MAT 1001 / 458 : Real Analysis II
Assignment 9, due March 29, 2022

1. Prove the *Hausdorff-Young inequality*

$$||\hat{f}||_q \leq ||f||_p\,, \quad 1 \leq p \leq 2$$

for a suitable value of q (depending on p). Here, $\hat{f}$ is the Fourier transform of f on $\mathbb{R}^n$.

2. *(Folland, Exercise 8.16.)* By the Riemann-Lebesgue lemma, the Fourier transform $\mathcal{F} : f \mapsto \hat{f}$ defines a bounded linear transformation from $L^1(\mathbb{R}^n)$ to the space

$$C_0(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{C} \,\middle|\, f \text{ continuous and } \lim_{|x|\to\infty} f(x) = 0 \right\}$$

with the topology of uniform convergence. Convince yourself that $C_0(\mathbb{R}^n)$ is complete.

Let $n = 1$, $t > 0$, and consider the function $f_t = \mathcal{X}_{[-1,1]} * \mathcal{X}_{[-t,t]}$.

(a) Show that $f_t \in C_0$ and $||f_t||_\infty \leq 2$. (Please make a sketch.)

(b) But $\lim_{t\to\infty} ||\hat{f}_t||_1 = \infty$, and likewise for $\check{f}_t$.

(c) Argue that $\mathcal{F} : L^1 \to C_0$ cannot be surjective.

3. *Heisenberg's uncertainty principle*

Let f be a complex-valued Schwarz function on $\mathbb{R}$. If $||f||_{L^2(\mathbb{R})} = 1$, show that

$$\left(\int_{\mathbb{R}} x^2\, |f(x)|^2\, dx \right) \left(\int_{\mathbb{R}} k^2 |\hat{f}(k)|^2\, dk \right) \geq \frac{1}{16\pi^2}\,.$$

What are the equality cases?

4. Solve the *heat equation*

$$\partial_t u = \Delta u\,, \quad (x \in \mathbb{R}^n, t > 0)$$

with initial values $u(x, 0) = f(x)$ for $x \in \mathbb{R}^n$, by deriving a differential equation for the Fourier transform $\hat{u}(k,t) = \int e^{-2\pi i k\cdot x} u(x,t)\, dx$. Here, $\Delta u = \sum_j \partial^2_{x_j} u$ is the Laplacian, and $f \in \mathcal{S}$ (the Schwarz space).

Don't forget to transform back ...

5. (*Fractional integration; Exercise 2.6.61 in Folland.*)
If f is continuous on $[0, \infty)$, for $\alpha > 0$ and $x \geq 0$ let

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt\,.$$

The map $f \mapsto I_\alpha f$ is linear and preserves positivity.

(a) *Semigroup property.* Show that $I_{\alpha+\beta} f = I_\alpha(I_\beta f)$ for all $\alpha, \beta > 0$.

(b) If n is a positive integer, show that $I_n(f)$ is an n-th order antiderivative of f, i.e., f can be recovered from $I_n f$ by differentiating n times.

(Nothing to hand in)

6. *Von Neumann's alternating projection theorem.* Let P_1 and P_2 be orthogonal projections onto closed subspaces V_1 and V_2 of a Hilbert space H, respectively, and let P be the orthogonal projection onto the intersection $V = V_1 \cap V_2$. You will show that $(P_1P_2)^n x$ converges to Px for all $x \in H$. (Please make a sketch!)

(a) Show that $P_1P_2x = x$ if and only if $x \in V$. (Consider $||P_1P_2x||^2$.)

(b) Prove that $||x - P_1P_2x||^2 \leq 2(||x||^2 - ||P_1P_2x||^2)$ for all $x \in H$.

(c) *Kakutani's lemma.* Let $x_n = (P_1P_2)^n x$. Show that $\lim ||x_n - x_{n+1}|| = 0$.

(d) Conclude that $\lim ||x_n - Px|| = 0$.

Remarks. See

[1] Netanyun and Solmon, MAA Monthly **113(7):** p.644-648, 2006 (p.645)

[2] Bauschke, Matouskova, and Reich, Nonl. Anal. **46:** p.715-738, 2004 (p.721)

The alternating projection theorem can be viewed as a special case of Trotter's formula:

$$e^{-t(A+B)} = \lim \left(e^{-\frac{t}{n}A} e^{-\frac{t}{n}B} \right)^n, \quad (t > 0)$$

for any pair of (possibly unbounded, non-commuting) positive semidefinite self-adjoint operators A, B on H. Here, $S : t \mapsto e^{-tA}$ is the semigroup generated by A (defined by solving the linear equation $\dot{x} + Ax = 0$ in H.)

In the special case of a projection, the associated semigroup is $S(t) := P$ for $t > 0$ and $S(0) = I$, and the semigroup property $S(s+t) = S(s)S(t)$ just says that $P^2 = P$.