Chapter 5

The Fourier Transform

The Fourier transform is a versatile tool in analysis, much loved by analysts, scientists and engineers. (In fact, in our definition below we use the engineer's convention about the placement of 2π, which eliminates the annoyance of having to multiply integrals by 2π.) The virtue of the Fourier transform is that it converts the operations of differentiation and convolution into multiplication operations. In particular it allows us to define the relativistic operators $\sqrt{-\Delta}$ and $\sqrt{-\Delta + m^2}$ and the space $H^{1/2}(\mathbb{R}^n)$ in Chapter 7. Some references for the Fourier transform are [Hörmander], [Rudin, 1991], [Reed–Simon, Vol. 2], [Schwartz] and [Stein–Weiss].

5.1 DEFINITION OF THE L^1 FOURIER TRANSFORM

Let f be a function in $L^1(\mathbb{R}^n)$. The Fourier transform of f, denoted by $\widehat{f}$, is the function on $\mathbb{R}^n$ given by

$$\widehat{f}(k) = \int_{\mathbb{R}^n} e^{-2\pi\imath(k,x)} f(x)\, \mathrm{d}x \tag{1}$$

where

$$(k, x) := \sum_{\imath=1}^{n} k_\imath x_\imath .$$

The following algebraic properties are the main motivation for studying the Fourier transform. They are very easy to prove.

$$\text{The map } f \mapsto \widehat{f} \text{ is linear in } f, \tag{2}$$

$$\widehat{\tau_h f}(k) = e^{-2\pi i(k,h)}\widehat{f}(k), \quad h \in \mathbb{R}^n, \tag{3}$$

$$\widehat{\delta_\lambda f}(k) = \lambda^n \widehat{f}(\lambda k), \quad \lambda > 0, \tag{4}$$

where τ_h is the translation operator, $(\tau_h f)(x) = f(x-h)$, and δ_λ is the scaling operator, $(\delta_\lambda f)(x) = f(x/\lambda)$.

Two other easy to prove facts are

$$\widehat{f} \in L^\infty(\mathbb{R}^n) \quad \text{and} \quad \|\widehat{f}\|_\infty \leq \|f\|_1, \tag{5}$$

$$\widehat{f} \text{ is a continuous (and hence measurable) function.} \tag{6}$$

The latter follows from dominated convergence. In fact it is part of the **Riemann–Lebesgue lemma**, which also states that $\widehat{f}(k) \to 0$ as $|k| \to \infty$ (see Exercise 2). Note that $\|\widehat{f}\|_\infty$ equals $\|f\|_1$ whenever f is any nonnegative function; in that case

$$\|\widehat{f}\|_\infty = \widehat{f}(0) = \int f = \|f\|_1.$$

Recall from Sect. 2.15 that the convolution of two functions f and g, both in $L^1(\mathbb{R}^n)$, is given by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y. \tag{7}$$

By Fubini's theorem $f * g \in L^1(\mathbb{R}^n)$, and also by Fubini's theorem

$$\begin{aligned}\widehat{(f * g)}(k) &= \int_{\mathbb{R}^n} e^{-2\pi i(k,x)} \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} e^{-2\pi i(k,y)}g(y) \int_{\mathbb{R}^n} e^{-2\pi i(k,(x-y))}f(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \widehat{f}(k)\widehat{g}(k).\end{aligned} \tag{8}$$

The following is an important example.

5.2 THEOREM (Fourier transform of a Gaussian)

For $\lambda > 0$, *denote by* g_λ **the Gaussian function** *on* $\mathbb{R}^n$ *given by*

$$g_\lambda(x) = \exp[-\pi\lambda|x|^2] \tag{1}$$

for $x \in \mathbb{R}^n$. *Then*

$$\widehat{g}_\lambda(k) = \lambda^{-n/2}\exp[-\pi|k|^2/\lambda].$$

REMARK. This is a special case of Exercise 4.4.

PROOF. By 5.1(4) it suffices to consider $\lambda = 1$. Since

$$g_1(x) = \prod_{i=1}^{n}\exp[-\pi(x_i)^2],$$

it suffices to consider $n = 1$. By definition (since $g_1 \in L^1(\mathbb{R})$)

$$\widehat{g}_1(k) = \int_{\mathbb{R}} e^{-2\pi i(x,k)}\exp[-\pi x^2]\,\mathrm{d}x = g_1(k)f(k),$$

where

$$f(k) = \int_{\mathbb{R}}\exp[-\pi(x+ik)^2]\,\mathrm{d}x. \tag{2}$$

A simple limiting argument using the dominated convergence theorem allows us to differentiate (2) under the integral sign as many times as we like. Therefore $f \in C^\infty(\mathbb{R})$ and

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}k}(k) &= -2\pi i\int_{\mathbb{R}}(x+ik)\exp[-\pi(x+ik)^2]\,\mathrm{d}x\\ &= i\int_{\mathbb{R}}\frac{\mathrm{d}}{\mathrm{d}x}\exp[-\pi(x+ik)^2]\,\mathrm{d}x\\ &= i\exp[-\pi(x+ik)^2]\Big|_{-\infty}^{\infty} = 0,\end{aligned}$$

i.e., $f(k)$ is constant. But $f(0) = \int_{\mathbb{R}}\exp[-\pi x^2]\,\mathrm{d}x = 1$. ■

● The Fourier transform can be defined for functions for which 5.1(1) does not make sense. In particular, it is important for quantum mechanics to define $\widehat{f}$ for $f \in L^2(\mathbb{R}^n)$. One route to this definition goes via the Schwartz space $\mathcal{S}$ (which we will not discuss here). The method below uses only Theorem 2.16 (approximation by C^∞-functions). We begin by considering functions in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, which are dense in $L^2(\mathbb{R}^n)$.

5.3 THEOREM (Plancherel's theorem)

If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\widehat{f}$ is in $L^2(\mathbb{R}^n)$ and the following formula of Plancherel holds:

$$\|\widehat{f}\|_2 = \|f\|_2. \tag{1}$$

The map $f \mapsto \widehat{f}$ has a unique extension to a continuous, linear map from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ which is an **isometry**, *i.e., Plancherel's formula (1) holds for this extension. We continue to denote this map by $f \mapsto \widehat{f}$ (even if $f \notin L^1(\mathbb{R}^n)$).*

If f and g are in $L^2(\mathbb{R}^n)$, then Parseval's formula holds,

$$(f, g) := \int_{\mathbb{R}^n} \overline{f}(x) g(x)\, \mathrm{d}x = \int_{\mathbb{R}^n} \overline{\widehat{f}}(k) \widehat{g}(k)\, \mathrm{d}k = (\widehat{f}, \widehat{g}). \tag{2}$$

PROOF. For $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, the function $\widehat{f}(k)$ is bounded, by 5.1(5), and hence

$$\int_{\mathbb{R}^n} |\widehat{f}(k)|^2 \exp[-\varepsilon\pi|k|^2]\, \mathrm{d}k \tag{3}$$

is defined. Since $f \in L^1(\mathbb{R}^n)$, the function $\overline{f}(x) f(y) \exp[-\varepsilon\pi|k|^2]$ of three variables is in $L^1(\mathbb{R}^{3n})$. Using Fubini's theorem and Theorem 5.2 we can express (3) as

$$\begin{aligned} &\int_{\mathbb{R}^{3n}} \overline{f}(x) f(y) e^{2\pi i(k,(x-y))} \exp[-\varepsilon\pi k^2]\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}k \\ &\quad = \int_{\mathbb{R}^{2n}} \varepsilon^{-n/2} \exp\left[-\frac{\pi(x-y)^2}{\varepsilon}\right] \overline{f}(x) f(y)\, \mathrm{d}x\, \mathrm{d}y. \end{aligned} \tag{4}$$

Using Theorem 2.16 (approximation by C^∞-functions)

$$\varepsilon^{-n/2} \int_{\mathbb{R}^n} \exp\left[-\frac{\pi(x-y)^2}{\varepsilon}\right] f(y)\, \mathrm{d}y \to f(x)$$

in $L^2(\mathbb{R}^n)$ as $\varepsilon \to 0$, and hence (using Fubini's theorem again) (3) tends to $\int_{\mathbb{R}^n} |f(x)|^2\, \mathrm{d}x$. This shows that (3) is uniformly bounded in ε and the monotone convergence theorem therefore shows that $\widehat{f} \in L^2(\mathbb{R}^n)$ with

$$\|\widehat{f}\|_2 = \|f\|_2. \tag{5}$$

Now let f be in $L^2(\mathbb{R}^n)$ but not in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Since $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, there exists a sequence $f^j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $\|f - f^j\|_2 \to 0$. By (5) $\|\widehat{f^j} - \widehat{f^m}\|_2 = \|f^j - f^m\|_2$ and hence $\widehat{f^j}$

is a Cauchy sequence in $L^2(\mathbb{R}^n)$ that converges to some function in $L^2(\mathbb{R}^n)$, which we call $\widehat{f}$. It is obvious from (5) that $\widehat{f}$ does not depend on the choice of the sequence f^j. Moreover,

$$\|\widehat{f}\|_2 = \lim_{j\to\infty} \|\widehat{f^j}\|_2 = \lim_{j\to\infty} \|f^j\|_2 = \|f\|_2.$$

The continuity (in $L^2(\mathbb{R}^n)$) and the linearity of this map is left to the reader.

Relation (2) follows from (1) by **polarization**, i.e., the identity

$$(f, g) = \frac{1}{2}\{\|f + g\|_2^2 - i\|f + ig\|_2^2 - (1 - i)\|f\|_2^2 - (1 - i)\|g\|_2^2\}.$$

Applying (1) to each of these four norms yields (2). ■

5.4 DEFINITION OF THE L^2 FOURIER TRANSFORM

For each f in $L^2(\mathbb{R}^n)$, the $L^2(\mathbb{R}^n)$-function $\widehat{f}$ defined by the limit given in Theorem 5.3 is called the Fourier transform of f.

Theorem 5.3 is remarkable because it states that for any given $f \in L^2(\mathbb{R}^n)$ one can compute its Fourier transform $\widehat{f}$ by using any $L^1(\mathbb{R}^n)$-approximating sequence whatsoever and one always obtains, as an $L^2(\mathbb{R}^n)$ limit, a function $\widehat{f}$ which is independent of the approximation. Here are two examples with the index $j = 1, 2, 3, \ldots$:

$$\widehat{f^j}(k) = \int_{|x|<j} e^{-2\pi i(k,x)} f(x)\, dx, \tag{1}$$

$$\widehat{h^j}(k) = \int_{\mathbb{R}^n} \cos(|x|^2/j) \exp[-|x|^2/j] e^{-2\pi i(k,x)} f(x)\, dx. \tag{2}$$

The assertion is that there is an $L^2(\mathbb{R}^n)$-function $\widehat{f}$ such that $\|\widehat{f^j} - \widehat{f}\|_2 \to 0$, $\|\widehat{h^j} - \widehat{f}\|_2 \to 0$ and $\|\widehat{f^j} - \widehat{h^j}\|_2 \to 0$ as $j \to \infty$. No assertion is made that the sequences $\widehat{f^j}(k)$ and $\widehat{h^j}(k)$ converge for any k as $j \to \infty$. However, by Theorem 2.7 (completeness of L^p-spaces), there is always a subsequence $j(l)$ with $l = 1, 2, 3, \ldots$ such $\widehat{f^{j(l)}}(h)$ and $\widehat{h^{j(l)}}(k)$ converge for almost every $k \in \mathbb{R}^n$ to $\widehat{f}(k)$.

As we show next, the map $f \mapsto \widehat{f}$ is not just an isometry but it is, in fact, a **unitary transformation**, that is, an invertible isometry. The following is an explicit formula for the inverse.

5.5 THEOREM (Inversion formula)

For $f \in L^2(\mathbb{R}^n)$, we use definition 5.4 to define

$$f^\vee(x) := \widehat{f}(-x) \tag{1}$$

(*which amounts to changing i to $-i$ in* 5.1(1)). *Then*

$$f = (\widehat{f})^\vee. \tag{2}$$

(*Note that the right side is well defined by Theorem* 5.3.)

PROOF. For $f \in L^2(\mathbb{R}^n)$ the following formula holds:

$$\int_{\mathbb{R}^n} \widehat{g}_\lambda(y-x) f(y)\, \mathrm{d}y = \int_{\mathbb{R}^n} g_\lambda(k) \widehat{f}(k) e^{2\pi i (k,x)}\, \mathrm{d}k, \tag{3}$$

where $g_\lambda(k) = \exp[-\lambda\pi|k|^2]$ and hence $\widehat{g}_\lambda(y-x) = \lambda^{-n/2}\exp[-\pi|x-y|^2/\lambda]$. To verify (3), approximate f by a sequence of functions f^j in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. For each of these functions formula (3) follows by Fubini's theorem. By Theorem 5.3 (Plancherel's theorem) we know that $f^j \to f$ in $L^2(\mathbb{R}^n)$ implies that $\widehat{f^j} \to \widehat{f}$ in $L^2(\mathbb{R}^n)$. Because g_λ and $\widehat{g}_\lambda$ are in $L^2(\mathbb{R}^n)$ the integrals converge to those in (3), and thus (3) is established in the general case.

As $\lambda \to 0$ the left side of (3) tends to $f(x)$ in $L^2(\mathbb{R}^n)$ by Theorem 2.16 (approximation by C^∞-functions). Since $g_\lambda \widehat{f} \to \widehat{f}$ in $L^2(\mathbb{R}^n)$ as $\lambda \to 0$ (by dominated convergence), we know, on account of Theorem 5.3, that $(g_\lambda \widehat{f})^\vee \to (\widehat{f})^\vee$ in $L^2(\mathbb{R}^n)$. Equating the $\lambda \to 0$ limit of the two sides of (3) gives us (2). ■

5.6 THE FOURIER TRANSFORM IN $L^p(\mathbb{R}^n)$

The Fourier transform has been defined for $L^1(\mathbb{R}^n)$-functions (with range in $L^\infty(\mathbb{R}^n)$) and $L^2(\mathbb{R}^n)$-functions (with range in $L^2(\mathbb{R}^n)$). Can it be extended to some other $L^p(\mathbb{R}^n)$-space so that its range is in some $L^q(\mathbb{R}^n)$-space?

Let us recall the properties that have been proved so far.

$$f \in L^1(\mathbb{R}^n) \Rightarrow \widehat{f} \in L^\infty(\mathbb{R}^n) \quad \text{with} \quad \|\widehat{f}\|_\infty \le \|f\|_1, \tag{A}$$

but *the L^1 Fourier transform is not an invertible mapping* (i.e., not every $L^\infty(\mathbb{R}^n)$-function is the Fourier transform of some $L^1(\mathbb{R}^n)$-function; the constant function is an example).

$$f \in L^2(\mathbb{R}^n) \Rightarrow \widehat{f} \in L^2(\mathbb{R}^n) \quad \text{with} \quad \|\widehat{f}\|_2 = \|f\|_2 \tag{B}$$

and the Fourier transform is invertible with $f = (\widehat{f})^{\vee}$.

One way to extend the Fourier transform for $p < \infty$ would be to imitate the $L^2(\mathbb{R}^n)$ construction. The goal would then be to find a constant $C_{p,q}$ such that for every $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ the Fourier transform is in $L^q(\mathbb{R}^n)$ and satisfies

$$\|\widehat{f}\|_q \leq C_{p,q}\|f\|_p. \tag{1}$$

Using the continuity argument of Theorem 5.3 (and the density of $L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$) one can then extend the Fourier transform to all of $L^p(\mathbb{R}^n)$ and (1) will continue to hold.

The first remark is that q cannot be arbitrary, in fact q must be p' (with $1/p + 1/p' = 1$). This is a simple consequence of the scaling property 5.1(4); if $q \neq p'$, then $\|\widehat{f}\|_q/\|f\|_p$ can be made arbitrarily large—even for $f \in L^1(\mathbb{R}^n)$. The second remark is that counterexamples show that no bound of type (1) can hold when $p > 2$; see Exercise 9. When $1 \leq p \leq 2$, however, (1) is true, as the following theorem (which is usually called the **Hausdorff–Young inequality**) states.

5.7 THEOREM (The sharp Hausdorff–Young inequality)

Let $1 < p < 2$ and let $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then, with $1/p + 1/p' = 1$,

$$\|\widehat{f}\|_{p'} \leq C_p^n\|f\|_p \tag{1}$$

with

$$C_p^2 = [p^{1/p}(p')^{-1/p'}]. \tag{2}$$

Furthermore, equality is achieved in (1) if and only if f is a Gaussian function of the form

$$f(x) = A\exp[-(x, Mx) + (B, x)] \tag{3}$$

with $A \in \mathbb{C}$, M any symmetric, real, positive-definite matrix and B any vector in $\mathbb{C}^n$.

Using the construction in Theorem 5.3, together with (1), $\widehat{f}$ can be extended to all of $L^p(\mathbb{R}^n)$ but, in contrast to the $p = 2$ case, this map is not invertible, i.e., the map is not onto all of $L^{p'}(\mathbb{R}^n)$.

REMARK. The proof of Theorem 5.7 is lengthy and we shall not attempt to give it here. The shortest proof is probably the one in [Lieb, 1990]; the basic idea is similar to that in the proof of Theorem 4.2 (Young's inequality), but the details are more involved. Inequality (1) was first proved with $C_p = 1$ by [Hausdorff] and [W. H. Young] for Fourier *series* by using the Riesz–Thorin interpolation theorem (see [Reed–Simon, Vol. 2]). It was extended to Fourier integrals by [Titchmarsh] with $C_p = 1$. [Babenko] derived (2) as the sharp constant for $p' = 4, 6, 8, \ldots$ and [Beckner] proved (2) for all $1 < p < 2$. The fact that equality holds in (1) *only* when f is a Gaussian as in (3) was proved in [Lieb, 1990]. Note that $C_p = 1$ if $p = 1$ or $p = 2$, in agreement with our earlier results, but in those two cases there are many functions that give equality in (1); indeed *all* $L^2(\mathbb{R}^n)$-functions give equality when $p = 2$.

5.8 THEOREM (Convolutions)

Let $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$, *and let* $1 + 1/r = 1/p + 1/q$. *Suppose* $1 \le p, q, r \le 2$. *Then*

$$\widehat{f * g}(k) = \widehat{f}(k)\widehat{g}(k). \tag{1}$$

PROOF. By Young's inequality, Theorem 4.2, $f * g \in L^r(\mathbb{R}^n)$. By Theorem 5.7, $\widehat{f} \in L^{p'}(\mathbb{R}^n)$ and $\widehat{g} \in L^{q'}(\mathbb{R}^n)$, so $\widehat{f}\widehat{g} \in L^{r'}(\mathbb{R}^n)$ by Hölder's inequality. Since $h := f * g$ is in $L^r(\mathbb{R}^n)$, $\widehat{h} \in L^{r'}(\mathbb{R}^n)$ by Theorem 5.7. If both f and g are also in $L^1(\mathbb{R}^n)$, then (1) is true by 5.1(8). The theorem follows by an approximation argument that is left to the reader. ■

● The function $|x|^{2-n}$ on $\mathbb{R}^n$ with $n \ge 3$ is very important in potential theory (Chapter 9) and as the Green's function in Sect. 6.20. Hence, it is useful to know its 'Fourier transform', even though this function is not in any $L^p(\mathbb{R}^n)$ for any p. However, its action in convolution or as a multiplier on nice functions *can* be expressed easily in terms of Fourier transforms.

5.9 THEOREM (Fourier transform of $|x|^{\alpha-n}$)

Let f *be a function in* $C_c^\infty(\mathbb{R}^n)$ *and let* $0 < \alpha < n$. *Then, with*

$$c_\alpha := \pi^{-\alpha/2}\Gamma(\alpha/2), \tag{1}$$

$$c_\alpha(|k|^{-\alpha}\widehat{f}(k))^\vee(x) = c_{n-\alpha}\int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\,\mathrm{d}y\ . \tag{2}$$

REMARK. Since $f \in C_c^\infty(\mathbb{R}^n)$, the Fourier transform $\widehat{f}$ is a very nice function; it is in $C^\infty(\mathbb{R}^n)$ (it is analytic, in fact) and, as $|k| \to \infty$, it, and all its derivatives, decay faster than the inverse of any polynomial in k. (The verification of these two facts is recommended as an exercise using integration by parts and dominated convergence.) Therefore, the function $|k|^{-\alpha}\widehat{f}(k)$ is in $L^1(\mathbb{R}^n)$, and thus it has a Fourier transform. The function on the right side of (2) is well defined and is also in $C^\infty(\mathbb{R}^n)$, but it decays, as $|x| \to \infty$, only as $|x|^{\alpha-n}$ (in general). Thus, generally speaking, the right side of (2) is *not* in $L^p(\mathbb{R}^n)$ for any $p \le 2$, *unless* $\alpha < n/2$ and, therefore, it does not generally have a well-defined Fourier transform. Nevertheless, (2) is true.

PROOF. Our starting point is the elementary formula

$$c_\alpha |k|^{-\alpha} = \int_0^\infty \exp[-\pi |k|^2 \lambda]\, \lambda^{\alpha/2-1}\, \mathrm{d}\lambda. \tag{3}$$

Since $|k|^{-\alpha}\widehat{f}(k)$ is integrable, we have, by Fubini's theorem,

$$\begin{aligned} c_\alpha(|k|^{-\alpha}\widehat{f}(k))^\vee(x) &= \int_{\mathbb{R}^n} e^{2\pi i(k,x)} \left\{ \int_0^\infty \exp[-\pi|k|^2\lambda]\lambda^{\alpha/2-1}\,\mathrm{d}\lambda \right\} \widehat{f}(k)\,\mathrm{d}k \\ &= \int_0^\infty \left\{ \int_{\mathbb{R}^n} e^{2\pi i(k,x)} \exp[-\pi|k|^2\lambda]\widehat{f}(k)\,\mathrm{d}k \right\} \lambda^{\alpha/2-1}\,\mathrm{d}\lambda \\ &= \int_0^\infty \lambda^{-n/2}\lambda^{\alpha/2-1} \left\{ \int_{\mathbb{R}^n} \exp[-\pi|x-y|^2/\lambda] f(y)\,\mathrm{d}y \right\} \mathrm{d}\lambda \\ &= c_{n-\alpha} \int_{\mathbb{R}^n} |x-y|^{-n+\alpha} f(y)\,\mathrm{d}y. \end{aligned}$$

In the penultimate equation we have used Theorem 5.2 and the convolution theorem 5.8(1). The last equation holds by Fubini's theorem. ■

5.10 COROLLARY (Extension of 5.9 to $L^p(\mathbb{R}^n)$)

If $0 < \alpha < n/2$ *and if* $f \in L^p(\mathbb{R}^n)$ *with* $p = 2n/(n+2\alpha)$, *then* $\widehat{f}$ *exists* (*by Theorem* 5.7). *Moreover, with* c_α *defined in* 5.9(1), *the function*

$$g := c_{n-\alpha}|x|^{\alpha-n} * f$$

is an $L^2(\mathbb{R}^n)$*-function* (*by Theorem* 4.3 (*HLS inequality*)) *and hence has a Fourier transform* $\widehat{g}$.

Our new result is that the relation between $\widehat{g}$ *and* $\widehat{f}$ *is given by*

$$c_\alpha |k|^{-\alpha}\widehat{f}(k) = \widehat{g}(k). \tag{1}$$

Moreover,

$$c_{2\alpha}\int_{\mathbb{R}^n} |k|^{-2\alpha}|\widehat{f}(k)|^2\,\mathrm{d}k = c_{n-2\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \overline{f}(x) f(y)|x-y|^{2\alpha-n}\,\mathrm{d}x\,\mathrm{d}y. \tag{2}$$

REMARK. The case $\alpha = 1$ and $n \geq 3$ is especially important for potential theory (Chapter 9) and for the Green's function of the Laplacian (before 6.20). The right side of (2), without $c_{n-2\alpha}$, is twice the Coulomb potential energy of the 'charge distribution' f, 9.1(2).

PROOF. By Theorem 2.16 (approximation by C^∞-functions) we can find a sequence $f^1, f^2, \dots$ of functions in $C_c^\infty(\mathbb{R}^n)$ such that $f^j \to f$ strongly in $L^p(\mathbb{R}^n)$. By Theorem 4.3 (HLS inequality) the functions g and

$$g^j := |x|^{\alpha-n} * f^j$$

are in $L^2(\mathbb{R}^n)$; this follows from Fubini's theorem and the fact that, for $0 < \alpha < n$, $0 < \beta < n$ and $0 < \alpha + \beta < n$, we have

$$\begin{aligned}(|x|^{\alpha-n} * |x|^{\beta-n})(y) &:= \int_{\mathbb{R}^n} |z|^{\alpha-n}|y-z|^{\beta-n}\,\mathrm{d}z \\ &= \frac{c_{n-\alpha-\beta}\,c_\alpha\,c_\beta}{c_{\alpha+\beta}\,c_{n-\alpha}\,c_{n-\beta}}|y|^{\alpha+\beta-n},\end{aligned} \tag{3}$$

which can be verified by a tedious but instructive computation using 5.9(3).

Since $f^j \to f$, we have $\widehat{f^j} \to \widehat{f}$ in $L^q(\mathbb{R}^n)$ with $q = 2n/(n-2\alpha)$ (by Theorem 5.7). By the HLS inequality $g^j \to g$ in $L^2(\mathbb{R}^n)$, and hence $\widehat{g^j} \to \widehat{g}$ in $L^2(\mathbb{R}^n)$ (by Theorem 5.3 (Plancherel)). By Theorem 5.9, we also know that

$$\widehat{g^j}(k) = c_\alpha |k|^{-\alpha}\widehat{f^j}(k).$$

Our problem is to show that

$$\widehat{g}(k) = c_\alpha |k|^{-\alpha}\widehat{f}(k).$$

To do this, we pass to a subsequence so that $\widehat{g^j}(k) \to \widehat{g}(k)$ and $\widehat{f^j}(k) \to \widehat{f}(k)$ pointwise a.e. (by Theorem 2.7(ii) (completeness of L^p-spaces)). Thus,

$$\widehat{g}(k) = \lim_{j\to\infty} c_\alpha|k|^{-\alpha}\widehat{f^j}(k) = c_\alpha|k|^{-\alpha}\lim_{j\to\infty}\widehat{f^j}(k) = c_\alpha|k|^{-\alpha}\widehat{f}(k)$$

for almost every k. This proves (1).

Formula (2) is just an application of Plancherel's theorem to (1), together with Fubini's theorem and (3). ∎

Exercises for Chapter 5

1. Prove that the Fourier transform has properties 5.1(2), (3) and (4).

2. Prove the Riemann–Lebesgue lemma mentioned in Sect. 5.1, i.e., for $f \in L^1(\mathbb{R}^n)$, $\widehat{f}(k) \to 0$ as $|k| \to \infty$.

   ▶ *Hint.* 5.1(3) is useful.

3. Show that the definition of the Fourier transform for functions in $L^2(\mathbb{R}^n)$, given in Sect. 5.4, does not depend on the approximating sequence.

4. Show that the definition of the Fourier transform for functions in $L^2(\mathbb{R}^n)$ gives rise to a linear map $f \mapsto \widehat{f}$.

5. Complete the proof of Theorem 5.8, i.e., work out the approximation argument mentioned at the end of Sect. 5.8.

6. For $f \in C_c^\infty(\mathbb{R}^n)$ show that its Fourier transform $\widehat{f}$ is also in C^∞ (in fact $\widehat{f}$ is analytic). Show also that $g_a(k) := |\, |k|^a \widehat{f}(k)|$ is a bounded function for each $a > 0$.

7. Verify formula 5.10(3).

8. This concerns an example of an extension of Theorem 5.8 (convolution) to the case in which $r > 2$. Suppose that f and g are $L^2(\mathbb{R}^n)$. Then we know that $f * g \in L^\infty(\mathbb{R}^n)$ and $\widehat{f}\,\widehat{g} \in L^1(\mathbb{R}^n)$. Although $\widehat{f * g}$ may not be obviously well defined, show that 5.1(8) holds, nevertheless, in the sense of inverse Fourier transforms, i.e.,

$$f * g = (\widehat{f}\,\widehat{g})^\vee.$$

9. Verify that 5.6(1) cannot hold when $p > 2$ by considering Gaussian functions, as in 5.2(1), with $\lambda = a + ib$ and with $a > 0$.