MAT 1001 / 458 : Real Analysis II
Assignment 8, due March 22, 2022

1. *(Stein & Shakarchi, Exercise 4.32)*
   Prove that the operator $T : L^2([0,1]) \to L^2([0,1])$ defined by $(Tf)(x) = xf(x)$ is bounded and self-adjoint, but not compact. Moreover, T has no eigenvectors.

2. *(Stein & Shakarchi, Exercise 4.34)*
   Let H be a Hilbert space (assume for simplicity that H is separable). Prove the following variants of the spectral theorem:

   (a) *Simultaneous diagonalization of commmuting operators.* If A_1 and A_1 are two compact self-adjoint operators on H with $A_1 A_2 = A_2 A_1$, show that there exists an orthonormal basis for H consisting of eigenvectors for both A_1 and A_2.

   (b) *Spectral theorm for normal operators.* A linear operator T is called **normal**, if it commutes with its adjoint ($TT^\dagger = T^\dagger T$). Prove that if T is normal and compact, then it can be diagonalized.

   (c) *Spectral theorem for unitary operators.* Recall that an operator U is **unitary** if $UU^\dagger = I$. Prove that if U is unitary, and $U = \gamma I - T$ where T is compact, then U can be diagonalized.

3. *Hilbert-Schmidt operators*
   Let $K(x, y)$ be a complex-valued function in $L^2(\mathbb{R}^2)$. Set

$$Tf(x) = \int_{\mathbb{R}} K(x,y) f(y)\, dy\,.$$

   (a) Show that $f \mapsto Tf$ defines a bounded linear operator on $L^2(\mathbb{R})$.

   (b) Moreover, T is compact.

   (c) Find a formula for its adjoint, $T^\dagger$.

4. *The unit sphere is weakly dense in the unit ball (Folland 5.63).*
   Let H be an infinite-dimensional Hilbert space. Prove that ...

   (a) ...every orthonormal sequence in H converges weakly to zero;

   (b) ...for every a with $||a|| \leq 1$ there exists a sequence $(x_n)_{n\geq 1}$ with

$$||x_n|| = 1 \quad \text{(for all } n \geq 1)\,, \qquad x_n \rightharpoonup a \quad \text{(as } n \to \infty)\,.$$

5. Stereographic projection of S^1 (*Exercise 4.7.9 in Stein & Shakarchi*)

   Let $H_1 = L^2([-\pi, \pi])$ be the Hilbert space of functions $F(e^{i\theta})$ on the unit circle with inner product
   $$\langle F, G \rangle = \frac{1}{2\pi} F(e^{i\theta})\overline{G(e^{i\theta})}\, d\theta\,.$$
   Let H_2 be the space $L^2(\mathbb{R})$ with the usual inner product.

   (a) Using the mapping $x \mapsto \frac{i-x}{i+x}$ of $\mathbb{R}$ to the unit circle, show that the corresponding transformation $U : F \mapsto f$, with
   $$f(x) = \tfrac{1}{\sqrt{\pi}(i+x)} F\left(\tfrac{i-x}{i+x}\right)$$
   defines a unitary mapping of H_1 to H_2. What is its inverse?

   (b) Conclude that
   $$\left\{ \tfrac{1}{\sqrt{\pi}(i+x)} \ \left(\tfrac{i-x}{i+x}\right)^n \right\}_{n\in\mathbb{Z}}$$
   is an orthonormal basis of $L^2(\mathbb{R})$.

   (Not to be handed in.)

6. *Commuting projections (Stein & Shakarchi, Exercise 4.7.13*)

   Let H be a Hilbert space, let P_1 and P_2 be a pair of orthogonal projections onto closed subspaces S_1 and S_2, respectively.

   Prove that $P := P_1 P_2$ is an orthogonal projection, if and only if $P_1P_2 = P_2P_1$.
   In that case, what is the range of P?