MAT 1001 / 458 : Real Analysis II
Midterm Test, March 9, 2022

(Four problems; 20 points each. Time: 1 hour 50 minutes.)

Please be brief but justify your answers, citing relevant theorems.

1. For $t > 0$ and $x, y \in \mathbb{R}^n$, let $K(x, y; t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$.
   Given an integrable function f on $\mathbb{R}^n$, set
   $$u(x; t) := \int_{\mathbb{R}^n} f(y)K(x, y; t)\, dy\,, \qquad t > 0, x \in \mathbb{R}^n\,.$$
   (a) Fix $t > 0$. Prove that the function $u(\cdot; t)$ lies in $L^p(\mathbb{R}^n)$ for every $p \in [1, \infty]$, and give a bound on its norm. (*Hint:* First find $\int_{\mathbb{R}^n} K(x, y; t)\, dy$ and $\max_{x,y} K(x, y; t)$.)
   (b) Determine a sufficient condition on p such that $u \in L^p(\mathbb{R}^n \times (0, 1))$.

2. Let ϕ be a proper convex lower continuous functional on a Banach space X.
   (a) Define the Fenchel-Legendre transform of ϕ, and state the Fenchel-Moreau theorem.
   (b) Let $(x_n)_{n\geq 1}$ be a sequence in X. If $x_n \rightharpoonup a$ weakly, prove that
   $$\phi(a) \;\leq\; \liminf_{n\to\infty} \phi(x_n)\,.$$
   (*Remark:* There are many different ways to argue this; I suggest to use Part (a).)

3. Let $f \in C^\infty(\mathbb{R})$ be a smooth real-valued function. Suppose that that for each $x \in \mathbb{R}$ there exists some integer $n = n_x \in \mathbb{N}$ such that $f^{(n)}(x) = 0$. Show that there exists a nonempty open interval $(a, b) \subset \mathbb{R}$ such that the restriction of f to (a, b) is a polynomial.

4. Let X be a reflexive Banach space, with dual space X^*.
   (a) Let $V \subset X$ be a closed subspace. Prove that V is reflexive.
   How is V^* related to X^*?
   (b) State the Banach-Alaoglu theorem.
   (c) Let $(x_n)_{n\geq 1}$ be a bounded sequence in X. Explain in detail how to prove that (x_n) has a weakly convergent subsequence.
   Hint: This will require several lines of argument. Start by considering the separable, closed subspace $V = \overline{\text{span}\,\{x_n, n = 1, 2, \ldots\}}$.