MAT 1001 / 458 : Real Analysis II
Midterm Test, March 4, 2020

(Four problems; 20 points each. Time: 2 hours.)

Please be brief but justify your answers, citing relevant theorems.

1. *Equivalent norms.* Let X be a Banach space with norm $||\cdot||_1$. Suppose $||\cdot||_2$ is another norm that also makes X into a Banach space. You will prove that that there are constants C_1, C_2 such that

$$\text{for all } x \in X: \qquad ||x||_2 \le C_1 ||x||_1 \text{ and } ||x||_1 \le C_2 ||x||_2 \,.$$

   (a) Prove the claim in the special case where $||x||_2 \le ||x||_1$.

   (b) Prove the claim in general.
   Hint: Compare both norms with their sum $||\cdot|| = ||\cdot||_1 + ||\cdot||_2$.
   You will need to verify that this is a norm that also makes X into a Banach space.

2. In class, we have defined the Fourier coefficients of a 2π-periodic integrable function by

$$\hat{f}(k) = \frac{1}{2\pi} \int_{i\pi}^{\pi} f(x) e^{-ikx} \, dx \,, \qquad (k \in \mathbb{Z}) \,.$$

   Let p, q be dual Hölder exponents.

   (a) For each $p \in [1, 2]$, the map $f \mapsto (\hat{f}(k))_{k \in \mathbb{Z}}$ defines a linear transformation from $L^p(-\pi, \pi)$ (the space of p-integrable functions) to ℓ^q (the space of q-summable sequences). Show that this transformation is bounded, and give a bound on its norm.

   (b) In the special case $p = 1$, show that this map is not surjective, by proving that

$$\lim_{k \to \pm\infty} \hat{f}(k) = 0$$

   for all $f \in L^1(-\pi, \pi)$. (*Hint:* L^2 is dense in L^1.)

3. Construct two subsets M, N of the unit interval such that

   (a) $M \cup N = [0, 1]$;

   (b) M is a countable union of nowhere dense subsets, and N has Lebesgue measure zero.

   (*Hint:* Consider the rationals in $[0, 1]$.)

4. Let A be a bounded linear operator on a Hilbert space.

   (a) If $x_n \rightharpoonup x$ weakly in H, prove that $Ax_n \rightharpoonup Ax$ weakly in H.

   (b) If $x_n \rightharpoonup x$ weakly, and $y_n \to y$ (strongly) in H, prove that $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

   Assume that A has the following three properties:

   - *Hermitian:* $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in H$;
   - *positive definite:* $\langle Ax, x \rangle > 0$ for all $x \neq 0$;
   - *compact:* If (x_n) is a bounded sequence, then (Ax_n) has a convergent subsequence.

   Define $\overline{\lambda} := \sup_{||x||=1} \langle Ax, x \rangle$.

   (c) Prove that the supremum is attained, i.e., there exists $x^* \in H$ with $||x^*|| = 1$ such that $\langle Ax^*, x^* \rangle = \overline{\lambda}$. (Consider a maximizing sequence (x_n).)