MAT 1001 / 458 : Real Analysis II
Midterm Test, March 5, 2014

(Four problems; 20 points each. Time: 2 hours.)

Please be brief but justify your answers, citing relevant theorems.

1. Let (x_n) be a sequence in a Hilbert space $\mathcal{H}$.

   (a) Define *weak convergence* ($x_n \rightharpoonup$) and *strong convergence* ($x_n \to x$) in $\mathcal{H}$.

   (b) Assume that $x_n \rightharpoonup x$. Prove that

$$x_n \to x \quad \text{(strongly in } \mathcal{H}) \quad \Longleftrightarrow \quad \lim ||x_n|| = ||x|| \, .$$

2. (a) Consider the operator $T : L^2([0,1]) \to L^2([0,1])$ defined by $(Tf)(x) = xf(x)$. Prove (from the definitions) that T is bounded and self-adjoint, but not compact.

   (b) Show that T has no eigenvalues. Why does this not contradict the Spectral Theorem? Please state the theorem, and give a short explanation.

3. (a) Given two functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, with $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, consider their convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy \, .$$

   Prove that the integral exists for each $x \in \mathbb{R}^n$, and that $f * g$ is bounded.

   (b) Furthermore, $f * g$ is continuous, and $\lim_{|x|\to\infty} f * g(x) = 0$.

4. Let $f\ \mathbb{R}_+ \to \mathbb{R}$ be a continuous function with the property that

$$\lim_{n\to\infty} f(nx) = 0$$

   for all $x > 0$. (Here, n runs over the non-negative integers). Prove that

$$\lim_{x\to\infty} f(x) = 0 \, .$$

   Hint: Fix $\delta > 0$, and consider $A_n = \{x : \forall m \geq n : |f(mx)| \leq \delta\}$.