h) = m(C(h)$ be its d-dimensional Lebesgue measure.

(a) Prove that $\log \gamma$ is concave in h, i.e.,

$$\log \gamma((1 - t)h_0 + th_1)) \leq (1 - t) \log \gamma(h_0) + t\gamma(h_1)$$

for all h_0, h_1 with $\gamma(h_i) > 0)$ and all $t \in (0, 1)$.
Hint: Brunn-Minkowski. (A sketch will help.)

(b) If, in addition, C is symmetric under the reflection $x \mapsto -x$, conclude that γ assumes its maximum at $h = 0$, i.e., the central cross section has the largest area.

— *end* —